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On subclasses of uniformly starlike and convex functions defined by Struve functions

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Abstract

The purpose of the present paper is to derive the necessary and sufficient condition for the generalized Bessel function (struve function) belonging to the classes $\Omega^*_\lambda(g;\alpha,\beta)$ ***and*** $\Upsilon^*_\lambda(g;\alpha,\beta)$.

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1 Introduction

Let A be the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \tag{1}$$

As usual, we denote by S the subclass of A consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in U . Denote by T the subclass of A consisting of functions whose non-zero coefficients from second on, is given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Also, for functions $f \in A$ given by (1) and $g \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad z \in U.$$

A function $f \in A$ is said to be starlike of order α $(0 \le \alpha < 1)$, if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in U)$. This function class is denoted by $S^*(\alpha)$. We also write $S^*(0) \equiv S^*$, where S^* denotes the class of functions $f \in A$ that $f(U)$ is starlike with respect to the origin. A function $f \in A$ is said to be convex of order α $(0 \le \alpha < 1)$ if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U)$. This class is denoted by $K(\alpha)$. Further, $K(0) = K$, the well-known standard class of convex functions. It is an established fact that $f \in K(\alpha) \Leftrightarrow zf' \in S^*(\alpha)$.

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [10] of the famous Bieberbach conjecture. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [9, 11, 16] and the Bessel functions [4, 5, 6, 12].

We recall here the Struve function of order p (see [14, 19]), denoted by H_p is given by

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{3}{2})\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in U$$

which is the particular solution of the second order non-homogeneous differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - p^2)w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}$$

where p is unrestricted real (or complex) number. The solution of the non-homogeneous differential equation

$$z^2 w''(z) + zw'(z) - (z^2 + p^2)w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathfrak{L}_p(z) = -ie^{-ip\pi/2}H_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{3}{2})\Gamma(p+n+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in U.$$

Let the second order non-homogeneous linear differential equation [16] (also see [11] and references cited therein),

$$z^2w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{b}{2})} \tag{3}$$

where $b, p, c \in \mathbb{C}$ which is natural generalization of Struve equation. It is of interest to note that when $b = c = 1$, then we get the Struve function (1)and for $c = -1, b = 1$ the modified Struve function (1).This permit us to study Struve and modified Struve functions. Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(c)^n}{\Gamma(n+\frac{3}{2})\Gamma(p+n+\frac{b+2}{2})}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in U$$

which is the particular solution of the differential equation (3).Although the series defined above is convergent everywhere, the function $w_{p,b,c}$ is generally not univalent in U. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)z^{\frac{-p-1}{2}}w_{p,b,c}\left(\sqrt{z}\right).$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1,2,3,...\}) \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned} u_{p,b,c}(z) &= z + \sum_{n=2}^{\infty} \frac{(-c/4)^n}{(m)_n(3/2)_n} z^n \\ &= b_0 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots, \end{aligned}$$

where $m = \left(p+\frac{b+2}{2}\right) \neq 0, -1, -2, \ldots$. This function is analytic on U and satisfies the second-order in homogeneous linear differential equation

$$4z^2u''(z) + 2(2p+b+3)zu'(z) + (cz+2p+b)u(z) = 2p+b.$$

For convenience, throughout in the sequel, we use the following notations

$$w_{p,b,c}(z) = w_p(z),$$

$$u_{p,b,c}(z) = u_p(z),$$

$$m = p + \frac{b+2}{2}$$

and for if $c < 0, m > 0 (m \neq 0, -1, -2, \ldots)$ let,

$$zu_p(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n = z + \sum_{n=2}^{\infty} b_{n-1} z^n$$

and

$$\Psi(z) = z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n \tag{4}$$

For $f(z)$ defined by (1) Aouf et al. [1] defined the classes $\Omega_\lambda(\alpha, \beta)$ and $\Upsilon_\lambda(\alpha, \beta)$ as follows :

Definition 1.1 *The function $f(z)$ defined by (1) is said to be in the class $\Omega_\lambda(\alpha, \beta)$ if and only if*

$$\Re\left\{\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - \alpha\right\} > \beta \left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right|,$$

for some $(0 \leq \lambda < 1)$, $(0 \leq \alpha < 1)$ and $(\beta > 0)$, and is in the class $\Upsilon_\lambda(\alpha, \beta)$ if and only if

$$\Re\left\{\frac{zf''(z) + f'(z)}{f'(z) + \lambda z f''(z)} - \alpha\right\} > \beta \left|\frac{zf''(z) + f'(z)}{f'(z) + \lambda z f''(z)} - 1\right|,$$

for some $(0 \leq \lambda < 1)$, $(0 \leq \alpha < 1)$ and $(\beta > 0)$. Denote $\Omega^*_\lambda(\alpha, \beta) = \Omega_\lambda(\alpha, \beta) \cap T$ and $\Upsilon^*_\lambda(\alpha, \beta) = \Upsilon_\lambda(\alpha, \beta) \cap T$, the subclasses of T.

Note that if $\lambda = 0$ then $\Omega_\lambda(\alpha, \beta) = \Omega_0(\alpha, \beta)$ and $\Upsilon_\lambda(\alpha, \beta) = \Upsilon_0(\alpha, \beta)$ [7], also we note $\Omega_0(\alpha, 0) \equiv T^*(\alpha)$ and $\Upsilon_0(\alpha, 0) \equiv C(\alpha)$ [15], further $\Omega_0(0, \beta) \equiv \Omega_0$ (β) and $\Upsilon_0(0, \beta) \equiv \Upsilon_0(\beta)$ [17]. Suitably specializing the parameters we get the various subclasses studied in [13] and see the references cited therein.(also see [1, 2, 8, 7, 18]). Recently, Yagmur and Orhan [19] (see [14]) have determined various sufficient conditions for the parameters p, b and c such that the functions $u_{p,b,c}(z)$ or $z \to zu_{p,b,c}(z)$ to be univalent, starlike, convex and close to convex in the open unit disk. Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [9, 11, 16]) and by work of Baricz [4, 5, 6]. In this paper, we obtain sufficient condition for function $h(z)$, given by

$$\begin{aligned} h_\mu(z) &= (1-\mu)zu_p(z) + \mu z u_p'(z) \\ &= z + \sum_{n=2}^{\infty}(1+n\mu-\mu)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n, \end{aligned} \tag{5}$$

where $0 \le \mu \le 1$, belonging to the classes $\Omega_\lambda(\alpha,\beta)$ and $\Upsilon_\lambda(\alpha,\beta)$.

2 Main results and their consequences

We recall the following necessary and sufficient conditions for the functions $f \in \Omega_\lambda^*(\alpha,\beta)$, $f \in \Upsilon_\lambda^*(\alpha,\beta)$.

Lemma 2.1 *A function $f(z)$of the form* (2)*is in*

(i) the class $\Omega_\lambda(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty} n(1+\beta) - (\alpha+\beta)[1+\lambda(n-1)]\,|a_n| \le 1-\alpha. \tag{6}$$

(ii) the class $\Upsilon_\lambda(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)(1+\lambda(n-1))]\,|a_n| \le 1-\alpha. \tag{7}$$

The above sufficient conditions are also necessary for functions f of the form (2).

Lemma 2.2 *A function $f(z)$ of the form* (2) *is in*

(i) the class $\Omega_0(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\beta) - (\alpha+\beta)]\,|a_n| \le 1-\alpha.$$

(ii) the class $\Upsilon_0(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)]\,|a_n| \le 1-\alpha.$$

Theorem 2.3 *If* $c < 0, m > 0 (m \neq 0, -1, -2, \dots$ *then* $h_\mu(z) \in \Omega_\lambda(\alpha, \beta)$ *if*

$$\begin{aligned} & [\mu(1+\lambda)(1+\beta)]u_p''(1) + [(1+\beta)(2\mu(1+\lambda)+1) \\ & -(\alpha+\beta)(\mu+\lambda(1-\mu))]u_p'(1)] \\ & +[(1+\beta) - (\alpha+\beta)(1+\mu(1-\lambda))]u_p(1) \\ \leq \; & [2(1-\alpha) - \mu(1-\lambda)(\alpha+\beta)]. \end{aligned} \tag{8}$$

Proof Since

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n,$$

then

$$u_p(1) - 1 = \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}, \tag{9}$$

and differentiating $zu_p(z)$ with respect to z and taking $z = 1$ we have

$$zu_p'(z) + u_p(z) = 1 + \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^{n-1}$$

$$u_p'(1) + u_p(1) - 1 = \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{10}$$

Further, differentiating $zu_p'(z) + u_p(z)$ with respect to z and taking $z = 1$, we get

$$zu_p''(z) + 2u_p'(z) = \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^{n-2}$$

$$u_p''(1) + 2u_p'(1) = \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{11}$$

Since $h_\mu(z) \in \Omega_\lambda(\alpha, \beta)$, by virtue of Lemma 2.1 and (5) it suffices to show that

$$\sum_{n=2}^{\infty} (1+n\mu-\mu) \left[n(1+\beta) - (\alpha+\beta)(1+\lambda(n-1))\right] \left(\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \right) \leq 1-\alpha. \tag{12}$$

Now, let

$$\begin{aligned} S(n, \lambda, \beta, \alpha) \;\; = \;\; & \sum_{n=2}^{\infty} (1 + n\mu - \mu) \\ & . \left\{ n(1+\beta) - (\alpha+\beta)[1 + \lambda(n-1)] \right\} \left(\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \right) \end{aligned}$$

$$\begin{aligned} &= \mu(1+\beta)\sum_{n=2}^{\infty} n^2 \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \\ &\quad -\lambda\mu(\alpha+\beta)\sum_{n=2}^{\infty} n(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \\ &\quad +[(1-\mu)(1+\beta)-[(\alpha+\beta)(\mu+\lambda(1-\mu)]\sum_{n=2}^{\infty} n\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \\ &\quad +[(1-\mu)(\alpha+\beta)(\lambda-1)]\sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \end{aligned}$$

Writing $n^2 = n(n-1)+n$, we get

$$\begin{aligned} S(n,\lambda,\beta,\alpha) &= [\mu(1+\beta)+\lambda\mu(\alpha+\beta)]\sum_{n=2}^{\infty} n(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \\ &\quad +(1+\beta)-[(\alpha+\beta)(\mu+\lambda(1-\mu))]\sum_{n=2}^{\infty} n\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \\ &\quad +\lambda[(1-\mu)(\alpha+\beta)(\lambda-1)]\sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \end{aligned}$$

From $(9), (10), (11)$ and taking $z = 1$, we get

$$\begin{aligned} S(n,\lambda,\beta,\alpha) &\leq [\mu(1+\beta)+\lambda\mu(\alpha+\beta)][u_p''(1)+2u_p'(1)] \\ &\quad +(1+\beta)-[(\alpha+\beta)(\mu+\lambda(1-\mu))](u_p'(1)+u_p(1)-1) \\ &\quad +\lambda[(1-\mu)(\alpha+\beta)(\lambda-1)](u_p(1)-1). \end{aligned}$$

$$\begin{aligned} S(n,\lambda,\beta,\alpha) &= [\mu(1+\beta)+\lambda\mu(\alpha+\beta)]u_p''(1) \\ &\quad +(1+\beta)(2\mu+1)+[(\alpha+\beta)(\lambda(3\mu-1)-\mu))]u_p'(1)] \\ &\quad +[(1+\beta)-(\alpha+\beta)(\mu+(1-\mu)(2\lambda-1))](u_p(1)-1). \end{aligned}$$

But this expression is bounded above by $1-\alpha$ if (8) holds.

Thus, the proof is complete.

Theorem 2.4 *If* $c < 0, m > 0(m \neq 0, -1, -2, \ldots$ *then* $zu_p(z) \in \Omega_\lambda(\alpha, \beta)$ *if*

$$[(1+\beta)-\lambda(\alpha+\beta)(1-\mu)]u_p'(1)+[(1+\beta)-(\alpha+\beta)]u_p(1) \leq 2(1-\alpha). \quad (13)$$

Proof By virtue of Lemma 2.1of (6), it suffices to show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)(1+\lambda(n-1))]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\right) \leq 1-\alpha.$$

Since $h_0(z) = zu_p(z)$, hence by taking $\mu = 0$ in (12) we get the above inequality. Hence by taking $\mu = 0$ in the Theorem 2.3, we get the desired result given in (13).

Theorem 2.5 *If $c < 0, m > 0(m \neq 0, -1, -2, \ldots$ then $zu_p(z) \in \Upsilon_\lambda(\alpha, \beta)$ if*

$$[(1+\lambda)(1+\beta)]u_p''(1) + [(1+\beta)(2(1+\lambda)+1)) - (\alpha+\beta)]u_p'(1)] \\ +[(1+\beta) - (\alpha+\beta)(1+(1-\lambda))]u_p(1) \leq [2(1-\alpha) - (1-\lambda)(\alpha+\beta)] \quad (14)$$

Proof By virtue of Lemma 2.1 of (7), it suffices to show that

$$\sum_{n=2}^{\infty} \{n[n(1+\beta) - (\alpha+\beta)(1+\lambda(n-1))]\} \left(\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \right) \leq 1 - \alpha.$$

By definition $zu_p(z) \in \Upsilon_\lambda(\alpha, \beta) \Leftrightarrow zu_p(z) \in \Omega_\lambda(\alpha, \beta)$. That is by taking $\mu = 1$ we have $h_1(z) = zu_p'(z)$, hence by taking $\mu = 1$ in the Theorem 2.3, we get the desired result given in (14).

Remark 2.6 *The above conditions* (8)*and* (14) *are also necessary for functions $\Psi(z)$ given by* (4)*and of the form*

$$\begin{aligned} h_\mu^*(z) &= (1-\mu)\Psi(z) + \mu\Psi'(z) \\ &= z - \sum_{n=2}^{\infty} (1 + n\mu - \mu) \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n, \end{aligned}$$

is in the classes $\Omega_\lambda(\alpha, \beta)$ and $\Upsilon_\lambda(\alpha, \beta)$ respectively.

Further, by taking $\lambda = 0$ (or) $\lambda = 1$ in Theorems 2.4 and 2.5, we state the following corollaries without proof.

Corollary 2.7 *If $c < 0, m > 0(m \neq 0, -1, -2, \ldots$ then $z(2 - u_p(z))$,*

(i) is in $\Omega_0(\alpha, \beta)$ if and only if

$$(1+\beta)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha).$$

(ii) is in $\Upsilon_0(\alpha, \beta)$ if and only if

$$(1+\beta)u_p''(1) + (3 - 2\beta - \alpha)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha).$$

3 Open problem

The authors suggest to study sufficient condition for function $h(z)$, given by

$$\begin{aligned} h_\mu(z) &= (1-\mu)zu_p(z) + \mu z u_p'(z) \\ &= z + \sum_{n=2}^{\infty}(1+n\mu-\mu)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}z^n, \end{aligned}$$

where $0 \le \mu \le 1$, belongs to the class

$$\Re\left\{1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\right\} > \beta\left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right|$$

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