Int. J. Open Problems Complex Analysis, Vol. 4, No. 2, July 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

On Generalizations of Quasi-Hadamard Products of p-valent Functions

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Abstract

The purpose of the present paper is to establish some interesting results on the generalizations of quasi-Hadamard product of functions belonging to the classes of p-valent starlike and p-valent convex functions of order α in the open unit disc U. Our results improve the results of previous authors to the case when r and s are any positive real numbers such that $s > 1$. It is worth noting that the technique employed by us is entirely different from the previous authors.

Keywords: *Analytic, p-valent, Modified-Hadamard product.*
2000 Mathematical Subject Classification: 30C45.

1 Introduction

Let $T(n,p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} |a_{k+P}|\, z^{k+p}, \qquad (p,n \in N = \{1,2,3,..\}). \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. Also let $T_n(p,\alpha)$ and $C_n(p,\alpha)$ denote the subclasses of $T(n,p)$ consisting of functions which satisfy the inequalities

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (0 \le \alpha < p), \tag{2}$$

and

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (0 \le \alpha < p), \tag{3}$$

respectively. Clearly, the functions in $T_n(p,\alpha)$ and $C_n(p,\alpha)$ are p-valent starlike and p-valent convex of order α respectively.

These classes $T_n(p,\alpha)$ and $C_n(p,\alpha)$ were studied by Owa [7].

By specializing the parameters in the classes $T_n(p,\alpha)$ and $C_n(p,\alpha)$, we obtain the following classes studied by various authors.

(i) $T_1(p,\alpha) \equiv T^*(p,\alpha)$ and $C_1(p,\alpha) \equiv C(p,\alpha)$ were studied by Owa [7] and Sekine [10].

(ii) $T_n(1,\alpha) \equiv T_n(\alpha)$ and $C_n(1,\alpha) \equiv C_n(\alpha)$ were studied by Domokos [4] and Srivastava et al. [12].

(iii) $T_1(1,\alpha) \equiv T^*(\alpha)$ and $C_1(1,\alpha) \equiv C(\alpha)$ were studied by Silverman [11].

Let $f_j(z) \quad (j = 1,2)$ in $T(n,p)$ be given by

$$f_j(z) = z^p - \sum_{k=n}^{\infty} |a_{k+P,j}|\, z^{k+p}, \quad (j = 1,2;\, p,n \in N). \tag{4}$$

Then the modified-Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n}^{\infty} |a_{k+p,1}||a_{k+p,2}| z^{k+p}. \tag{5}$$

For any real number r and s, we define the generalized modified-Hadamard product $(f_1 \Delta f_2)(r,s;z)$ by

$$(f_1 \Delta f_2)(r,s;z) = z^p - \sum_{k=n}^{\infty} |a_{k+P,1}|^r \, |a_{k+P,2}|^s \, z^{k+p}. \tag{6}$$

If we take $r = s = 1$, then we have

$$(f_1 \Delta f_2)(1,1;z) = (f_1 * f_2)(z), (z \in U).$$

Recently Darwish and Aouf [3] (See also the work of Choi et al. [1], Darwish [2], Nishiwaki and Owa [5], Nishiwaki et al. [6], Owa [8], Schild and Silverman [9] and Sekine [10]) studied the generalized Hadamard product to the case when $r, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. The classical technique used by these authors fails when r and s are any positive real numbers. It is therefore natural to ask whether their result can be improved for any positive real numbers r and s.

The purpose of the present paper is to establish the result on generalized quasi-Hadamard product which improves the results of previous authors to the case when r and s are any positive real numbers such that $s > 1$. It is worth noting that the technique employed by us is entirely different from the previous authors.

2 Main Results

In order to prove that our results for functions belonging to the classes $T_n(p,\alpha)$ and $C_n(p,\alpha)$, we shall need the following lemmas given by Owa [7].

Lemma 2.1. *Let the function $f(z)$ be defined by 1. Then $f(z)$ is in the class $T_n(p,\alpha)$ if and only if*

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha)}{p-\alpha} |a_{k+p}| \le 1, \qquad (p,n \in N). \tag{7}$$

Lemma 2.2. *Let the function $f(z)$ be defined by 1. Then $f(z)$ is in the class $C_n(p,\alpha)$, if and only if*

$$\sum_{k=n}^{\infty} \frac{(k+p)(k+p-\alpha)}{p(p-\alpha)} |a_{k+p}| \le 1, \qquad (p,n \in N). \tag{8}$$

Theorem 2.3. *For $0 \le \alpha_1 \le \alpha_2 < p$, $r > 0$, $s > 1$ and let $f_j \in T_n(p,\alpha_j)$ for each j, then*

$$(f_1 \Delta f_2)(r,s;z) \in T_n(p,\alpha_2) \subseteq T_n(p,\alpha_1). \tag{9}$$

Proof Since $f_j(z) \in T_n(p, \alpha_j)$, $(j = 1, 2)$.
By using Lemma 2.1, we have

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha_j)}{p-\alpha_j} |a_{k+p,j}| \leq 1, (j = 1, 2;\ p, n \in N). \tag{10}$$

Moreover

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha_1)}{p-\alpha_1} |a_{k+p,1}| \leq 1,$$

or, $|a_{k+p,1}| \leq 1, \qquad k \geq n.$
equivalently, $|a_{k+p,1}|^r \leq 1,\ k \geq n$
and

$$\left\{\sum_{k=n}^{\infty} \frac{(k+p-\alpha_2)}{p-\alpha_2} |a_{k+p,2}|\right\}^s \leq 1.$$

Now

$$\begin{aligned}
&\sum_{k=n}^{\infty} \left(\frac{k+p-\alpha_2}{p-\alpha_2}\right) |a_{k+p,1}|^r |a_{k+p,2}|^s \\
\leq &\sum_{k=n}^{\infty} \left(\frac{k+p-\alpha_2}{p-\alpha_2}\right) |a_{k+p,2}|^s \\
\leq &\sum_{k=n}^{\infty} \left(\frac{k+p-\alpha_2}{p-\alpha_2}\right)^s |a_{k+p,2}|^s, (s > 1) \\
\leq &\left\{\sum_{k=n}^{\infty} \left(\frac{k+p-\alpha_2}{p-\alpha_2}\right) |a_{k+p,2}|\right\}^s \\
\leq &1.
\end{aligned}$$

Therefore $(f_1 \Delta f_2)(r, s; z) \in T_n(p, \alpha_2)$.
Taking $\alpha_j = \alpha$ in Theorem 2.3, we obtain

Corollary 2.4. *If the functions* $f_j(z)$, $(j = 1, 2)$, *defined by 4 are in the class* $T_n(p, \alpha)$.
Then $(f_1 \Delta f_2)(r, s; z) \in T_n(p, \alpha)$.

Remark 2.5. *Darwish and Aouf [3] obtain Corollary 2.4 only for* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$ *whereas our result holds for* $r > 0$ *and* $s > 1$.

Theorem 2.6. *For* $0 \leq \alpha_1 \leq \alpha_2 < p$, $r > 0$, $s > 1$ *and let* $f_j \in C_n(p, \alpha_j)$ *for each* j, *then*

$$(f_1 \Delta f_2)(r, s; z) \in C_n(p, \alpha_2) \subseteq C_n(p, \alpha_1).$$

Proof The proof of above theorem is much akin to that of Theorem 2.3 so we omit the details involved.

Taking $\alpha_j = \alpha$ in Theorem 2.6, we obtain

Corollary 2.7. *If the functions$f_j(z)$, $(j = 1, 2)$, defined by 4 are in the class $C_n(p, \alpha)$.*

Then $(f_1 \Delta f_2)(r, s; z) \in C_n(p, \alpha)$.

Remark 2.8. *Darwish and Aouf [3] obtained Corollary 2.7 only for $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, whereas our result holds for $r > 0$, $s > 1$.*

Remark 2.9. *If we put $p = 1$ in Theorem 2.3, Theorem 2.6, Corollary 2.4 and Corollary 2.7, we improve the results of Choi et al. [1].*

3 Open Problem

Here we give some open problems for the readers.

1. Find inf $r, s \in R$ and sup $\alpha \in [0, p)$ such that Theorem 2.3 and 2.6 holds.

2. The results of Theorem 2.3 and 2.6 are hold only for functions of the form 1 i.e. the coefficients of expansion are negative . Therefore it is natural to ask that what is the analogues results for the function of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+P} z^{k+p}, \qquad (p, n \in N = \{1, 2, 3, ..\}), \tag{11}$$

where $a_{k+p} \in C$.

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