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Majorization for Certain Classes of Analytic Multivalent Functions

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Abstract

In the present paper we investigate the majorization properties for certain classes of multivalent analytic functions defined by extended multiplier transformation. Moreover, we point out some new or known consequences of our main result.

Keywords: *Analytic functions, Multivalent functions, Starlike functions, Subordination, Majorization property, Extended multiplier transformation.*
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1 Introduction

Let f and g be two analytic functions in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We say that f is majorized by g in $\mathcal{U}$ (see [20]), and we write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}),$$

if there exists a function φ, analytic in $\mathcal{U}$ such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \mathcal{U}). \tag{1.1}$$

It may be noted that the notion of majorization is closely related to the concept of quasi-subordination between analytic functions.

For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, and we write

$$f(z) \prec g(z),$$

if there exists a *Schwarz function* w, which (by definition) is analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathcal{U}$, such that

$$f(z) = g\,(w(z)) \quad (z \in \mathcal{U}).$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad \left(p \in \mathbb{N} := \{1, 2, 3, \dots\}\right), \tag{1.2}$$

which are analytic in the open unit disk $\mathcal{U}$. For simplicity, we write $\mathcal{A} := \mathcal{A}_1$.

Catas [1] extended the multiplier transformation and defined the operator

$$\mathcal{I}_p^{\mu}(n, \lambda) : \mathcal{A}_p \to \mathcal{A}_p\ ,$$

on $f \in \mathcal{A}_p$, defined by (1.2) by the following infinite series:

$$\mathcal{I}_p^{\mu}(n, \lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p + \lambda + \mu(k-p)}{p + \lambda} \right)^n a_k z^k,$$
$$\left(\lambda \geq 0;\ \mu \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N} \cup \{0\}\right), \tag{1.3}$$

where $f \in \mathcal{A}_p$ has the form (1.2). We note that:

$\mathcal{I}_p^0(0, 1) f(z) = f(z),\ \mathcal{I}_p^0(1, 1) = \frac{z f'(z)}{p},$ and $\mathcal{I}_p^0(2, 1) = \frac{z f'(z) + z^2 f''(z)}{p^2}$

By specializing the parameters λ, μ and p, we obtain the following operators studied by various authors:

(*i*) $\mathcal{I}_p^1(n, \lambda) f(z) = \mathcal{I}_p(n, \lambda) f(z),$ see [2, 3, 4]
(*ii*) $\mathcal{I}_p^1(n, 0) f(z) = D_p^n f(z)$ (see [5, 6, 7]
(*iii*) $\mathcal{I}_1^1(n, \lambda) f(z) = \mathcal{I}_{\lambda}^n f(z)$ (see [8, 9])
(*iv*) $\mathcal{I}_1^1(n, 0) f(z) = D^n f(z)$ (see [10, 11])
(*v*) $\mathcal{I}_1^{\mu}(n, 0) f(z) = D_{\mu}^n f(z)$ (see [12])
(*vi*) $\mathcal{I}_1^1(n, 1) f(z) = \mathcal{I}_n f(z)$ (see [13])
(*vii*) $\mathcal{I}_p^{\mu}(n, 0) f(z) = D_{p,\mu}^n f(z)$

where $D_{p,\mu}^n f(z)$ is defined by

$$D_{p,\mu}^n f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p + \mu(k-p)}{p} \right)^n a_k z^k.$$

Definition 1.1 *A function $f \in \mathcal{A}_p$ is said to be in the class $S_{p,q}^{n,\lambda}[A,B;\gamma]$, if and only if*

$$1+\frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right)^{(q+1)}}{\left(\mathcal{I}_p^{\mu}(n,\lambda)f(z)\right)^{(q)}}-p+q+n\right)\prec\frac{1+\frac{n}{\gamma}+Az}{1+Bz}, \tag{1.4}$$

with $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $n,q \in \mathbb{N}_0$, $\lambda \geq 0$; $\mu \geq 0$, $|p+\lambda| \geq |\gamma(A-B)+(p-n+\lambda)B|$, $\gamma \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$, where $\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)^{(q)}$ represents the q times derivative of $\mathcal{I}_p^{\mu}(n,\lambda)f$.

Clearly, we have the following relationships:

$$\begin{aligned}
&(i) && S_{p,q}^{n,0}[1,-1;\gamma]=S_{p,q}^{n}(\gamma);\\
&(ii) && S_{p,0}^{n,0}[1,-1;\gamma]=S^{n}(p,\gamma);\\
&(iii) && S_{p,q}^{0,0}[1,-1;\gamma]=S_{p,q}(\gamma);\\
&(iv) && S_{1,0}^{1,0}[1,-1;\gamma]=S(\gamma);\\
&(v) && S_{1,0}^{0,0}[1,-1;1-\alpha]=S^*(\alpha), \quad \text{for} \quad 0\leq\alpha<1.
\end{aligned}$$

The class $S_{p,q}(\gamma)$ was introduced by O. Altintaş and H. M. Srivastava [15]. The class $S(\gamma)$ is said to be class of *starlike functions of complex order* $\gamma \in \mathbb{C}^*$ in $\mathcal{U}$, which were considered by Nasr and Aouf [21] and Wiatrowski [22], while $S^*(\alpha)$ denotes the class of *starlike functions of order* α in $\mathcal{U}$.

A majorization problem for the class $S(\gamma)$ has recently been investigated by Altintaş *et al.* [16], and majorization problems for the class $S^* = S^*(0)$ have been investigated by MacGregor [20]. Very recently, Goyal and Goswami [17] generalized these results for the fractional derivative operator. In the present paper we investigate a majorization problem for the class $S_{p,q}^{n,\lambda}[A,B;\gamma]$, and we give some special cases of our main result obtained for appropriate choices of the parameters A, B and λ. Further results on majorization problems can be found in [18], [19] etc.

2 Majorization problem for the class $S_{p,q}^{n,\lambda}[A,B;\gamma]$

We begin by stating and proving the following main result.

Theorem 2.1 *Let the function $f \in \mathcal{A}_p$, and suppose that $g \in S_{p,q}^{n,\lambda}[A,B;\gamma]$. If $\left(\mathcal{I}_p^{\mu}(n,\lambda)f\right)^{(q)}$ is majorized by $\left(\mathcal{I}_p^{\mu}(n,\lambda)g\right)^{(q)}$ in $\mathcal{U}$, then*

$$\left|\left(\mathcal{I}_p^{\mu}(n+1,\lambda)f(z)\right)^{(q)}\right| \leq \left|\left(\mathcal{I}_p^{\mu}(n+1,\lambda)g(z)\right)^{(q)}\right| \quad \textit{for} \quad |z|\leq R, \tag{2.1}$$

where $R = R(p, \gamma, \lambda, \mu, n, A, B)$ *is the smallest positive root of the equation*

$$\begin{aligned} &\Big|\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}\Big| r^3 \\ &- |p+\lambda|\,(1+2|B|)\,r^2 \\ &- \Big[|\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}| + 2|p+\lambda|\Big] r \\ &+ |p+\lambda| = 0 \end{aligned} \tag{2.2}$$

and $-1 \le B < A \le 1$; $p \in \mathbb{N}$; $n, q \in \mathbb{N}_0$; $\lambda \ge 0$; $|p+\lambda| \ge |\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu$
$\gamma \in \mathbb{C}^*$.

Proof.
Since $g \in S^{n,\lambda}_{p,q}[A, B; \gamma]$, we find from (1.4) that

$$1 + \frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q+1)}}{\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q)}} - p + q + n\right) = \frac{1 + \frac{n}{\gamma} + Aw(z)}{1 + Bw(z)}, \tag{2.3}$$

where ω is analytic in $\mathcal{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. From (2.3), we get

$$\frac{z\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q+1)}}{\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q)}} = \frac{p - q + [\gamma(A-B) + B(p-q-n)]w(z)}{1 + Bw(z)}. \tag{2.4}$$

Now, making use of the relation

$$\begin{aligned} &\mu z\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q+1)} = (p+\lambda)\left(\mathcal{I}^{\mu}_p(n+1,\lambda)g(z)\right)^{(q)} \\ &- [p(1-\mu) + q\mu + \lambda]\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q)}, \end{aligned} \tag{2.5}$$

and after some simple computations, we get from (2.4) the following relation.

$$\begin{aligned} &\frac{\left(\mathcal{I}^{\mu}_p(n+1,\lambda)g(z)\right)^{(q)}}{\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q)}} = \\ &\frac{p + \lambda + [\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}]w(z)}{(p+\lambda)(1 + Bw(z)} \end{aligned}$$

which, after some easy calculations gives

$$\begin{aligned} &\left|\left(\mathcal{I}^{\mu}_p(n,\lambda)g(z)\right)^{(q)}\right| \le \\ &\frac{|p+\lambda|[1 + |B||z|]\left|\left(\mathcal{I}^{\mu}_p(n+1,\lambda)g(z)\right)^{(q)}\right|}{|p+\lambda| - |\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}|\,|z|}. \end{aligned} \tag{2.6}$$

Next, since $\left(\mathcal{I}_p^\mu(n,\lambda)f\right)^{(q)}$ is majorized by $\left(\mathcal{I}_p^\mu(n,\lambda)g\right)^{(q)}$ in the open unit disk $\mathcal{U}$, from (1.1) we have

$$\left(\mathcal{I}_p^\mu(n,\lambda)f(z)\right)^{(q)} = \varphi(z)\left(\mathcal{I}_p^\mu(n,\lambda)g(z)\right)^{(q)}.$$

Differentiating the last equality with respect to z, and then multiplying by z, we get

$$z\left(\mathcal{I}_p^\mu(n,\lambda)f(z)\right)^{(q+1)} = z\varphi'(z)\left(\mathcal{I}_p^\mu(n,\lambda)g(z)\right)^{(q)} + z\varphi(z)\left(\mathcal{I}_p^\mu(n,\lambda)g(z)\right)^{(q+1)}, \tag{2.7}$$

and using (2.5), we easily obtain

$$\begin{aligned}(p+\lambda)\left(\mathcal{I}_p^\mu(n+1,\lambda)f(z)\right)^{(q)} &= z\varphi'(z)\left(\mathcal{I}_p^{\mu,(q)}(n,\lambda)g(z)\right)^{(q)} \\ &+ (p+\lambda)\varphi(z)\left(\mathcal{I}_p^\mu(n+1,\lambda)g(z)\right)^{(q)}.\end{aligned} \tag{2.8}$$

Thus, by noting that the Schwarz function φ satisfies the inequality (see, e.g. Nehari [14])

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad (z\in\mathcal{U}) \tag{2.9}$$

and using (2.6) and (2.9) in (2.8), we get

$$\begin{aligned}&\left|\left(\mathcal{I}_p^\mu(n+1,\lambda)f(z)\right)^{(q)}\right| \le \\ &\left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{|p+\lambda|(1+|B||z|)|z|}{|p+\lambda| - |\mu\gamma(A-B)+B\{\mu(p-q-n)+p(1-\mu)+\mu q+\lambda\}|\,|z|}\right) \\ &\cdot\left|\left(\mathcal{I}_p^\mu(n+1,\lambda)g(z)\right)^{(q)}\right|,\end{aligned}$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho, \quad (0\le\rho\le 1)$$

leads us to the inequality

$$\begin{aligned}&\left|\left(\mathcal{I}_p^\mu(n+1,\lambda)f(z)\right)^{(q)}\right| \le \\ &\frac{\Phi(r,\rho)\left|\left(\mathcal{I}_p^\mu(n+1,\lambda)g(z)\right)^{(q)}\right|}{(1-r^2)\Big[|p+\lambda| - \big|\mu\gamma(A-B)+B\{\mu(p-q-n)+p(1-\mu)+\mu q+\lambda\}\big|r\Big]},\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}\Phi(r,\rho) &= -|p+\lambda|r(1+|B|r)\rho^2 \\ &+ (1-r^2)\Big[|p+\lambda| - \big|\mu\gamma(A-B)+B\{\mu(p-q-n)+p(1-\mu)+\mu q+\lambda\}|r\Big]\rho \\ &+ |p+\lambda|r(1+|B|r).\end{aligned}$$

If we denote

$$\Psi(r,\rho) = \frac{\Phi(r,\rho)}{(1-r^2)\Big[|p+\lambda| - \big|\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}\big| r\Big]},$$

then (2.10) becomes

$$\left|\left(\mathcal{I}_p^{\mu}(n+1,\lambda)f(z)\right)^{(q)}\right| \le \Psi(r,\rho)\left|\left(\mathcal{I}_p^{\mu}(n+1,\lambda)g(z)\right)^{(q)}\right|.$$

Now, in order to prove Theorem 2.1, we have to determine

$$\begin{aligned} R &= \max\{r \in [0,1] : \Psi(r,\rho) \le 1, \forall \rho \in [0,1]\} \\ &= \max\{r \in [0,1] : \chi(r,\rho) \ge 0, \forall \rho \in [0,1]\}, \end{aligned}$$

where

$$\begin{aligned} \chi(r,\rho) &= (1-\rho)(1-r^2)\Big[|p+\lambda| - \big|\mu\gamma(A-B) + B\{\mu(p-q-n) \\ &+ p(1-\mu) + \mu q + \lambda\}\big| r\Big] \\ &- (1-\rho^2)|p+\lambda|(1+|B|r)r. \end{aligned}$$

which shows that the inequality $\chi(r,\rho) \ge 0$ is equivalent to

$$\begin{aligned} \xi(r,\rho) &= \Big[|p+\lambda| - \big|\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) \\ &+ \mu q + \lambda\}\big| r\Big]\left(1-r^2\right) \\ &- |p+\lambda|\,(1+|B|r)\,r(1+\rho) \ge 0, \end{aligned}$$

while the function $\xi(r,\rho)$ takes its minimum value at $\rho = 1$ with $R = R(p,\gamma,\lambda,\mu,n,A,B)$, the smallest positive root of the equation (2.5), i.e.

$$\min\{\xi(r,\rho) : \rho \in [0,1]\} = \xi(r,1) = \phi(r),$$

where

$$\begin{aligned} \phi(r) &= \big|\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}\big| r^3 \\ &- |p+\lambda|\,(1+2|B|)\,r^2 \\ &- \Big[|\mu\gamma(A-B) + B\{\mu(p-q-n) + p(1-\mu) + \mu q + \lambda\}| + 2|p+\lambda|\Big] r \\ &+ |p+\lambda| = 0. \end{aligned}$$

Thus it follows that $\phi(r) \ge 0$ for all $r \in [0,R]$, which proves the desired result (2.1).

Setting $\mu = 1$ in (2.1) and (2.2), we get the following results of [2]:

Corollary 2.1 *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S_{p,q}^{n,\lambda}(\gamma)$. If $(\mathcal{I}_p(n,\lambda)f)^{(q)}$ is majorized by $(\mathcal{I}_p(n,\lambda)g)^{(q)}$ in $\mathcal{U}$, then*

$$\left|(\mathcal{I}_p(n+1,\lambda)f(z))^{(q)}\right| \leq \left|(\mathcal{I}_p(n+1,\lambda)g(z))^{(q)}\right| \quad \text{for} \quad |z| \leq r_1,$$

where r_1 is the smallest positive root of the following cubic equation

$$\begin{aligned} &\left|\gamma(A-B)+B(p-n+\lambda)\right|r^3 - |p+\lambda|\,(1+2|B|)\,r^2 \\ &-\Big[|\gamma(A-B)+B(p-n+\lambda)|+2|p+\lambda|\Big]r+|p+\lambda|=0 \end{aligned} \tag{2.11}$$

and $-1 \leq B < A \leq 1$; $p \in \mathbb{N}$; $n,q \in \mathbb{N}_0$; $\lambda \geq 0$; $|p+\lambda| \geq |\gamma(A-B)+B(p-n+\lambda)|$; $\gamma \in \mathbb{C}^*$.

Setting $A = 1$ and $B = -1$ in Corollary 2.1, the equation (2.2) becomes

$$\Big|2\gamma-(p-n+\lambda)\Big|r^3-3|p+\lambda|r^2-\Big[|2\gamma-(p-n+\lambda)|+2|p+\lambda|\Big]r+|p+\lambda|=0. \tag{2.12}$$

We see that $r = -1$ is one of the roots of this equation, and the other two roots are given by

$$|2\gamma-p+n-\lambda|r^2-\Big[|2\gamma-p+n-\lambda|+3|p+\lambda|\Big]r+|p+\lambda|=0,$$

so we may easily find the smallest positive root of (2.12). Hence, we have the following result:

Corollary 2.2 *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S_{p,q}^{n,\lambda}(\gamma)$. If $(\mathcal{I}_p(n,\lambda)f)^{(q)}$ is majorized by $(\mathcal{I}_p(n,\lambda)g)^{(q)}$ in $\mathcal{U}$, then*

$$\left|(\mathcal{I}_p(n+1,\lambda)f(z))^{(q)}\right| \leq \left|(\mathcal{I}_p(n+1,\lambda)g(z))^{(q)}\right| \quad \text{for} \quad |z| \leq r_1,$$

where

$$r_1 = r_1(p,\gamma,\lambda,n) = \frac{\eta-\sqrt{\eta^2-4|2\gamma-p+n-\lambda||p+\lambda|}}{2|2\gamma-p+n-\lambda|},$$

with $\eta = 3|p+\lambda|+|2\gamma-p+n-\lambda|$, and $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \geq 0$, $|p+\lambda| \geq |2\gamma-p+n-\lambda|$, $\gamma \in \mathbb{C}^*$.

Substituting $\lambda = 0$ in (2.11), we obtain a Sălăgean-type operator for multivalent function defined by

$$D_p^n f(z) = \mathcal{I}_p(n,0) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)^n a_k z^k, \quad (n \in \mathbb{N}_0),$$

where $f \in \mathcal{A}_p$ is given by (1.2).

Putting $A = 1$, $B = -1$, and $\lambda = 0$ in Corollary 2.2, we obtain the following result for the operator D_p^n:

Corollary 2.3 *Let the function* $f \in \mathcal{A}_p$ *and suppose that* $g \in S_{p,q}^{n}(\gamma)$. *If* $\left(D_p^n f\right)^{(q)}$ *is majorized by* $\left(D_p^n g\right)^{(q)}$, *then*

$$\left|\left(D_p^{n+1} f(z)\right)^{(q)}\right| \leq \left|\left(D_p^{n+1} g(z)\right)^{(q)}\right| \quad \text{for} \quad |z| \leq r_2,$$

where

$$r_2 = r_2(p, \gamma, n) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4p|2\gamma - p + n|}}{2|2\gamma - p + n|},$$

with $\eta_1 = 3p + |2\gamma - p + n|$, *and* $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\gamma \in \mathbb{C}^*$.

Further, putting $n = q = 0$, $p = 1$ in Corollary 2.3, we obtain the result of Altinas *et. al.* [16]:

Corollary 2.4 *Let the function* $f \in \mathcal{A}$ *be univalent in the open unit disk* Δ, *and suppose that* $g \in S(\gamma)$. *If* f *is majorized by* g *in* Δ, *then*

$$|f'(z)| \leq |g'(z)| \quad \text{for} \quad |z| \leq r_3,$$

where

$$r_3 = r_3(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|},$$

with $\gamma \in \mathbb{C}^*$.

Also, putting $\gamma = 1$ in Corollary 2.4, we obtain the well-known result given by MacGregor [20]:

Corollary 2.5 *Let the function* $f \in \mathcal{A}$ *be univalent in the open unit disk* Δ, *and suppose that* $g \in S^*$. *If* f *is majorized by* g *in* Δ, *then*

$$|f'(z)| \leq |g'(z)| \quad \text{for} \quad |z| \leq 2 - \sqrt{3}.$$

3 Open Problem

In this paper we have solved majorization problem for the class of analytic functions. Are these results can be obtained for meromorphic multivalent function?

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