Int. J. Open Problems Complex Analysis, Vol. 5, No. 1, March 2013
ISSN 2074-2827; Copyright ©ICSRS Publication, 2013
www.i-csrs.org

On certain generalized class of non-Bazilevič functions

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Abstract

In this paper, a subclass $N(\lambda, \alpha, A, B, g(z))$ ***of analytic functions is introduced, which is a generalized class of non-Bazilevič functions. The subordination relations, inclusion relations, distortion theorems and inequality properties are discussed by applying differential subordination method***.

Keywords: *non-Bazilevič function; subordination; distortion theorem.*
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Let H denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \tag{1}$$

that are analytic in the unit disk $U = \{z : |z| < 1\}$ and let S be the class of all the univalent functions in H. Further, let S^* and C denote the classes of the well-known starlike functions and convex functions, respectively.

For $0 < \alpha < 1$, a function $f(z) \in N(\alpha)$ if and only if $f(z) \in H$ and

$$\mathrm{Re}\Big\{\frac{zf'(z)}{f(z)}\Big(\frac{z}{f(z)}\Big)^{\alpha}\Big\} > 0 \tag{2}$$

$N(\alpha)$ was introduced by M.Obradović [1] recently, and he called this class of functions to be non-Bazilevič type.Until now, this class was studied in a direction of finding necessary conditions over α that embeds this class into the class of univalent functions or its subclasses, which is still an open problem.

Let $f(z)$ and $F(z)$ be analytic in U, then we say that the function $f(z)$ is subordinate to $F(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)| \leq |z|$, and $f(z) \equiv F(w(z))$, denoted $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in U, then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$.

Assume that $0 < \alpha < 1, \lambda \in \mathbb{C}, -1 \leq B \leq 1, A \neq B, A \in R, g(z) \in S^*$, we define the following subclass $\mathcal{N}(\lambda, \alpha, A, B, g(z))$ of H:

$$\Big\{f(z) \in H\colon \Big(1 + \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \prec \frac{1 + Az}{1 + Bz}, z \in U\Big\}. \tag{3}$$

Clearly, the class $\mathcal{N}(-1, \alpha, 1, -1, z)$ is the class of non-Bazilevič functions and the class $\mathcal{N}(-1, \alpha, 1, -2\beta, z)$ is the class of non-Bazilevič functions of order $\beta(0 \leq \beta < 1)$. In this paper, we will discuss the subordination relations, inclusion relations, distortion theorems and inequality properties of $\mathcal{N}(\lambda, \alpha, A, B, g(z))$.

2 Some lemmas

To prove our main result, we need the following lemmas:

Lemma 1 [2]. Let $F(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in U, $h(z)$ be analytic and convex in U, $h(0) = 1$. If

$$F(z) + \frac{1}{c}zF'(Z) \prec h(z) \tag{4}$$

where $c \neq 0$ and Re $c \geq 0$, then

$$F(z) \prec cz^{-c}\int_0^z t^{c-1}h(t)dt \prec h(z)$$

and $cz^{-c}\int_0^z t^{c-1}h(t)dt$ is the best dominant for differential subordination (4).

Lemma 2. Let $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then

$$\frac{1 + A_2 z}{1 + B_2 z} \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

Lemma 3 [3]. Let $F(z)$ be analytic and convex in U, $f(z) \in H, g(z) \in H$, and $f(z) \prec F(z), f(z) \prec F(z)$, then

$$\lambda f(z) + (1-\lambda)g(z) \prec F(z), 0 \le \lambda \le 1.$$

Lemma 4 [4]. Let $g(z) \in S^*$, for $|z| = r < 1$, then

$$\frac{r}{(1+r)^2} \le |g(z)| \le \frac{r}{(1-r)^2}, \tag{5}$$

and inequality (5) is sharp, with the extremal function defined by

$$g(z) = \frac{z}{(1-z)^2}.$$

3 Main Results

Theorem 1. Let $0 < \alpha < 1, \lambda \ge 0, -1 \le B \le 1, A \ne B, A \in \mathbf{R}$. If $f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

$$\Big(\frac{g(z)}{f(z)}\Big)^\alpha \prec \frac{1+Az}{1+Bz}. \tag{6}$$

Proof. If $\lambda = 0$, we obtain the result from the definition of $\mathcal{N}(\lambda, \alpha, A, B, g(z))$.

If $\lambda > 0$. Let $F(z) = (\frac{g(z)}{f(z)})^\alpha$, then $F(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in U. By taking the derivatives in the both sides, we have

$$\Big(1 + \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{g(z)}{f(z)}\Big)^\alpha - \lambda\frac{zf'(z)}{f(z)}\Big(\frac{g(z)}{f(z)}\Big)^\alpha = F(z) + \frac{\lambda}{\alpha}zF'(z). \tag{7}$$

Since $f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, we have

$$F(z) + \frac{\lambda}{\alpha}zF'(z) \prec \frac{1+Az}{1+Bz}.$$

It is obvious that $h(z) = (1+Az)/(1+Bz)$ is analytic, convex in U, $h(0) = 1$. Since $\mathrm{Re}\lambda \ge 0$, we have $\mathrm{Re}(\alpha/\lambda) \ge 0$. Therefore it follows from Lemma 1 that

$$\begin{aligned}\Big(\frac{g(z)}{f(z)}\Big)^\alpha = F(z) \quad &\prec \tfrac{\alpha}{\lambda} z^{-\frac{\alpha}{\lambda}} \int_0^1 \tfrac{1+At}{1+Bt} t^{\frac{\alpha}{\lambda}-1} dt \\ &\prec \tfrac{\alpha}{\lambda} \int_0^1 \tfrac{1+Azu}{1+Bzu} u^{\frac{\alpha}{\lambda}-1} du \\ &\prec \tfrac{1+Az}{1+Bz}.\end{aligned}$$

Corollary 1. Let $0 < \alpha < 1, \lambda > 0, \beta \ne 1$, if

$$\Big(1 + \lambda\frac{zg'(z)}{g(z)}\Big)\Big(\frac{g(z)}{f(z)}\Big)^\alpha - \lambda\frac{zf'(z)}{f(z)}\Big(\frac{g(z)}{f(z)}\Big)^\alpha \prec \frac{1+(1-2\beta)z}{1-z}, z \in U,$$

then

$$\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \beta + \frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1+zu}{1+zu} u^{\frac{\alpha}{\lambda}-1} du, z \in U.$$

Corollary 2. Let $0 < \alpha < 1, \lambda \geq 0$, then

$$\mathcal{N}(\lambda, \alpha, A, B, g(z)) \subset \mathcal{N}(0, \alpha, A, B, g(z)).$$

Theorem 2. Let $0 < \alpha < 1, \lambda_2 \geq \lambda_1 \geq 0, -1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then $\mathcal{N}(\lambda_2, \alpha, A_2, B_2, g(z)) \subset \mathcal{N}(\lambda_1, \alpha, A_1, B_1, g(z))$.

Proof. Let $f(z) \in \mathcal{N}(\lambda_2, \alpha, A_2, B_2, g(z))$, we have $f(z) \in H$ and

$$\left(1 + \lambda_2 \frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \lambda_2 \frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \frac{1+A_2 z}{1+B_2 z}, z \in U.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, it follows from Lemma 2 that

$$\left(1 + \lambda_2 \frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \lambda_2 \frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \frac{1+A_1 z}{1+B_1 z}, z \in U. \tag{8}$$

That is $f(z) \in \mathcal{N}(\lambda_2, \alpha, A_1, B_1, g(z))$. So Theorem 2 is proved when $\lambda_2 = \lambda_1 \geq 0$.

When $\lambda_2 > \lambda_1 \geq 0$, it follows from Corollary 2 that $f(z) \in \mathcal{N}(0, \alpha_2, A_1, B_1, g(z))$. That is

$$\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \frac{1+A_1 z}{1+B_1 z}, z \in U. \tag{9}$$

But

$$\begin{aligned} &\left(1 + \lambda_1 \tfrac{zg'(z)}{g(z)}\right)\left(\tfrac{g(z)}{f(z)}\right)^{\alpha} - \lambda_1 \tfrac{zf'(z)}{f(z)}\left(\tfrac{g(z)}{f(z)}\right)^{\alpha} \\ &= \left(1 - \tfrac{\lambda_1}{\lambda_2}\right)\left(\tfrac{g(z)}{f(z)}\right)^{\alpha} + \tfrac{\lambda_1}{\lambda_2}\left[\left(1 + \lambda_2 \tfrac{zg'(z)}{g(z)}\right)\left(\tfrac{g(z)}{f(z)}\right)^{\alpha} - \lambda_2 \tfrac{zf'(z)}{f(z)}\left(\tfrac{g(z)}{f(z)}\right)^{\alpha}\right], z \in U. \end{aligned}$$

Note that $h_1(z) = (1 + A_1 z)/(1 + B_1 z)$ is analytic and convex in U. So we obtain from Lemma 3 and differential subordinations (8) and (9) that

$$\left(1 + \lambda_1 \frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \lambda_1 \frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \frac{1+A_1 z}{1+B_1 z}, z \in U.$$

That is $f(z) \in \mathcal{N}(\lambda_1, \alpha, A_1, B_1, g(z))$. Thus we have

$$\mathcal{N}(\lambda_2, \alpha, A_2, B_2, g(z)) \subset \mathcal{N}(\lambda_1, \alpha, A_1, B_1, g(z)).$$

Corollary 3. Let $0 < \alpha < 1, \lambda_2 \geq \lambda_1 \geq 0, 1 > \beta_2 \geq \beta_1 \geq 0$, then

$$\mathcal{N}(\lambda_2, \alpha, A_2, B_2, g(z)) \subset \mathcal{N}(\lambda_1, \alpha, A_1, B_1, g(z)).$$

Theorem 3. Let $0 < \alpha < 1, \lambda > 0, -1 \leq B < A \leq 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

$$\frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du, z \in U. \quad (10)$$

and inequality (10) is sharp with the extremal function defined by

$$f_{\lambda,\alpha,A,B}(z, g(z)) = g(z)\Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Auz}{1+Buz}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}}, z \in U. \quad (11)$$

Proof. Since $f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, by Theorem 1, we have

$$\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \prec \frac{\alpha}{\lambda}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du.$$

Therefore it follows from the definition of the subordination and $A > B$ that

$$\begin{aligned}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} &< \sup_{z\in U}\mathrm{Re}\Big[\tfrac{\alpha}{\lambda}\textstyle\int_0^1 \tfrac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du\Big] \\ &\leq \tfrac{\alpha}{\lambda}\textstyle\int_0^1 \sup_{z\in U}\mathrm{Re}\Big(\tfrac{1+Azu}{1+Bzu}\Big)u^{\frac{\alpha}{\lambda}-1}du \\ &< \tfrac{\alpha}{\lambda}\textstyle\int_0^1 \tfrac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} &> \inf_{z\in U}\mathrm{Re}\Big[\tfrac{\alpha}{\lambda}\textstyle\int_0^1 \tfrac{1+Azu}{1+Bzu}u^{\frac{\alpha}{\lambda}-1}du\Big] \\ &\geq \tfrac{\alpha}{\lambda}\textstyle\int_0^1 \inf_{z\in U}\mathrm{Re}\Big(\tfrac{1+Azu}{1+Bzu}\Big)u^{\frac{\alpha}{\lambda}-1}du \\ &> \tfrac{\alpha}{\lambda}\textstyle\int_0^1 \tfrac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du.\end{aligned}$$

The inequality (10) is sharp by taking the function in (11).

By applying the similar method as in Theorem 3, we have

Theorem 4. Let $0 < \alpha < 1, \lambda > 0, -1 \leq A < B \leq 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

$$\frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du, z \in U, \quad (12)$$

and inequality (12) is sharp with the extremal function defined by eqnarray (11).

Corollary 4. Let $0 < \alpha < 1, \lambda > 0, 0 \le \beta < 1, f(z) \in \mathcal{N}(\lambda, \alpha, \beta, g(z))$, then

$$\frac{\alpha}{\lambda}\int_0^1 \frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{\lambda}-1}du < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \frac{\alpha}{\lambda}\int_0^1 \frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{\lambda}-1}du, z \in U, \tag{13}$$

and inequality (13) is equivalent to

$$\beta + \frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{\lambda}-1}du < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \beta + \frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{\lambda}-1}du, z \in U.$$

Corollary 5. Let $0 < \alpha < 1, \lambda > 0, \beta > 1, g(z) \in S^*$, $f(z) \in H$, and

$$\mathrm{Re}\Big[(1+\lambda\frac{zg'(z)}{g(z)})(\frac{g(z)}{f(z)})^{\alpha} - \lambda\frac{zf'(z)}{f(z)}(\frac{g(z)}{f(z)})^{\alpha}\Big] < \beta, z \in U,$$

then

$$\frac{\alpha}{\lambda}\int_0^1 \frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{\lambda}-1}du < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \frac{\alpha}{\lambda}\int_0^1 \frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{\lambda}-1}du, z \in U, \tag{14}$$

and inequality (14) is equivalent to

$$\beta + \frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{\lambda}-1}du < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \beta + \frac{(1-\beta)\alpha}{\lambda}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{\lambda}-1}du, z \in U.$$

Note that if $\mathrm{Re}w \ge 0$, then $(\mathrm{Re}w)^{\frac{1}{2}} \le \mathrm{Re}w^{\frac{1}{2}} \le |\mathrm{Re}w|^{\frac{1}{2}}$. Thus, we have the following theorem.

Theorem 5. Let $\alpha > 0, \lambda > 0, -1 \le B < A \le 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

$$\Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du\Big)^{\frac{1}{2}} < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\frac{\alpha}{2}} < \Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du\Big)^{\frac{1}{2}}, z \in U, \tag{15}$$

and inequality (15) is sharp with the extremal function defined by eqnarray (11).

Proof. According to Theorem 1, we have

$$\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \prec \frac{1+Az}{1+Bz}.$$

Since $-1 \leq B < A \leq 1$, we have

$$0 \leq \frac{1-A}{1-B} < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} < \frac{1+A}{1+B}.$$

Hence the result follows by Theorem 3.

By applying the similar method as in Theorem 6, we have

Theorem 6. Let $0 < \alpha < 1, \lambda > 0, -1 \leq A < B \leq 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

$$\Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{\lambda}-1}du\Big)^{\frac{1}{2}} < \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\frac{\alpha}{2}} < \Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{\lambda}-1}du\Big)^{\frac{1}{2}}, z \in U, \tag{16}$$

and inequality (16) is sharp with the extremal function defined by eqnarray (11).

Theorem 7. Let $0 < \alpha < 1, \lambda \geq 0, -1 \leq B < A \leq 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

(i) If $\lambda = 0$, when $|z| = r < 1$, we have

$$|g(z)| \Big(\frac{1+Br}{1+Ar}\Big)^{\frac{1}{\alpha}} \leq |f(z)| \leq |g(z)| \Big(\frac{1-Br}{1-Ar}\Big)^{\frac{1}{\alpha}}, \tag{17}$$

and inequality (17) is sharp with the extremal function defined by

$$f(z) = g(z)\Big(\frac{1+Bz}{1+Az}\Big)^{\frac{1}{\alpha}}. \tag{18}$$

(ii) If $\lambda \neq 0$, when $|z| = r < 1$, we have

$$|g(z)| \Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}} \leq |f(z)| \leq |g(z)| \Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}} \tag{19}$$

and inequality (19) is sharp with the extremal function defined by eqnarray (11).

Proof. (i) If $\lambda = 0$ and $f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z)), -1 \leq B < A \leq 1$. We obtain from the definition of $\mathcal{N}(\lambda, \alpha, A, B, g(z))$ that

$$\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of the subordination that

$$\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} = \frac{1+Aw(z)}{1+Bw(z)},$$

where $w(z)$ is analytic in U. By applying Schwarz Lemma we obtain that $w(z) = c_1 z + c_2 z^2 + \cdots$ and $|w(z)| \leq |z|$, so when $|\, z \,| = r < 1$, we have

$$\Big|\frac{g(z)}{f(z)}\Big|^{\alpha} = \Big|\frac{1 + Aw(z)}{1 + Bw(z)}\Big| \leq \frac{1 + A|w(z)|}{1 + B|w(z)|} \leq \frac{1 + Ar}{1 + Br}$$

and

$$\Big|\frac{g(z)}{f(z)}\Big|^{\alpha} \geq \mathrm{Re}\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \geq \frac{1 - Ar}{1 - Br}.$$

It is obvious that inequality (17) is sharp with the extremal function defined by (18).

(ii) If $\lambda \neq 0$, according to Theorem 1 we have

$$\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \prec \frac{\alpha}{\lambda}\int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{\alpha}{\lambda}-1} du.$$

Therefore it follows from the definition of the subordination that

$$\Big(\frac{g(z)}{f(z)}\Big)^{\alpha} = \frac{\alpha}{\lambda}\int_0^1 \frac{1 + Auw(z)}{1 + Buw(z)} u^{\frac{\alpha}{\lambda}-1} du,$$

where $w(z) = c_1 z + c_2 z^2 + \cdots$ is analytic in U and $|w(z)| \leq |z|$. So when $|\, z \,| = r < 1$, we have

$$\begin{aligned}\Big|\frac{g(z)}{f(z)}\Big|^{\alpha} &\leq \tfrac{\alpha}{\lambda}\int_0^1 \Big|\tfrac{1+Auw(z)}{1+Buw(z)}\Big| u^{\frac{\alpha}{\lambda}-1} du \\ &\leq \tfrac{\alpha}{\lambda}\int_0^1 \tfrac{1+Au|w(z)|}{1+Bu|w(z)|} u^{\frac{\alpha}{\lambda}-1} du \\ &\leq \tfrac{\alpha}{\lambda}\int_0^1 \tfrac{1+Aur}{1+Bur} u^{\frac{\alpha}{\lambda}-1} du\end{aligned}$$

and

$$\Big|\frac{g(z)}{f(z)}\Big|^{\alpha} \geq \Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \geq \frac{\alpha}{\lambda}\int_0^1 \frac{1 - Aur}{1 - Bur} u^{\frac{\alpha}{\lambda}-1} du.$$

By taking $f_{\lambda,\alpha,A,B}(z, g(z)) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$ defined by (11), we can see that inequality (19) is sharp.

By applying the similar method as in Theorem 7, we have

Theorem 8. Let $0 < \alpha < 1, \lambda \geq 0, -1 \leq A < B \leq 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

(i) If $\lambda = 0$, when $|\, z \,| = r < 1$, we have

$$|\, g(z) \,| \Big(\frac{1 - Br}{1 - Ar}\Big)^{\frac{1}{\alpha}} \leq |\, f(z) \,| \leq |\, g(z) \,| \Big(\frac{1 + Br}{1 + Ar}\Big)^{\frac{1}{\alpha}}, z \in U \tag{20}$$

and inequality (20) is sharp, with the extremal function defined by (18).

(ii) If $\lambda \neq 0$, when $| z |= r < 1$, we have

$$| g(z) | \Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}} \leq | f(z) | \leq | g(z) | \Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}} \quad (21)$$

and inequality (21) is sharp with the extremal function defined by eqnarray (11).

Corollary 6. Let $0 < \alpha < 1, \lambda > 0, -1 \leq B < A \leq 1, f(z) \in \mathcal{N}(\lambda, \alpha, A, B, g(z))$, then

(i) If $\lambda = 0$, when $| z |= r < 1$, we have

$$\frac{r}{(1+r)^2}\Big(\frac{1+Br}{1+Ar}\Big)^{\frac{1}{\alpha}} \leq | f(z) | \leq \frac{r}{(1-r)^2}\Big(\frac{1-Br}{1-Ar}\Big)^{\frac{1}{\alpha}} \quad (22)$$

and inequality (22) is sharp, with the extremal function defined by

$$f(z) = \frac{z}{(1-z)^2}\Big(\frac{1+Bz}{1+Az}\Big)^{\frac{1}{\alpha}} \quad (23)$$

(ii) If $\lambda \neq 0$, when $| z |= r < 1$,we have

$$\frac{r}{(1+r)^2}\Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}} \leq | f(z) | \leq \frac{r}{(1-r)^2}\Big(\frac{\alpha}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\alpha}{\lambda}-1}du\Big)^{-\frac{1}{\alpha}} \quad (24)$$

and inequality (24) is sharp, with the extremal function defined by eqnarray (11).

4 Open Problem

In our last section, we suggest an open problem as follows:

Let $p, h \in H$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \quad (25)$$

then p is a solution of the differential superordination (4.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (4.1). A univalent subordinant Q that satisfies $q \prec Q$ for all subordinants q of (4.1) is said to be the best subordinant.

Recently Miller and Mocanu [5] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{26}$$

Using the results of Miller and Mocanu [5], We can consider sufficient conditions h, q_1, q_2 and ϕ for which the following implication holds:

$$q_1(z) \prec \Big(\frac{g(z)}{f(z)}\Big)^{\alpha} \prec q_2(z), \tag{27}$$

or

$$q_1(z) \prec \Big(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\Big)^{\alpha} \prec q_2(z), \tag{28}$$

where $f(z) \in H$, $g(z) \in \mathcal{S}^*$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.

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