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Subclass of Meromorphic Functions with Positive Coefficients Defined by Frasin and Darus Operator

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Abstract: In this paper we introduce and study new class $F^n(\alpha,\beta,\gamma)$ *of meromorphic univalent functions defined in* $U^* = \{z : z \in \mathbb{C}$ *and* $0 < |z| < 1\} = U\backslash\{0\}$. *We obtain coefficients inequalities, distortion theorems, extreme points, closure theorems, radius of convexity estimates, and many results for the modified Hadamard products. Finally, we obtain application involving an integral transforms for the class* $F^n(\alpha,\beta,\gamma)$.

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1. Introduction

Let Σ^* denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \ (\alpha_k \geq 0), \tag{1.1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. Let $g(z) \in \Sigma^*$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \Sigma^*$ is meromorphic starlike of order β $(0 \leq \beta < 1)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ (z \in \mathbb{U}). \tag{1.4}$$

The class of all such functions is denoted by $\Sigma^*(\beta)$. A function $f \in \Sigma^*$ is meromorphic convex of order β $(0 \leq \beta < 1)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in \mathbb{U}). \tag{1.5}$$

The class of such functions is denoted by $\Sigma_k^*(\beta)$. The classes $\Sigma^*(\beta)$ and $\Sigma_k^*(\beta)$ were introduced and studied by Pommerenke [8], Miller [6], Mogra et al. [7], Cho [3] and Aouf ([1] and [2]).

For a funcion $f(z) \in \Sigma^*$, Frasin and Darus [5] defined an operator $I^n : \Sigma^* \to \Sigma^*$ as follows:

$$I^0 f(z) = f(z),$$
$$I^1 f(z) = zf^{'}(z) + \frac{2}{z},$$
$$I^2 f(z) = z(I^1 f(z))^{'} + \frac{2}{z},$$

and for $n \in \mathbb{N} = \{1, 2, ...\}$, we have

$$\begin{aligned} I^n f(z) &= z(I^{n-1} f(z))^{'} + \frac{2}{z} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} k^n a_k z^k \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \mathbb{U}^*). \end{aligned}$$

For $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \frac{1}{2} \leq \gamma \leq 1$ and $n \in \mathbb{N}_0$, we denote by $F^n(\alpha, \beta, \gamma)$ the subclass of Σ^* consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$\left| \frac{z^2 (I^n f(z))' + 1}{(2\gamma - 1) z^2 (I^n f(z))' + (2\alpha\gamma - 1)} \right| < \beta \ (z \in \mathbb{U}^*). \tag{1.6}$$

Choosing different values of β, γ and n, we have
(i) $F^0(\alpha, 1, 1) = F^0(\alpha) = \left\{ f(z) \in \Sigma^* : Re\{-z^2(f(z))'\} > \alpha \ (0 \leq \alpha < 1) \right\};$
(ii) $F^n(\alpha, 1, 1) = F^n(\alpha) = \left\{ f(z) \in \Sigma^* : Re\{-z^2(I^n f(z))'\} > \alpha \ (n \in \mathbb{N}_0,\ 0 \leq \alpha < 1) \right\};$
(iii)$F^n(\alpha, \beta, 1) = F^n(\alpha, \beta)$

$$= \left\{ f(z) \in \Sigma^* : \left| \frac{z^2 (I^n f(z))' + 1}{z^2 (I^n f(z))' + (2\alpha - 1)} \right| < \beta \ (n \in \mathbb{N}_0,\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ z \in \mathbb{U}^*) \right\}.$$

1 2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \frac{1}{2} \leq \gamma \leq 1$, $n \in \mathbb{N}_0$ and $z \in \mathbb{U}^*$.

Theorem 1. The function $f(z) \in F^n(\alpha, \beta, \gamma)$ if and only if

$$\sum_{k=1}^{\infty} k^{n+1} (1 + 2\beta\gamma - \beta) a_k \leq 2\beta\gamma(1 - \alpha). \tag{2.1}$$

Proof. Suppose (2.1) holds, so

$$\left| z^2 (I^n f(z))' + 1 \right| - \beta \left| (2\gamma - 1) z^2 (I^n f(z))' + (2\alpha\gamma - 1) \right|$$

$$= \left| \sum_{k=1}^{\infty} k^{n+1} a_k z^{k+1} \right| - \beta \left| 2\gamma(\alpha - 1) + \sum_{k=1}^{\infty} k^{n+1} a_k z^{k+1} \right|$$

$$\leq \sum_{k=1}^{\infty} k^{n+1} a_k r^{k+1} - \beta \left\{ 2\gamma(1 - \alpha) - \sum_{k=1}^{\infty} k^{n+1} (2\gamma - 1) a_k r^{k+1} \right\}$$

$$= \sum_{k=1}^{\infty} k^{n+1} (1 + 2\beta\gamma - \beta) a_k r^{k+1} - 2\beta\gamma(1 - \alpha).$$

Since the above inequality holds for all r, $0 < r < 1$, letting $r \to 1^-$, we have

$$\sum_{k=1}^{\infty} k^{n+1} (1 + 2\beta\gamma - \beta) a_k - 2\beta\gamma(1 - \alpha) \leq 0,$$

by (2.1). Hence $f(z) \in F^n(\alpha, \beta, \gamma)$.

Conversely, suppose that $f(z)$ is in the class $F^n(\alpha, \beta, \gamma)$. Then

$$\left|\frac{z^2(I^n f(z))' + 1}{(2\gamma-1)z^2(I^n f(z))' + (2\alpha\gamma - 1)}\right|$$

$$\left|\frac{\sum_{k=1}^{\infty} k^{n+1} a_k z^{k+1}}{2\gamma(1-\alpha) - \sum_{k=1}^{\infty} k^{n+1}(2\gamma-1) a_k z^{k+1}}\right| \leq \beta.$$

Using the fact that $Re(z) \leq |z|$ for all z, we have

$$\left|\frac{z^2(I^n f(z))' + 1}{(2\gamma-1)z^2(I^n f(z))' + (2\alpha\gamma - 1)}\right|$$

$$\leq \; Re\left\{\frac{\sum_{k=1}^{\infty} k^{n+1} a_k z^{k+1}}{2\gamma(1-\alpha) - \sum_{k=1}^{\infty} k^{n+1}(2\gamma-1) a_k z^{k+1}}\right\} \leq \beta \; (z \in \mathbb{U}^*).2.2 \qquad (1)$$

If we choose z to be real so that $z^2(I^n f(z))'$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through positive values, we obtain

$$\sum_{k=1}^{\infty} k^{n+1}(1 + 2\beta\gamma - \beta) a_k \leq 2\beta\gamma(1-\alpha).$$

This completes the proof of Theorem 1.

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $F^n(\alpha, \beta, \gamma)$. Then

$$a_k \leq \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1 + 2\beta\gamma - \beta)} \; (k \geq 1),$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1 + 2\beta\gamma - \beta)} z^k. \qquad (2.3)$$

Putting $\beta = \gamma = 1$ and $n = 0$ in Theorem 1, we have

Corollary 2. The function $f(z) \in F^0(\alpha)$ $(0 \leq \alpha < 1)$ if and only if

$$\sum_{k=1}^{\infty} k a_k \leq (1-\alpha), \qquad (2.4)$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{k} z^k \; (k \geq 1). \qquad (2.5)$$

3. Distortion theorems

Theorem 2. Let the function $f(z) \in F^n(\alpha, \beta, \gamma)$, then for $0 < |z| = r < 1$, we have

$$\frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}r \leq |f(z)| \leq \frac{1}{r} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}r, \tag{3.1}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}z. \tag{3.2}$$

Proof. It is easy to see from Theorem 1 that

$$(1+2\beta\gamma-\beta)\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} k^{n+1}(1+2\beta\gamma-\beta)a_k$$

$$\leq 2\beta\gamma(1-\alpha).$$

Then

$$\sum_{k=1}^{\infty} a_k \leq \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}. \tag{3.3}$$

Making use of (3.3), we have

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - |z|\sum_{k=1}^{\infty} a_k \\ &\geq \frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}r, 3.4 \end{aligned} \tag{2}$$

and

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|} + |z|\sum_{k=1}^{\infty} a_k \\ &\leq \frac{1}{r} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}r, 3.5 \end{aligned} \tag{3}$$

which proves the assertion (3.1). The proof is completed.

Theorem 3. Let the function $f(z) \in F^n(\alpha, \beta, \gamma)$, then for $0 < |z| = r < 1$, we have

$$\frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}, \tag{3.6}$$

with equality for the function $f(z)$ given by (3.2).

Proof. From Theorem 1 and (3.3), we have

$$\sum_{k=1}^{\infty} k a_k \leq \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}. \tag{3.7}$$

The remaining part of the proof is similar to the proof of Theorem 2 so, we omit the details.

4. Closure theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \ (a_{k,j} \geq 0). \tag{4.1}$$

Theorem 4. Let $f_j(z) \in F^n(\alpha, \beta, \gamma)$ $(j = 1, 2, ..., m)$. Then the function $h(z)$,

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) z^k, \tag{4.2}$$

is in $F^n(\alpha, \beta, \gamma)$.
Proof. Since $f_j(z) \in F^n(\alpha, \beta, \gamma)$ $(j = 1, 2, ..., m)$, it follows from Theorem 1, that

$$\sum_{k=1}^{\infty} k^{n+1}(1+2\beta\gamma-\beta) a_{k,j} \leq 2\beta\gamma(1-\alpha),$$

for every $j = 1, 2, ..., m$. Hence

$$\sum_{k=1}^{\infty} k^{n+1}(1+2\beta\gamma-\beta) \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} k^{n+1}(1+2\beta\gamma-\beta) a_{k,j} \right) \leq 2\beta\gamma(1-\alpha).$$

From Theorem 1, it follows that $h(z) \in F^n(\alpha, \beta, \gamma)$. This completes the proof.

Theorem 5. The class $F^n(\alpha, \beta, \gamma)$ is closed under convex linear combinations.
Proof. Let $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $F^n(\alpha, \beta, \gamma)$. Then it is sufficient to show that

$$h(z) = \eta f_1(z) + (1-\eta) f_2(z) \ (0 \leq \eta \leq 1), \tag{4.3}$$

is in the class $F^n(\alpha, \beta, \gamma)$. Since

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [\eta a_{k,1} + (1-\eta) a_{k,2}] z^k, \tag{4.4}$$

then, we have from Theorem 1 that

$$\sum_{k=1}^{\infty} k^{n+1}(1+2\beta\gamma-\beta)[\eta a_{k,1} + (1-\eta) a_{k,2}]$$

$$\begin{aligned} &\leq 2\eta\beta\gamma(1-\alpha) + 2\beta\gamma(1-\eta)(1-\alpha) \\ &= 2\beta\gamma(1-\alpha), \end{aligned}$$

so, $h(z) \in F^n(\alpha, \beta, \gamma)$.

Theorem 6. Let $0 \leq \sigma < 1$, then

$$F^n(\alpha, \beta, \gamma) \subseteq F^n(\sigma, \beta, 1) = F^n(\sigma, \beta)$$

where

$$\sigma = 1 - \frac{\gamma(1+\beta)(1-\alpha)}{(1+2\beta\gamma-\beta)}. \tag{4.5}$$

Proof. Let $f(z) \in F^n(\alpha, \beta, \gamma)$, then

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_k \leq 1. \tag{4.6}$$

We need to find the value of σ such that

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+\beta)}{2\beta(1-\sigma)} a_k \leq 1. \tag{4.7}$$

In view of (4.6) and (4.7) we have

$$\frac{k^{n+1}(1+\beta)}{2\beta(1-\sigma)} \leq \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}.$$

That is

$$\sigma \leq 1 - \frac{\gamma(1+\beta)(1-\alpha)}{(1+2\beta\gamma-\beta)},$$

which completes the proof of Theorem 6.

Theorem 7. Let $f_0(z) = \dfrac{1}{z}$ and

$$f_k(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1+2\beta\gamma-\beta)} z^k \ (k \geq 1). \tag{4.8}$$

Then $f(z)$ is in the class $F^n(\alpha, \beta, \gamma)$ if and only if can be expressed in the form

$$f(z) =_{k=0}^{\infty} \mu_k f_k(z), \tag{4.9}$$

where $\mu_k \geq 0$ and ${}_{k=0}^{\infty}\mu_k = 1$.
Proof . Assume that

$$\begin{aligned} f(z) &= {}_{k=0}^{\infty}\mu_k f_k(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1+2\beta\gamma-\beta)} \mu_k z^k .4.10 \end{aligned} \tag{4}$$

Then it follows that

$$\sum_{k=1}^{\infty} \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1+2\beta\gamma-\beta)} \mu_k . \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}$$

$$= \sum_{k=1}^{\infty} \mu_k = 1 - \mu_0 \leq 1.$$

which implies that $f(z) \in F^n(\alpha, \beta, \gamma)$.

Conversely, assume that the function $f(z)$ defined by (1.1) be in the class $F^n(\alpha, \beta, \gamma)$. Then

$$a_k \leq \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1+2\beta\gamma-\beta)}.$$

Setting

$$\mu_k = \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_k, k \geq 1,$$

and

$$\mu_0 = 1 -_{k=1}^{\infty} \mu_k \ ,$$

we can see that $f(z)$ can be expressed in the form (4.9). This completes the proof of Theorem 7.

Corollary 2. The extreme points of the class $F^n(\alpha, \beta, \gamma)$ are the functions $f_0(z) = \dfrac{1}{z}$ and

$$f_k(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1+2\beta\gamma-\beta)} z^k \ (k \geq 1). \tag{4.11}$$

5. Integral operators

Theorem 8. Let the functions $f(z) \in F^n(\alpha, \beta, \gamma)$. Then the integral operator

$$F_c(z) = c_0^1 u^c \ f(uz)du \ (0 < u \leq 1; \ c > 0), \tag{5.1}$$

is in the class $F^0(\zeta)$, where

$$\xi = 1 - \frac{2\beta\gamma c(1-\alpha)}{(1+2\beta\gamma-\beta)(c+2)}. \tag{5.2}$$

The result is sharp for the function $f(z)$ given by (3.2).
Proof. Let $f(z) \in F^0(\zeta)$, then

$$\begin{aligned} F_c(z) &= c_0^1 u^c \ f(uz)du \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{c}{k+c+1} a_k z^k .5.3 \end{aligned} \tag{5}$$

In view of Corollary 2, it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{kc}{(k+c+1)(1-\zeta)} a_k \leq 1. \tag{5.4}$$

Since $f(z) \in F^n(\alpha, \beta, \gamma)$, then

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_k \leq 1. \tag{5.5}$$

From (5.4) and (5.5), we have

$$\frac{kc}{(k+c+1)(1-\zeta)} \leq \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}.$$

Then

$$\xi \leq 1 - \frac{2\beta\gamma c(1-\alpha)}{k^n(1+2\beta\gamma-\beta)(c+k+1)}.$$

Since

$$Y(k) = 1 - \frac{2\beta\gamma c(1-\alpha)}{k^n(1+2\beta\gamma-\beta)(c+k+1)},$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\xi \leq Y(1) = 1 - \frac{2\beta\gamma c(1-\alpha)}{(1+2\beta\gamma-\beta)(c+2)},$$

and hence the proof of Theorem 8 is completed.

6. Radii of convexity

Theorem 9. Let the function $f(z) \in F^n(\alpha, \beta, \gamma)$. Then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $0 < |z| < r$, where

$$r \leq \left\{ \frac{k^n(1+2\beta\gamma-\beta)(1-\delta)}{2\beta\gamma(k+2-\delta)(1-\alpha)} \right\}^{\frac{1}{k+1}}. \tag{6.1}$$

The result is sharp.
Proof. We must show that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta \; for \;\; 0 < |z| < r, \tag{6.2}$$

where r_1 is given by (6.1). Indeed we find from (1.1) that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq \frac{\sum_{k=1}^{\infty} k(k+1)a_k |z|^{k+1}}{1 - \sum_{k=1}^{\infty} k a_k |z|^{k+1}}.$$

Thus

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta,$$

if

$$\sum_{k=1}^{\infty} \frac{k(k+2-\delta)}{1-\delta} a_k r^{k+1} \leq 1. \tag{6.3}$$

But by using Theorem 1, (6.3) will be true if

$$\frac{k(k+2-\delta)}{1-\delta} r^{k+1} \leq \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}.$$

Then

$$r \leq \left\{ \frac{k^n(1+2\beta\gamma-\beta)(1-\delta)}{2\beta\gamma(k+2-\delta)(1-\alpha)} \right\}^{\frac{1}{k+1}} \quad (k \geq 1).$$

This completes the proof of Theorem 9.

7. Modified Hadamard products

For $f_j(z)$ $(j = 1, 2)$ defined by (4.1), the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{7.1}$$

Theorem 10. Let $f_j(z) \in F^n(\alpha, \beta, \gamma)$ $(j = 1, 2)$. Then $(f_1 * f_2)(z) \in F^n(\phi, \beta, \gamma)$, where

$$\phi = 1 - \frac{2\beta\gamma(1-\alpha)^2}{(1+2\beta\gamma-\beta)}. \tag{7.2}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)} z \ (j = 1, 2). \tag{7.3}$$

Proof. Using the techinque for Schild and Silverman [9], we need to find the largest ϕ such that

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\phi)} a_{k,1}a_{k,2} \leq 1. \tag{7.4}$$

Since $f_j(z) \in F^n(\alpha, \beta, \gamma)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_{k,1} \leq 1, \tag{7.5}$$

and

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_{k,2} \leq 1. \tag{7.6}$$

By the Cauchy Schwarz inequality we have

$$\sum_{k=1}^{\infty} \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\phi)} a_{k,1}a_{k,2} \leq \frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \sqrt{a_{k,1}a_{k,2}}, \tag{7.8}$$

or equilvalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\phi)}{(1-\alpha)}. \tag{7.9}$$

Connecting with (7.7), it is sufficient to prove that

$$\frac{2\beta\gamma(1-\alpha)}{k^{n+1}(1+2\beta\gamma-\beta)} \leq \frac{(1-\phi)}{(1-\alpha)}. \tag{7.10}$$

It follows from (7.10) that

$$\phi \leq 1 - \frac{2\beta\gamma(1-\alpha)^2}{k^{n+1}(1+2\beta\gamma-\beta)}. \tag{7.11}$$

Now defining the function $E(k)$ by

$$E(k) = 1 - \frac{2\beta\gamma(1-\alpha)^2}{k^{n+1}(1+2\beta\gamma-\beta)}. \tag{7.12}$$

We see that $E(k)$ is an increasing function of k ($k \geq 1$). Therefore, we conclude that

$$\phi \leq E(1) = 1 - \frac{2\beta\gamma(1-\alpha)^2}{(1+2\beta\gamma-\beta)}, \tag{7.13}$$

which evidently completes the proof of Theorem 10.

Using arguments similar to those in the proof of Theorem 10, we obtain the following theorem.
Theorem 11. Let $f_1(z) \in F^n(\alpha,\beta,\gamma)$. Suppose also that $f_2(z) \in F^n(\phi,\beta,\gamma)$. Then $(f_1 * f_2)(z) \in F^n(\zeta,\beta,\gamma)$ where

$$\zeta = 1 - \frac{2\beta\gamma(1-\alpha)(1-\phi)}{(1+2\beta\gamma-\beta)}. \tag{7.14}$$

The result is sharp for the functions $f_j(z)$ ($j = 1,2$) given by

$$f_1(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}z \ , \tag{7.15}$$

and

$$f_2(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\phi)}{(1+2\beta\gamma-\beta)}z \ . \tag{7.16}$$

Theorem 12. Let $f_j(z) \in F^n(\alpha,\beta,\gamma)$ ($j = 1,2$). Then

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{7.17}$$

belong to the class $F^n(\varepsilon,\beta,\gamma)$, where

$$\varepsilon = 1 - \frac{4\beta\gamma(1-\alpha)^2}{(1+2\beta\gamma-\beta)}. \tag{7.18}$$

The result is sharp for the functions $f_j(z)$ ($j = 1,2$) defined by (7.3).
Proof . By using Theorem 1, we obtain

$$\sum_{k=1}^{\infty}\left\{\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}\right\}^2 a_{k,1}^2 \leq \left\{\sum_{k=1}^{\infty}\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}a_{k,1}\right\}^2 \leq 1, \tag{7.19}$$

and

$$\sum_{k=1}^{\infty}\left\{\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}\right\}^2 a_{k,2}^2 \leq \left\{\sum_{k=1}^{\infty}\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}a_{k,2}\right\}^2 \leq 1. \tag{7.20}$$

It follow from (7.19) and (7.20) that

$$\sum_{k=1}^{\infty}\frac{1}{2}\left\{\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}\right\}^2 (a_{k,1}^2+a_{k,2}^2) \leq 1. \tag{7.21}$$

Therefore, we need to find the largest ε such that

$$\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\varepsilon)} \leq \frac{1}{2}\left\{\frac{k^{n+1}(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}\right\}^2, \tag{7.22}$$

that is

$$\varepsilon \leq 1-\frac{4\beta\gamma(1-\alpha)^2}{k^{n+1}(1+2\beta\gamma-\beta)}, \tag{7.23}$$

since

$$G(k) = 1-\frac{4\beta\gamma(1-\alpha)^2}{k^{n+1}(1+2\beta\gamma-\beta)}, \tag{7.24}$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\varepsilon \leq G(1) = 1-\frac{4\beta\gamma(1-\alpha)^2}{(1+2\beta\gamma-\beta)}, \tag{7.25}$$

and hence the proof of Theorem 12 is completed.

Putting $n = 0$ in Theorem 12 we obtain the following corollary:
Corollary 3. Let the functions $f_j(z) \in F^0(\alpha,\beta,\gamma) = \Sigma_p(\alpha,\beta,\gamma)$ $(j = 1, 2)$. Then the function $h(z)$ defined by (7.17) belong to the class $F^n(\varepsilon,\beta,\gamma) = \Sigma_p(\varepsilon,\beta,\gamma)$, where

$$\varepsilon = 1-\frac{4\beta\gamma(1-\alpha)^2}{(1+2\beta\gamma-\beta)}. \tag{7.26}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ defined by (7.3).

Remark 1. The corollary 3 corrects the result obtained by Cho et al. [4, Theorem 5, with $p = 1$].

Corollary 4. If $f_1(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1}z^k \in F^n(\alpha,\beta,\gamma)$, and $f_2(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,2}z^k$ with $0 \leq a_{k,2} \leq 1$, $k \geq 1$ then $(f_1 * f_2)(z) \in F^n(\alpha,\beta,\gamma)$.

8. Open Problem

The authors suggest to study the properties of the same class

$$\left| \frac{z^2 f'(z) + 1}{B z^2 f"(z) + [B + (A - B)(1 - \alpha)]} \right| < \beta.$$

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