Some preserving subordination and superordination results of certain integral operator

M. K. Aouf
Department of Mathematics, Faculty of Science, Mansoura University
Mansoura 35516, Egypt.
e-mail:mkaouf127@yahoo.com

T. M. Seoudy
Department of Mathematics, Faculty of Science, Fayoum University
Fayoum 63514, Egypt.
e-mail:tms00@fayoum.edu.eg

Abstract

In this paper, we obtain some subordination and superordination-preserving results of certain integral operator. Sandwich-type result is also obtained.

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1 Introduction

Let \( H(U) \) be the class of functions analytic in \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( H[a, n] \) be the subclass of \( H(U) \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \), with \( H_0 = H[0, 1] \) and \( H = H[1, 1] \). Let \( A(p) \) denote the class of all analytic functions of the form

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N}, \{1, 2, 3, \ldots \}; z \in U)
\]  

(1.1)

and let \( A(1) = A \). Let \( f \) and \( F \) be members of \( H(U) \). The function \( f(z) \) is said to be subordinate to \( F(z) \), or \( F(z) \) is said to be superordinate to \( f(z) \), if there exists a function \( \omega(z) \) analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 (z \in U) \), such that \( f(z) = F(\omega(z)) \). In such a case we write \( f(z) \prec F(z) \). If \( F \) is univalent, then \( f(z) \prec F(z) \) if and only if \( f(0) = F(0) \) and \( f(U) \subset F(U) \) (see [5] and [6]).
Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the first order differential subordination:
\[
\phi \left( p(z), zp'(z); z \right) \prec h(z),
\] (1.2)
then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi \left( p(z), zp'(z); z \right)$ are univalent in $U$ and if $p(z)$ satisfies first order differential superordination:
\[
h(z) \prec \phi \left( p(z), zp'(z); z \right),
\] (1.3)
then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [5] and [6]).

Motivated essentially by Jung et al. [2], Shams et al. [8] introduced the integral operator $I_p^\alpha : A(p) \to A(p)$ as follows (see also Aouf et al. [1]):
\[
I_p^\alpha f(z) = \frac{(p + 1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left( \log \frac{z}{t} \right)^{\alpha-1} f(t) dt, \quad (\alpha > 0; p \in \mathbb{N}),
\] (1.4)
and
\[
I_0^\alpha f(z) = f(z), \quad (\alpha = 0; p \in \mathbb{N}).
\] (1.5)
For $f \in A(p)$ given by (1.1), then from (1.4) we deduce that
\[
I_p^\alpha f(z) = z^p + \sum_{n=k}^{\infty} \left( \frac{p + 1}{n + p + 1} \right)^\alpha a_{p+n} z^{p+n}, \quad (\alpha \geq 0; p \in \mathbb{N}).
\] (1.6)
Using the above relation, it is easy to verify the identity:
\[
z \left( I_p^\alpha f(z) \right)' = (p + 1) I_p^{\alpha-1} f(z) - I_p^\alpha f(z).
\] (1.7)
We note that the one-parameter family of integral operator $I_1^\alpha = I^\alpha$ was defined by Jung et al. [2].

To prove our results, we need the following definitions and lemmas.

**Definition 1** [5]. Denote by $F$ the set of all functions $q(z)$ that are analytic and injective on $U \setminus E(q)$ where
\[
E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\},
\]
and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $F$ for which $q(0) = a$ be denoted by $F(a)$, $F(0) \equiv F_0$ and $F(1) \equiv F_1$.

**Definition 2** [6]. A function $L(z, t) \ (z \in U, t \geq 0)$ is said to be a subordination chain if $L(0, t)$ is analytic and univalent in $U$ for all $t \geq 0$, $L(z, 0)$ is continuously differentiable on $[0; 1)$ for all $z \in U$ and $L(z, t_1) \prec L(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

**Lemma 1** [7]. The function $L(z, t) : U \times [0; 1) \rightarrow \mathbb{C}$ of the form

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots \ (a_1(t) \neq 0, t \geq 0),$$

and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\text{Re} \left\{ \frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t} \right\} > 0 \quad (z \in U, t \geq 0).$$

**Lemma 2** [3]. Suppose that the function $H : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the condition

$$\text{Re} \{ H(is; t) \} \leq 0$$

for all real $s$ and for all $t \leq -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \ldots$ is analytic in $U$ and

$$\text{Re} \left\{ H \left( p(z); zq'(z) \right) \right\} > 0 \quad (z \in U),$$

then $\text{Re} \{ p(z) \} > 0$ for $z \in U$.

**Lemma 3** [4]. Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $\text{Re} \{ \kappa h(z) + \gamma \} > 0 \ (z \in U)$, then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in $U$ and satisfies $\text{Re} \left\{ \kappa q(z) + \gamma \right\} > 0$ for $z \in U$.

**Lemma 4** [5]. Let $p \in F(a)$ and let $q(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots$ be analytic in $U$ with $q(z) \neq a$ and $n \geq 1$. If $q$ is not subordinate to $p$, then there exists two points $z_0 = r_0e^{i\theta} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ such that

$$q(U_n) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0p'(z_0) = m\zeta_0p(\zeta_0) \quad (m \geq n).$$

**Lemma 5** [6]. Let $q \in H[a; 1]$ and $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}$. Also set $\varphi \left( q(z), zq'(z) \right) = h(z)$. If $L(z, t) = \varphi \left( q(z), tzq'(z) \right)$ is a subordination chain and $q \in H[a; 1] \cap F(a)$, then

$$h(z) \prec \varphi \left( q(z), zq'(z) \right)$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi \left( q(z), zq'(z) \right) = h(z)$ has a univalent solution $q \in F(a)$, then $q$ is the best subordinant.

In the present paper, we aim at proving some subordination-preserving and superordination-preserving properties associated with the integral operator $I^p_q$. Sandwich-type result involving this operator is also derived.
2 Subordination, superordination and sandwich results involving the operator $I_\alpha^p$

Unless otherwise mentioned, we assume throughout this section that $\alpha \geq 1$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$.

**Theorem 1.** Let $f, g \in A(p)$ and let

$$\text{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta \quad (\phi(z) = \frac{I_\alpha^p g(z)}{z^{p-1}}; z \in U),$$

(2.1)

where $\delta$ is given by

$$\delta = \frac{1 + (p+1)^2 - |1 - (p+1)^2|}{4(p+1)}.$$ (2.2)

Then the subordination condition

$$\frac{I_\alpha^p f(z)}{z^p} \prec \frac{I_\alpha^p g(z)}{z^p}$$

implies that

$$\frac{I^\alpha_p f(z)}{z^p} \prec \frac{I^\alpha_p g(z)}{z^p}$$

and the function $\frac{I^\alpha_p g(z)}{z^p}$ is the best dominant.

**Proof.** Let us define the functions $F(z)$ and $G(z)$ in $U$ by

$$F(z) = \frac{I^\alpha_p f(z)}{z^p} \quad \text{and} \quad G(z) = \frac{I^\alpha_p g(z)}{z^p} \quad (z \in U),$$

(2.3)

we assume here, without loss of generality, that $G(z)$ is analytic and univalent on $\mathbb{U}$ and

$$G'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\mathbb{U}$, and we can use them in the proof of our result. Therefore, the results would follow by letting $\rho \to 1$.

We first show that, if

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U),$$

(2.4)

then

$$\text{Re} \left\{ q(z) \right\} > 0 \quad (z \in U).$$
From (1.6) and the definition of the functions $G, \phi$, we obtain that
\[ \phi(z) = G(z) + \frac{zG'(z)}{p+1}. \] (2.5)

Differentiating both side of (2.5) with respect to $z$ yields
\[ \phi'(z) = \left(1 + \frac{1}{p+1}\right)G'(z) + \frac{zG''(z)}{p+1}. \] (2.6)

Combining (2.4) and (2.6), we easily get
\[ 1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + p + 1} = h(z) \quad (z \in U) \] (2.7)

It follows from (2.1) and (2.7) that
\[ \Re \{ h(z) + p + 1 \} > 0 \quad (z \in U). \] (2.8)

Moreover, by using Lemma 2, we conclude that the differential equation (2.7) has a solution $q(z) \in H(U)$ with $h(0) = q(0) = 1$. Let
\[ H(u, v) = u + \frac{v}{u + p + 1} + \delta \]
where $\delta$ is given by (2.2). From (2.7) and (2.8), we obtain
\[ \Re \left\{ H(q(z); zq'(z)) \right\} > 0 \quad (z \in U). \]

To verify the condition that
\[ \Re \{ H(is; t) \} \leq 0 \quad \left(s \in \mathbb{R}; t \leq -\frac{1 + s^2}{2}\right), \] (2.9)
we proceed it as follows:
\[ \Re \{ H(is; t) \} = \Re \left\{ is + \frac{t}{is + p + 1} + \delta \right\} = \frac{t(p+1)}{s^2 + (p+1)^2} + \delta \]
\[ \leq -\frac{\Psi_p(\delta, s)}{2 \left[s^2 + (p+1)^2\right]}, \]
where
\[ \Psi_p(\delta, s) = [(p + 1) - 2\delta] s^2 - 2\delta (p + 1)^2 + (p + 1). \] (2.10)

For $\delta$ given by (2.2), we observe that the expression $\Psi_p(\delta, s)$ in (2.10) is a positive, which implies that (2.9) holds. Thus, by using Lemma 2, we conclude that
\[ \Re \{ q(z) \} > 0 \quad (z \in U). \]
By the definition of $q(z)$, we know that $G$ is convex. To prove $F \prec G$, let the function $L(z,t)$ be defined by
\[
L(z,t) = G(z) + \frac{(1 + t)zG'(z)}{p + 1} \quad (0 \leq t < \infty; z \in U).
\]  
(2.11)

Since $G$ is convex, then
\[
\frac{\partial L(z,t)}{\partial z}
\bigg|_{z=0} = G'(0) \left(1 + \frac{1 + t}{p + 1}\right) \neq 0 \quad (0 \leq t < \infty; z \in U)
\]
and
\[
Re \left\{ \frac{z\partial L(z,t)}{\partial z} \middle/ \partial t \right\} = Re \left\{ (p + 1) + (1 + t)q(z) \right\} > 0 \quad (0 \leq t < \infty; z \in U).
\]

Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that
\[
\phi(z) = G(z) + \frac{zG'(z)}{p + 1} = L(z,0),
\]
and
\[
L(z,0) \prec L(z,t) \quad (0 \leq t < \infty),
\]
which implies that
\[
L(\zeta,t) \notin L(U,t) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U).
\]  
(2.12)

If $F$ is not subordinate to $G$, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that
\[
F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0F'(z_0) = (1 + t)\zeta_0G'(\zeta_0) \quad (0 \leq t < \infty).
\]  
(2.13)

Hence, by virtue of (1.6) and (2.13), we have
\[
L(\zeta_0,t) = G(\zeta_0) + \frac{(1 + t)zG'(\zeta_0)}{p + 1} = F(z_0) + \frac{z_0F'(z_0)}{p + 1} = \frac{I_p^0 f(z_0)}{z^p} \in \phi(U).
\]

This contradicts to (2.12). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function $G$ is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

**Theorem 2.** Let $f, g \in A(p)$ and let
\[
Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta \quad \left( \phi(z) = \frac{I_p^{0-1} g(z)}{z^p}; z \in U \right),
\]  
(2.14)
where \( \delta \) is given by (2.2). If the function \( \frac{I^a_p g(z)}{z^p} \) is univalent in \( U \) and \( \frac{I^a_p g(z)}{z^p} \in F \), then the superordination condition

\[
\frac{I^a_p g(z)}{z^p} \prec \frac{I^a_p f(z)}{z^p}
\]

implies that

\[
\frac{I^a_p g(z)}{z^p} \prec \frac{I^a_p f(z)}{z^p}
\]

and the function \( \frac{I^a_p g(z)}{z^p} \) is the best subordinant.

\textbf{Proof.} Suppose that the functions \( F, G \) and \( q \) are defined by (2.3) and (2.4), respectively. By applying the similar method as in the proof of Theorem 1, we get

\[
\text{Re}\{q(z)\} > 0 \quad (z \in U).
\]

Next, to arrive at our desired result, we show that \( G \prec F \). For this, we suppose that the function \( L(z, t) \) be defined by (2.11). Since \( G \) is convex, by applying a similar method as in Theorem 1, we deduce that \( L(z, t) \) is subordination chain. Therefore, by using Lemma 5, we conclude that \( G \prec F \). Moreover, since the differential equation

\[
\phi (z) = G(z) + \frac{zG'(z)}{p+1} = \varphi \left( G(z), zG'(z) \right)
\]

has a univalent solution \( G \), it is the best subordinant. This completes the proof of Theorem 2.

Combining the above-mentioned subordination and superordination results involving the operator \( I^a_p \), the following "sandwich-type result" is derived.

\textbf{Theorem 3.} Let \( f, g_j \in A(p) \) \( (j = 1, 2) \) and let

\[
\text{Re} \left\{ 1 + \frac{z\phi''_j(z)}{\phi'_j(z)} \right\} > -\delta \quad \left( \phi_j(z) = \frac{I^a_p g_j(z)}{z^p} (j = 1, 2); z \in U \right),
\]

where \( \delta \) is given by (2.2). If the function \( \frac{I^a_p g_1(z)}{z^p} \) is univalent in \( U \) and \( \frac{I^a_p g_1(z)}{z^p} \in F \), then the condition

\[
\frac{I^a_p g_1(z)}{z^p} \prec \frac{I^a_p f(z)}{z^p} \prec \frac{I^a_p g_2(z)}{z^p}
\]
implies that
\[
\frac{I_\alpha^p g_1(z)}{z^p} < \frac{I_\alpha^p f(z)}{z^p} < \frac{I_\alpha^p g_2(z)}{z^p}
\]
and the functions \( \frac{I_\alpha^p g_1(z)}{z^p} \) and \( \frac{I_\alpha^p g_2(z)}{z^p} \) are, respectively, the best subordinant and the best dominant.

3 Open Problem

Find sufficient conditions for normalized analytic functions \( f, g_j \in A(p) \ (j = 1, 2) \) and \( \mu \) to satisfy the following sandwich-type result
\[
\left( \frac{z^p}{I_\alpha^p g_1(z)} \right)^\mu < \left( \frac{z^p}{I_\alpha^p f(z)} \right)^\mu < \left( \frac{z^p}{I_\alpha^p g_2(z)} \right)^\mu.
\]

References


