Meromorphic Functions That Share
One Small Function CM or IM
with Their First Derivative

Amer H. H. Al-Khaladi
Department of Mathematics
College of Science
Diyala University, Baquba, Iraq
email: ameralkhaladi@yahoo.com

Abstract

In this paper we obtain a unicity theorem of a meromorphic function and its first derivative that share one small function CM or IM. So we generalize some results given in [1], [2] and [3].

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1 Introduction

A meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, $m(r, f)$ etc (see [4], [5]). By $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of $r$ with finite linear measure. Then the meromorphic function $\beta$ is called a small function of $f$ if $T(r, \beta) = S(r, f)$. We say that two non-constant meromorphic functions $f$ and $g$ share a small function $\beta$ IM (ignoring multiplicities), if $f$ and $g$ have the same $\beta$-points. If $f$ and $g$ have the same $\beta$-points with the same multiplicities, we say that $f$ and $g$ share the small function $\beta$ CM (counting multiplicities). Let $k$ be a positive integer, and let $b$ be a small function of $f$ or $\infty$, we denote by $N_{k}(r, \frac{1}{f-b})$ the counting function
of \( b \)-points of \( f \) with multiplicity \( \leq k \) and by \( \bar{N}_k(r, \frac{1}{T-b}) \) the counting function of \( b \)-points of \( f \) with multiplicity \( > k \). In like manner we define \( \bar{N}_k(r, \frac{1}{T-b}) \) and \( \bar{N}_k(r, \frac{1}{T-b}) \) where in counting the \( b \)-points of \( f \) we ignore the multiplicities.

In [2] G. G. Gundersen proved the following theorem:

**Theorem 1.1** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share two distinct values \( 0, a \neq \infty \) CM, then \( f \equiv f' \)

In 2009, A. H. H. Al-Khaladi [1] proved the following theorems which are improvement and extension of Theorem 1.1:

**Theorem 1.2** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share the value \( a \neq 0, \infty \) CM and if \( \bar{N}(r, \frac{1}{f}) = S(r, f) \), then either \( f \equiv f' \) or
\[
f(z) = \frac{a + A}{1 + ce^{-2z}}, \quad \text{where} \quad A \text{ and } c \neq 0 \text{ are constants},
\]

**Theorem 1.3** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share the value \( a \neq 0, \infty \) IM and if \( \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r, f) \), then either \( f \equiv f' \) or
\[
f(z) = \frac{2a}{1 + ce^{-2z}}, \quad \text{(1)}
\]
where \( c \) is a nonzero constant.

On the other hand, Q. C. Zhang [3] proved the following theorem:

**Theorem 1.4** Let \( f \) be a non-constant meromorphic function, \( a \) be a nonzero finite complex constant. If \( f \) and \( f' \) share 0 CM, and share a IM, then \( f \equiv f' \) or \( f \) is given as (1).

In this paper we will generalize the above results (Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4).

# 2 Main Results

**Lemma 2.1** Let \( f' \) be a non-constant meromorphic function, and let \( \beta \) be a small function of \( f' \) such that \( \beta' \equiv \beta \neq 0, \infty \). Then
\[
m\left(r, \frac{1}{f' - \beta} \right) \leq 2\bar{N}\left(r, \frac{1}{f'} \right) + 2\bar{N}(r, f) + S(r, f')
\]

**Proof** Set
\[
W = \left( \frac{F'}{F} \right)^2 - 2\left( \frac{F'}{F} \right)' + 2\frac{F'}{F}, \quad \text{(2)}
\]
where \( F = \frac{f'}{\beta} \). Then from Nevanlinna’s fundamental estimate of the logarithmic derivative we obtain
\[
m(r, W) \leq 4m\left(r, \frac{F'}{F}\right) + S\left(r, \frac{F'}{F}\right) + O(1) = S(r, F') + S\left(r, \frac{F'}{F}\right).
\]
Since
\[
T\left(r, \frac{F'}{F}\right) = N\left(r, \frac{F'}{F}\right) + m\left(r, \frac{F'}{F}\right) \leq \tilde{N}(r, F') + \tilde{N}\left(r, \frac{1}{F}\right) + S(r, F)
\]
this means that
\[
m(r, W) = S(r, F) = S(r, f').
\]

(3)

Suppose that \( z_\infty \) is a simple pole of \( f \). Then the Laurent expansion of \( f \) about \( z_\infty \) is
\[
f(z) = a_{-1}(z - z_\infty)^{-1} + O(1)
\]
where \( a_{-1} \) be the residue of \( f \) at \( z_\infty \). Hence
\[
\frac{F'}{F} = -2(z - z_\infty)^{-1} - 1 + O(z - z_\infty).
\]
Substitution of this into (2) gives
\[
W(z_\infty) = O(1).
\]

(4)

It follows from (2) that the poles of \( f \) with multiplicity \( p \geq 2 \) are poles of \( W \) with multiplicity 2 at most. We can also conclude from (2) that the zeros of \( f' \) with multiplicity \( q \geq 1 \) are poles of \( W \) with multiplicity 2. Thus, from (4) we get
\[
N(r, W) \leq 2\tilde{N}(2(r, f) + 2\tilde{N}\left(r, \frac{1}{f'}\right).
\]

(5)

We distinguish the following the two cases:

**Case 1.** \( W \neq 0 \). We write (2) in the form
\[
\frac{1}{F - 1} = \frac{1}{W}\left(\frac{F'}{F} - \frac{F'}{F} - \frac{3F'}{F} + 2\frac{F''}{F'} + 2\right).
\]
Then it is clear that
\[
m\left(r, \frac{1}{F - 1}\right) \leq m\left(r, \frac{1}{W}\right) + S(r, F) \leq T(r, W) + S(r, F')
\]
\[
\leq m(r, W) + N(r, W) + S(r, f').
\]
Combining this with (3) and (5), we have
\[
m\left(r, \frac{1}{F - 1}\right) \leq 2\tilde{N}(2(r, f) + 2\tilde{N}\left(r, \frac{1}{f'}\right) + S(r, f').
\]
That is,
\[ m\left(r, \frac{1}{f' - \beta}\right) \leq 2\bar{N}(r, f) + 2\bar{N}\left(r, \frac{1}{f'}\right) + S(r, f'). \]

**Case 2.** \(W \equiv 0\). If \(\frac{F'}{F} \equiv 0\), then \(f' = c\beta\) and so \(T(r, f') = S(r, f')\) a contradiction. Therefore \(\frac{F'}{F} \not\equiv 0\) and (2) becomes

\[
\frac{y'}{y + 2} - \frac{y'}{y} = \frac{1}{2},
\]
where \(y = \frac{F'}{F}\). Integrating (6) twice we obtain

\[
f' = \beta A \left(c - e^{-\frac{1}{2}z}\right)^4,
\]
where \(A\) and \(c \neq 0\) are constants. So

\[ T(r, f') = 4T\left(r, e^{-\frac{1}{2}z}\right) + S(r, f'). \]

But

\[ T(r, \beta) = 2T\left(r, e^{-\frac{1}{2}z}\right) + O(1). \]

Therefore

\[ T(r, f') = 2T(r, \beta) + S(r, f') = S(r, f'). \]

This is a contradiction. \(\square\)

The following lemma belongs to [4].

**Lemma 2.2** Let \(f\) be a non-constant meromorphic function, and \(a_1, a_2, a_3\) be distinct small functions of \(f\). Then

\[ T(r, f) \leq \sum_{j=1}^{3} \bar{N}\left(\frac{1}{f - a_j}\right) + S(r, f). \]

**Theorem 2.3** Let \(f\) be a non-constant meromorphic function, and let \(\beta\) be a small meromorphic function of \(f\) such that \(\beta \not\equiv 0, \infty\). If \(f\) and \(f'\) share \(\beta\) CM and if \(\bar{N}(r, \frac{1}{f}) = S(r, f)\), then either \(f \equiv f'\) or

\[
f(z) = \frac{\int_{0}^{z} \beta(t)dt + A}{1 + ce^{-z}},
\]
where \(A\) and \(c \neq 0\) are constants.
Proof Suppose that \( f \neq f' \) and let \( \Omega \) be the function defined by

\[
\Omega = \frac{1}{f} \left[ \frac{(f'/\beta)' - (f/\beta)'}{f'/\beta - f/\beta} \right]
= \frac{1}{\beta^2} \left[ \frac{f'}{f} \left( \frac{(f'/\beta)'}{f'/\beta} - 1 \right) - \frac{(f/\beta)'}{(f'/\beta) - f/\beta} - \right. \\
\left. \frac{(f/\beta)'}{(f'/\beta) - f/\beta} - \frac{(f/\beta)'}{f'/\beta} \right].
\tag{8}
\]

Then from Nevanlinna’s fundamental estimate of the logarithmic derivative we obtain

\[
m(r, \Omega) \leq m\left(r, \frac{1}{\beta^2}\right) + m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{(f'/\beta)'}{f'/\beta} - 1\right) + \\
m\left(r, \frac{(f/\beta)'}{f'/\beta} - 1\right) + m\left(r, \frac{(f/\beta)'}{f'/\beta}\right) + O(1)
= S(r, f) + S(r, f').
\]

Since

\[
T(r, f') \leq 2T(r, f) + S(r, f),
\]

this means that

\[
m(r, \Omega) = S(r, f).
\tag{9}
\]

It follows from (8) that if \( z_\infty \) is a pole of \( f \) with multiplicity \( p \geq 1 \) and \( \beta(z_\infty) \neq 0, \infty \), then

\[
\Omega(z) = O\left((z - z_\infty)^{p-1}\right).
\tag{10}
\]

Since \( f \) and \( f' \) share \( \beta \) CM, we find from (8) that \( \Omega \) is holomorphic at the zeros of \( f - \beta \) and \( f' - \beta \). Thus the pole of \( \Omega \) can only occur at zeros of \( f \). However the zeros of \( f \) with multiplicity \( q \geq 2 \) are pole of \( \Omega \) with multiplicity 2. Thus, from \( \tilde{N}(r, \frac{1}{f}) = S(r, f) \) we get

\[
N(r, \Omega) \leq \tilde{N} \left( r, \frac{1}{f} \right) + \tilde{N}(2) \left( r, \frac{1}{f} \right) + S(r, f)
\leq 2\tilde{N} \left( r, \frac{1}{f} \right) + S(r, f) = S(r, f).
\]

Together with (9) we have

\[
T(r, \Omega) = m(r, \Omega) + N(r, \Omega) = S(r, f).
\tag{11}
\]

If \( \Omega \equiv 0 \), then from integration of (8) we get \( f - \beta = c(f' - \beta) \), where \( c \) is some nonzero constant. This implies that \( \tilde{N}(r, f) = S(r, f) \). If \( c = 1 \), then \( f \equiv f' \), a contradiction. Therefore \( c \neq 1 \) and so

\[
\frac{1}{f} = \frac{c}{\beta(c - 1)} \left( \frac{f'}{f} - 1 \right).
\]
Hence, we obtain
\[
T(r, f) \leq T\left(r, \frac{f'}{f}\right) + S(r, f) = N\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f}\right) + S(r, f)
\]
\[
\leq \tilde{N}\left(r, \frac{1}{f}\right) + \tilde{N}(r, f) + S(r, f) = S(r, f),
\]

which is impossible. Therefore, we obtain $\Omega \not\equiv 0$. Writing (8) as
\[
f = \frac{1}{\beta \Omega} \left[ \frac{(f'/\beta)'}{(f'/\beta) - 1} - \frac{(f/\beta)'}{(f/\beta) - 1} \right].
\]

Consequently, from (11),
\[
m(r, f) \leq m\left(r, \frac{1}{\beta}\right) + m\left(r, \frac{1}{\Omega}\right) + S(r, f) \leq m\left(r, \frac{1}{\Omega}\right) + S(r, f)
\]
\[
\leq T(r, \Omega) + S(r, f) = S(r, f).
\]

Furthermore, from (10) and (11) we deduce that
\[
N_{12}(r, f) - \tilde{N}_{12}(r, f) \leq N\left(r, \frac{1}{\Omega}\right) + S(r, f)
\]
\[
\leq T(r, \Omega) + S(r, f) = S(r, f),
\]
so that,
\[
N_{12}(r, f) = S(r, f).
\]

We set
\[
\omega = \frac{f' - f}{f(f - \beta)} = \frac{1}{f - \beta}\left(\frac{f'}{f} - 1\right).
\]

Then
\[
m(r, \omega) \leq m\left(r, \frac{1}{f - \beta}\right) + m\left(r, \frac{f'}{f}\right) + O(1)
\]
\[
= m\left(r, \frac{1}{f - \beta}\right) + S(r, f).
\]

Since $f$ and $f'$ share $\beta$ CM, from (15) we deduce that $\omega$ is holomorphic at the zeros of $f - \beta$. Also it is clear that the poles of $f$ being not the poles of $\omega$. Thus,
\[
N(r, \omega) \leq \tilde{N}\left(r, \frac{1}{f}\right) + S(r, f) = S(r, f).
\]

Further, if $z_\infty$ is a simple pole of $f$ and $\beta(z_\infty) \neq 0, \infty$, by a simple computation, we deduce from (8) and (15) that
\[
\Omega(z_\infty) = \frac{-1}{\beta(z_\infty)a_{-1}} \quad \text{and} \quad \omega(z_\infty) = \frac{-1}{a_{-1}},
\]
where \( a_{-1} \) be the residue of \( f \) at \( z_{\infty} \). In the following we shall treat two cases \( \beta \Omega \equiv \omega \) and \( \beta \Omega \not\equiv \omega \) separately.

**Case 1.** \( \beta \Omega \equiv \omega \). From (8) and (15) we know that if

\[
h = \frac{f' - \beta}{f - \beta} = \frac{f'/\beta - 1}{f/\beta - 1},
\]

\[
\beta \Omega = \frac{1}{f} \left( \frac{h'}{h} \right) \quad \text{and} \quad \omega = \frac{1}{f}(h - 1).
\]

Hence,

\[
\frac{h'}{h - 1} - \frac{h'}{h} = 1.
\]

By integration, we get \( h(z) = \frac{1}{1 - ce^z} \), where \( c \) nonzero constant. Combining this with (19) yields

\[
f' - \frac{1}{1 - ce^z} f = -c\beta e^z,
\]

which leads to

\[
\frac{d}{dz} \left[ f(z) \left( 1 - \frac{1}{c} e^{-z} \right) \right] = \beta(z).
\]

From this we arrive at (7).

**Case 2.** \( \beta \Omega \not\equiv \omega \). Then from (18), (11), (16) and (17) we see that

\[
N_1(r, f) \leq N \left( r, \frac{1}{\beta \Omega - \omega} \right) \leq T(r, \beta \Omega - \omega) + O(1)
\]

\[
\leq T(r, \Omega) + T(r, \omega) + S(r, f) \leq m \left( r, \frac{1}{f - \beta} \right) + S(r, f).
\]

Combining this, (14) and (13), we obtain

\[
T(r, f) = m(r, f) + N(r, f) = N_1(r, f) + S(r, f)
\]

\[
\leq m \left( r, \frac{1}{f - \beta} \right) + S(r, f).
\]

Hence, we find that

\[
N \left( r, \frac{1}{f - \beta} \right) = S(r, f).
\]

We define

\[
\mu = \frac{(f/\beta)'}{f(f - \beta)} = \frac{1}{\beta^2} \left[ \frac{(f/\beta)'}{(f/\beta) - 1} - \frac{(f/\beta)'}{f/\beta} \right].
\]

Then it is clear that

\[
m(r, \mu) = S(r, f).
\]
If $z_\infty$ is a simple pole of $f$ and $\beta(z_\infty) \neq 0, \infty$, by a simple calculation on the local expansions we find that

$$\mu(z_\infty) = \frac{-1}{\beta(z_\infty)a_{-1}}. \quad (23)$$

Thus, it can be obtained from (22), (23), (14), (20) and $\bar{N}(r, \frac{1}{f}) = S(r, f)$ that

$$T(r, \mu) = m(r, \mu) + N(r, \mu) = N(r, \mu) + S(r, f)$$

$$\leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-\beta}\right) + S(r, f) = S(r, f). \quad (24)$$

Further, from (23) and (18) we have

$$\Omega(z_\infty) = \mu(z_\infty). \quad (25)$$

If $\Omega \equiv \mu$, we know from (8) and (21) that

$$2 \frac{(f/\beta)'}{f/\beta - 1} = \frac{(f'/\beta)'}{f'/\beta - 1}. \quad (26)$$

By integration once, $(f - \beta)^2 = c\beta(f' - \beta)$, where $c$ is a nonzero constant. We rewrite this in the form

$$\frac{\beta' - \beta}{f - \beta} = \frac{c\beta}{f - \beta} - \frac{(f - \beta)'}{f - \beta}. \quad (26)$$

If $\beta' - \beta \neq 0$, from this, (13) and (20) we conclude that

$$T(r, f) = T\left(r, \frac{1}{f - \beta}\right) + S(r, f)$$

$$= m\left(r, \frac{1}{f - \beta}\right) + N\left(r, \frac{1}{f - \beta}\right) + S(r, f)$$

$$\leq m\left(r, \frac{1}{f' - \beta}\right) + m(r, f) + N\left(r, \frac{1}{f - \beta}\right) + S(r, f)$$

$$= S(r, f)$$

This is impossible. Therefore we have $\beta' - \beta \equiv 0$, and (26) becomes

$$\frac{(f' - \beta)'}{(f - \beta)^2} = \frac{\beta'}{c\beta^2}. \quad (26)$$

By integration, we get

$$\frac{-1}{f - \beta} = \frac{-1}{c\beta} + A,$$
Meromorphic Share One Small Function CM or IM

where \( A \) is a constant. So \( T(r, f) = S(r, f) \), a cotradiction. Thus \( \Omega \neq \mu \). It follows from this, (13), (14), (25), (11) and (24) that

\[
T(r, f) = N(r, f) + m(r, f) = N_{13}(r, f) + N_{12}(r, f) + m(r, f) \\
= N_{13}(r, f) + S(r, f) \leq N\left(r, \frac{1}{\Omega - \mu}\right) + S(r, f) \\
\leq T(r, \Omega) + T(r, \mu) + S(r, f) = S(r, f).
\]

This is impossible. The proof of Theorem 2.3 is complete. \( \square \)

**Theorem 2.4** Let \( f \) be a non-constant meromorphic function, and let \( \beta \) be a small meromorphic function of \( f \) such that \( \beta \neq 0, \infty \). If \( f \) and \( f' \) share \( \beta \) IM and if \( \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r, f) \), then either \( f \equiv f' \) or \( \beta \) is a constant and \( f \) is given as (1) when \( \beta = a \).

**Proof** In the following, we assume that \( f \neq f' \). Suppose \( z_0 \) is a zero of \( f - \beta \) with multiplicity \( n \geq 1 \) and \( \beta(z_\infty) \neq 0, \infty \). Then the Taylor expansion of \( f - \beta \) about \( z_0 \) is

\[
f(z) - \beta = a_n(z - z_0)^n + \ldots, \quad a_n \neq 0.
\]  

(27)

Since \( f \) and \( f' \) share \( \beta \) IM,

\[
f'(z) - \beta = b_m(z - z_0)^m + \ldots, \quad b_m \neq 0.
\]  

(28)

Differentiating (27) and then using (28), we obtain

\[
\beta(z) - \beta'(z) = na_n(z - z_0)^{n-1} - b_m(z - z_0)^m + \ldots.
\]  

(29)

We consider the following two cases.

**Case I.** \( \beta - \beta' \neq 0 \). Then we get from (29) that

\[
\bar{N}(2)\left(r, \frac{1}{f - \beta}\right) \leq N\left(r, \frac{1}{\beta' - \beta}\right) + S(r, f) \leq T(r, \beta' - \beta) + S(r, f) \\
\leq 3T(r, \beta) + S(r, f) = S(r, f).
\]  

(30)

If \( z_0 \) is a simple zero of \( f - \beta \) and \( f' - \beta \), from (8) we see that \( \Omega \) is holomorphic at \( z_0 \). It follows from this, (8), (10), \( f \) and \( f' \) share \( \beta \) IM, \( \bar{N}(r, \frac{1}{f}) = S(r, f) \) and (30) that

\[
N(r, \Omega) \leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f'}\right) + \bar{N}\left(r, \frac{1}{f' - \beta}\right) + S(r, f) \\
\leq \bar{N}(2)\left(r, \frac{1}{f' - \beta}\right) + S(r, f).
\]
Combining with (9) we obtain
\[ T(r, \Omega) \leq \tilde{N}(r, \frac{1}{f' - \beta}) + S(r, f). \] (31)

Also we know from (10), (12) and (31) that
\[ N_{(2)}(r, f) - \tilde{N}(r, f) \leq N\left(r, \frac{1}{\Omega}\right) + S(r, f) \]
\[ \leq T(r, \Omega) - m\left(r, \frac{1}{\Omega}\right) + S(r, f) \]
\[ \leq \tilde{N}(r, \frac{1}{f' - \beta}) - m(r, f) + S(r, f). \] (32)

We set
\[ H = \frac{(f'/\beta)'(f - \beta)}{f'(f' - \beta)} = \frac{f - \beta}{\beta^2} \left[ \frac{(f'/\beta)'}{(f'/\beta) - 1} - \frac{(f'/\beta)'}{f'/\beta} \right]. \] (33)

Then it is clear that
\[ m(r, H) \leq m(r, f) + S(r, f). \] (34)

From (33) we deduce that if \( z_\infty \) is a pole of \( f \) with multiplicity \( p \geq 1 \) and \( \beta(z_\infty) \neq 0, \infty \),
\[ H(z_\infty) = \frac{1}{\beta(z_\infty)} \left( \frac{p + 1}{p} \right). \] (35)

Substituting (27) and (28) into (33) gives
\[ H(z) = O\left( (z - z_\infty)^{n-1} \right). \] (36)

Thus the pole of \( H \) can only occur at zeros of \( f' \). However, the zeros of \( f' \)
with multiplicity \( s \geq 1 \) are poles of \( H \) with multiplicity 1. Therefore from this,
(35), (36) and \( \tilde{N}(r, \frac{1}{f'}) = S(r, f) \) we get
\[ N(r, H) \leq \tilde{N}\left(r, \frac{1}{f'}\right) + S(r, f) = S(r, f). \]

Together with (34) we have
\[ T(r, H) \leq m(r, f) + S(r, f). \] (37)

If \( z_\infty \) is a simple pole of \( f \), then by (35) there are two cases.

**Case 1.** \( H \equiv \frac{2}{\beta} \). This and (33) imply that
\[ \frac{1}{f' - \beta} = \frac{1}{2\beta} \left[ \frac{(f'/\beta)'}{(f'/\beta) - 1} - \frac{(f'/\beta)'}{f'/\beta} \right]. \] (38)
Obviously, by logarithmic derivative lemma $m(r, \frac{1}{f-\beta}) = S(r, f)$. Combining with (16) we get

$$m(r, \omega) = S(r, f).$$ (39)

From (38), (27) and (28) we know that if $z_0$ is a zero of $f - \beta$ with multiplicity $n \geq 1$ and $\beta(z_0) \neq 0, \infty$, then $n = 1$. In addition since $f$ and $f'$ share $\beta$ IM, from (15) we see $\omega$ is holomorphic at $z_0$. Also it is easily verified that the pole of $f$ being not the pole of $\omega$. Thus, from $\bar{N}(r, \frac{1}{f}) = S(r, f)$ we obtain

$$N(r, \omega) \leq \bar{N}(r, \frac{1}{f}) + S(r, f) = S(r, f).$$

Together with (39) we have

$$T(r, \omega) = S(r, f).$$ (40)

If $\omega \equiv 0$, then $f \equiv f'$ a contradiction. In the following we assume $\omega \neq 0$. Further, it can be obtained from (15), (40) and $\bar{N}(r, \frac{1}{f}) = S(r, f)$ that

$$T(r, f) \leq T\left(r, \frac{1}{\omega} \right) + T\left(r, \frac{f'}{f} \right) + S(r, f)$$

$$= N\left(r, \frac{f'}{f} \right) + m\left(r, \frac{f'}{f} \right) + S(r, f)$$

$$\leq \bar{N}\left(r, \frac{1}{f} \right) + \bar{N}(r, f) + S(r, f)$$

$$= \bar{N}(r, f) + S(r, f),$$

which shows that

$$T(r, f) = N_1(r, f) + S(r, f).$$ (41)

Using an argument similar to that in the proof of Theorem 2.3, we can conclude that $\beta \Omega \equiv \omega$ or $\beta \Omega \neq \omega$. If $\beta \Omega \equiv \omega$, then (see Case 1 in the proof of Theorem 2.3)

$$f(z) = \int_0^z \beta(t)dt + A \frac{1}{1 + ce^{-z}}.$$

Hence

$$f'(z) = \frac{\beta + \left(\beta + A + \int_0^z \beta(t)dt\right)ce^{-z}}{1 + ce^{-z}}.$$ 

Since $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r, f)$, we must have $\beta + A + \int_0^z \beta(z)dz \equiv 0$. Differentiating this we obtain $\beta' + \beta \equiv 0$. Integrating once, $\beta(z) = c_1e^{-z}$, where $c_1$ is a nonzero constant. So $\beta$ can not small function of $f$. Therefore,
we get $\beta \Omega \neq \omega$. It follows from this, (41), (18), (40) and (31) that

$$T(r, f) = N_1(r, f) + S(r, f) \leq N(r, \frac{1}{\beta \Omega - \omega}) + S(r, f)$$

$$\leq T(r, \beta) + T(r, \Omega) + T(r, \omega) + S(r, f)$$

$$\leq \tilde{N}(r, \frac{1}{f - \beta}) + S(r, f).$$  (42)

Since $f$ and $f'$ share $\beta$ IM,

$$\tilde{N}(r, \frac{1}{f - \beta}) = \tilde{N}(r, \frac{1}{f' - \beta}).$$

By this and (42) we have

$$N_1(r, \frac{1}{f' - \beta}) = S(r, f).$$  (43)

If we rewrite (38) and (15) in the form

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = 2\left(\frac{f' - \beta}{f - \beta}\right) \text{ and } f\omega + 1 = \frac{f' - \beta}{f - \beta}$$

respectively, and then elimination $\frac{f' - \beta}{f - \beta}$ between the last two equations we obtain

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = 2(f\omega + 1).$$  (44)

Let $z_0$ be a zero of $f' - \beta$ with multiplicity $m \geq 2$ and $\beta(z_0) \neq 0, \infty$. By (27), (28) and (44) we find that

$$\beta(z_0)\omega(z_0) + 1 = 0$$  (45)

If $\beta \omega + 1 \neq 0$, then from this, (42), (45) and (40) we see that

$$T(r, f) \leq \tilde{N}(r, \frac{1}{f' - \beta}) + S(r, f) \leq N(r, \frac{1}{\beta \omega + 1}) + S(r, f)$$

$$\leq T(r, \beta) + T(r, \omega) + S(r, f) = S(r, f)$$

This is impossible. Therefore, $\beta \omega + 1 \equiv 0$. Thus, from this, (44) and (15) we get

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = \frac{-2}{\beta}(f - \beta)$$

and

$$\frac{1}{\beta}(f - \beta) = \frac{f'}{f} - 1.$$  (46)
respectively. If we now eliminate $-\frac{1}{\beta}(f - \beta)$ between the last two equations leads to

$$\frac{f''}{f'} - \frac{\beta'}{\beta} = 2\left(\frac{f'}{f} - 1\right)$$

By integrating once,

$$f'(z) = c\beta(z)f^2e^{-2z}.$$ Substituting this into (46) gives

$$f(z) = \frac{2\beta(z)}{1 + c\beta^2(z)e^{-2z}}.$$ Hence,

$$f'(z) = \frac{2\beta' - 2\beta^2ce^{-2z}(\beta' - 2\beta)}{(1 + c\beta^2e^{-2z})^2}.$$ Since $\bar{N}(r, \frac{1}{f'}) = S(r, f)$, therefore we must have $\beta' \equiv 0$ and so $\beta$ is a constant. Thus (1) holds when $\beta = a$.

**Case 2.** $H \not\equiv \frac{2}{\beta}$. Then from (35) and (37) we have

$$N_1(r, f) \leq N\left(r, \frac{1}{H - \frac{2}{\beta}}\right) + S(r, f) \leq T(r, H) + S(r, f) \leq m(r, f) + S(r, f).$$ From Lemma 2.2 ($a_1 = 0$, $a_2 = \beta$ and $a_3 = \infty$), $\bar{N}(r, \frac{1}{f'}) = S(r, f)$, (47), (37) and (32) we get

$$T(r, f') \leq \bar{N}\left(r, \frac{1}{f'}\right) + \bar{N}\left(r, \frac{1}{f' - \beta}\right) + \bar{N}(r, f) + S(r, f) \leq \bar{N}\left(r, \frac{1}{f' - \beta}\right) + \bar{N}_1(r, f) + \bar{N}_2(r, f) + S(r, f) \leq \bar{N}\left(r, \frac{1}{f' - \beta}\right) + m(r, f) + \bar{N}_2(r, f) + S(r, f) \leq \bar{N}\left(r, \frac{1}{f' - \beta}\right) + \bar{N}_2\left(r, \frac{1}{f' - \beta}\right) + S(r, f)$$

which results in

$$N_3\left(r, \frac{1}{f' - \beta}\right) = S(r, f).$$

Writing (15) as

$$f = \beta + \frac{1}{\omega}\left(\frac{f'}{f} - 1\right),$$

which implies

$$m(r, f) \leq m(r, \beta) + m\left(r, \frac{1}{\omega}\right) + m\left(r, \frac{f'}{f}\right) + O(1) = m\left(r, \frac{1}{\omega}\right) + S(r, f).$$
Also we know from (15) that if $z_\infty$ is a pole of $f$ with multiplicity $p \geq 1$, then $z_\infty$ is a zero of $\omega$ with multiplicity $p - 1$. Thus

$$N(r, f) - \tilde{N}(r, f) \leq N\left(r, \frac{1}{\omega}\right) + S(r, f). \tag{50}$$

Combining (49), (50) and (16) we have

$$m(r, f) + N(r, f) - \tilde{N}(r, f) \leq T(r, \omega) + S(r, f) = m(r, \omega) + N(r, \omega) + S(r, f) \leq N(r, \omega) + m\left(r, \frac{1}{f - \beta}\right) + S(r, f). \tag{51}$$

From (15), (27), (28), (50) and $\tilde{N}(r, \frac{1}{f}) = S(r, f)$, we conclude that

$$N(r, \omega) \leq \tilde{N}\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{f - \beta}\right) + S(r, f) = N_2\left(r, \frac{1}{f - \beta}\right) + S(r, f).$$

Together with (51) we get

$$m(r, f) + N_2(r, f) - \tilde{N}_2(r, f) \leq N_2\left(r, \frac{1}{f - \beta}\right) + m\left(r, \frac{1}{f - \beta}\right) + S(r, f). \tag{52}$$

From (31), it is easily verified that $H \not\equiv 0$ and

$$m\left(r, \frac{1}{f - \beta}\right) \leq m\left(r, \frac{1}{H}\right) + S(r, f). \tag{53}$$

By (36),

$$N_2\left(r, \frac{1}{f - \beta}\right) - \tilde{N}_2\left(r, \frac{1}{f - \beta}\right) \leq N\left(r, \frac{1}{H}\right) + S(r, f).$$

Combining this, (51) and (37) we obtain

$$m\left(r, \frac{1}{f - \beta}\right) + N_2\left(r, \frac{1}{f - \beta}\right) - \tilde{N}_2\left(r, \frac{1}{f - \beta}\right) \leq T(r, H) + S(r, f) \leq m(r, f) + S(r, f).$$

Adding this, (52) and (30) we deduce that

$$N_2(r, f) = S(r, f), \tag{54}$$
and
\[ m(r, f) = N_2 \left( r, \frac{1}{f - \beta} \right) + m \left( r, \frac{1}{f - \beta} \right) + S(r, f). \] (55)

By (55) we see that
\[ m(r, f) + N_{11} \left( r, \frac{1}{f - \beta} \right) = T(r, f) + S(r, f). \]

Hence from (54) and (30) we get
\[ N_{11}(r) = N_1 \left( r, \frac{1}{f - \beta} \right) + S(r, f) = N \left( r, \frac{1}{f - \beta} \right) + S(r, f) \]
\[ = N \left( r, \frac{1}{f' - \beta} \right) + S(r, f). \]

From this, (47) and (32) we see
\[ \bar{N} \left( r, \frac{1}{f' - \beta} \right) = N_{11}(r, f) + S(r, f) \leq m(r, f) + S(r, f) \]
\[ \leq \bar{N}_2 \left( r, \frac{1}{f' - \beta} \right) + S(r, f), \]
which results in
\[ N_{11} \left( r, \frac{1}{f' - \beta} \right) = S(r, f) \] (56)

Set
\[ G = \frac{1}{f} \left[ \frac{(f'/\beta)^\prime}{f' - \beta} - 2 \frac{(f/\beta)^\prime}{f - \beta} \right]. \] (57)

Similarly as (8) we have
\[ m(r, G) = S(r, f). \] (58)

if \( z_0 \) is a zero of \( f - \beta \) and \( f' - \beta \) with multiplicity 1 and 2 respectively, and \( \beta(z_0) \neq 0, \infty \), then \( G \) is holomorphic. Also, if \( z_\infty \) is a simple pole of \( f \), then by (57) we see \( G(z_\infty) = 0 \). We discuss the following two cases.

**Case 1.** \( G \neq 0 \). Then from (58), (56), (48), (54), (30) and \( \bar{N}(r, \frac{1}{f}) = S(r, f) \), we deduce that
\[ N_{11}(r, f) \leq N \left( r, \frac{1}{G} \right) + S(r, f) \]
\[ \leq -m \left( r, \frac{1}{G} \right) + N(r, G) + m(r, G) + S(r, f) \]
\[ \leq -m \left( r, \frac{1}{G} \right) + N(r, G) + S(r, f) \]
\[ \leq -m \left( r, \frac{1}{G} \right) + S(r, f) \]
which shows that
\[ N_1(r, f) + m\left(r, \frac{1}{G}\right) = S(r, f). \] (59)

Noting that
\[ f = \frac{1}{G} \left( \frac{(f'/\beta)' - 2(f/\beta)'}{f' - \beta} \right), \]
by (57). This imply,
\[ m(r, f) \leq m\left(r, \frac{1}{G}\right) + S(r, f). \]

From this, (59) and (54), we can see that \( T(r, f) = S(r, f) \) a contradiction.

**Case 2.** \( G \equiv 0 \). Integrating of (57) we have
\[ \beta(f' - \beta) = c(f - \beta)^2, \] (60)
where \( c \) is a nonzero constant. This yields
\[
2m(r, f) = m(r, f') + S(r, f) \\
\leq m\left(r, \frac{f'}{f}\right) + m(r, f) + S(r, f) \\
= m(r, f) + S(r, f)
\]
which means \( m(r, f) + S(r, f) \). Together with (47) and (54) gives the contradiction \( T(r, f) = S(r, f) \).

**Case II.** \( \beta - \beta' \equiv 0 \). Then \( N_1(r, \frac{1}{f - \beta}) \equiv 0 \) and if \( z_0 \) is a zero of \( f - \beta \) with multiplicity \( n \), then \( z_0 \) is a zero of \( f' - \beta \) with multiplicity \( n - 1 \). Using an argument similar to that in the Case I, we can deduce from (8) and (33) that \( \Omega \neq 0, H \neq 0 \) and
\[
N_2(r, f) - \tilde{N}_2(r, f) \leq N\left(r, \frac{1}{\Omega}\right) + S(r, f) \\
\leq -m\left(r, \frac{1}{\Omega}\right) + N(r, \Omega) + S(r, f) \\
\leq -m\left(r, \frac{1}{\Omega}\right) + \tilde{N}_2\left(r, \frac{1}{f - \beta}\right) + S(r, f) \] (61)
and
\[
N_2\left(r, \frac{1}{f - \beta}\right) - \tilde{N}_2\left(r, \frac{1}{f - \beta}\right) \leq N\left(r, \frac{1}{H}\right) + S(r, f) \\
\leq -m\left(r, \frac{1}{H}\right) + m(r, H) + S(r, f) \\
\leq -m\left(r, \frac{1}{H}\right) + m(r, f) + S(r, f). \] (62)
Combining (61), (12), (62) and (53) we conclude
\[ N_2(r, f) + N_3 \left( r, \frac{1}{f - \beta} \right) + m \left( r, \frac{1}{f - \beta} \right) = S(r, f) \tag{63} \]
and
\[ m(r, f) = \bar{N}_2 \left( r, \frac{1}{f - \beta} \right) + S(r, f). \tag{64} \]

Since \( \bar{N}(r, \frac{1}{f}) = S(r, f) \), it follows from (63) and Lemma 2.1 that
\[ m \left( r, \frac{1}{f' - \beta} \right) = S(r, f). \tag{65} \]

Set
\[ \Delta = \frac{f - \beta}{f' - \beta}. \]

It is easy to see that \( \Delta \not\equiv 0 \), \( N(r, \Delta) = S(r, f) \) and
\[
\bar{N}(r, f) + \bar{N}_2 \left( r, \frac{1}{f - \beta} \right) \leq N \left( r, \frac{1}{\Delta} \right) \\
\leq T(r, \Delta) + O(1) \\
\leq m(r, \Delta) + O(1) \\
\leq m(r, f) + m \left( r, \frac{1}{f' - \beta} \right) + S(r, f).
\]

Together with (64) and (65) we have \( \bar{N}(r, f) = S(r, f) \). Finally, from this, (63) and (64) we find that
\[
T(r, f) = m(r, f) + S(r, f) = \bar{N}_2 \left( r, \frac{1}{f - \beta} \right) \\
\leq \frac{1}{2} \bar{N}_2 \left( r, \frac{1}{f - \beta} \right) + S(r, f) \leq \frac{1}{2} T(r, f) + S(r, f),
\]
which gives the contradiction \( T(r, f) = S(r, f) \). This completes the proof of Theorem 2.4. \( \square \)

From Theorem 2.3 and Theorem 2.4 we immediately deduce the following corollary:

**Corollary 2.5** Let \( f \) be a non-constant meromorphic function, and let \( \beta \) be a small meromorphic function of \( f \) such that \( \beta \not\equiv 0, \infty \). If \( f \) and \( f' \) share 0 and \( \beta \) CM, then \( f \equiv f' \).

**Corollary 2.6** Let \( f \) be a non-constant meromorphic function, and let \( \beta \) be a small meromorphic function of \( f \) such that \( \beta \not\equiv 0, \infty \). If \( f \) and \( f' \) share 0 CM and \( \beta \) IM, then either \( f \equiv f' \) or \( \beta \) is a constant and \( f \) is given as (1) when \( \beta = a \).
3 Open Problem

From Corollary 2.5 and Corollary 2.6 we establish the following:

**Conjecture 3.1** Let $f$ be a non-constant meromorphic function, $\beta$ and $\alpha$ two distinct small meromorphic functions of $f$ with $\beta \neq \infty$ and $\alpha \neq \infty$. If $f$ and $f'$ share $\alpha$ and $\beta$ CM, then $f \equiv f'$.

**Conjecture 3.2** Let $f$ be a non-constant meromorphic function, and let $\beta$ be a small meromorphic function of $f$ such that $\beta \neq 0, \infty$. If $f$ and $f'$ share 0 and $\beta$ IM, then either $f \equiv f'$ or $\beta$ is a constant and $f$ is given as (1) when $\beta = a$.

Corollary 2.5 shows that Conjecture 3.1 is valid when $\alpha \equiv 0$ and Corollary 2.6 shows that Conjecture 3.2 is true if 0 IM replaced by 0 CM.

References


