Int. J. Open Problems Complex Analysis, Vol. 8, No. 1, March 2016
ISSN 2074-2827; Copyright ©ICSRS Publication, 2016
www.i-csrs.org

Some Results on the Generalized Fourier Transform Associated with the Family of Differential-difference Operators

Hatem Mejjaoli

Department of Mathematics, PO BOX 30002 Al Madinah AL Munawarah
Taibah University, Saudi Arabia
e-mail: hatem.mejjaoli@yahoo.fr

Abstract

We consider a family of differential-difference operators $T_{A,\varepsilon}$ on the real line which generalizes at the same time the Cherednik and Dunkl type operators on $\mathbb{R}$. We establish some spectral theorems for the generalized Fourier transform on $\mathbb{R}$ tied to $T_{A,\varepsilon}$. Finally, the Roe's theorem is established in the context of the family of differential-difference operators.

Keywords: *Cherednik type operator, Dunkl type operator, family of differential-difference operators, generalized Fourier transform, Paley-Wiener theorems, Roe's theorem*

2000 Mathematical Subject Classification: 35L05, 22E30
1

1 Introduction

We consider the family of (A, ε) operators on $\mathbb{R}$:

$$T_{A,\varepsilon}f(x) = \frac{d}{dx}f(x) + \frac{A'(x)}{A(x)}(\frac{f(x) - f(-x)}{2}) - \varepsilon\rho f(-x), \tag{1}$$

where $\varepsilon \in \mathbb{R}$,

$$A(x) = |x|^{2k}B(x), \quad k > 0, \tag{2}$$

B being a positive C^∞ even function on $\mathbb{R}$, with $B(0) = 1$, and $\rho \geq 0$.
We suppose in addition that the function A satisfies the following conditions:

i) For all $x \geq 0, A(x)$ is increasing and $\lim_{x\to\infty} A(x) = \infty$.

ii) For all $x > 0$, $\dfrac{A'(x)}{A(x)}$ is decreasing and $\lim\limits_{x\to\infty} \dfrac{A'(x)}{A(x)} = 2\rho$.

iii) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty), x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x} D(x),$$

where D is a C^∞-function, bounded together with its derivatives.

For

$$\begin{cases} A(x) = |x|^{2k}, \quad k \geq 0 \\ \varepsilon \text{ arbitrary} \end{cases}$$

we have the differential-difference operator

$$T_k f(x) = \frac{d}{dx} f(x) + \frac{2k}{x}\{f(x) - f(-x)\},$$

which is refereed to as the Dunkl operator on $\mathbb{R}$ (see [11]).

For

$$\begin{cases} A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}, k \geq k' \geq 0, k \neq 0 \\ \rho = k + k' \\ \varepsilon = 0, \end{cases}$$

we have the differential-difference operator

$$T_{k,k'} f(x) = \frac{d}{dx} f(x) + (k \coth(x) + k' \tanh(x))\{f(x) - f(-x)\},$$

which is refereed to as the Jacobi-Dunkl operator (see [10, 6]).

For

$$\begin{cases} A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}, k \geq k' \geq 0, k \neq 0 \\ \rho = k + k' \\ \varepsilon = 1, \end{cases} \tag{3}$$

we have the differential-difference operator

$$T_{k,k'} f(x) = \frac{d}{dx} f(x) + (k \coth(x) + k' \tanh(x))\{f(x) - f(-x)\} - \rho f(-x), \tag{4}$$

which is refereed to as the Jacobi-Cherednik operator (see [13]).

For $\varepsilon = 0$, we have the differential-difference operator

$$T_{A,0} f(x) = \frac{d}{dx} f(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right), \tag{5}$$

which is refereed to as the Dunkl type operator (see [18, 24]).

For $\varepsilon = 1$, we have the differential-difference operator

$$T_{A,1} f(x) = \frac{d}{dx} f(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right) - \rho f(-x), \tag{6}$$

which is refereed to as the Cherednik type operator (see [19]).

In [5] the authors provides a new harmonic analysis on the real line corresponding to the differential-difference operators $T_{A,\varepsilon}$.

The purpose of the present paper is twofold. On one hand, we want to improve and generalize many results presented in [4, 16, 17].

On the other hand we want to prove a new characterisation for the spectrum of the Opdam-Cherednik transform under the generalized potential function.

We note that the subject of the spectral theorems was studied for many other integral transforms, for examples (cf. [1, 2, 3, 7, 9, 15, 16, 17, 20, 25]).

The remaining part of the paper is organized as follows. In §2 we recall the main results about the harmonic analysis associated with the family of differential-difference operators $T_{A,\varepsilon}$. The §3 is devoted to characterize the functions in the generalized Schwartz spaces such that their generalized Fourier transform vanishes outside a polynomial domain. In §4, we prove new versions of real Paley-Wiener theorems associated with the generalized Fourier transform. The §5 is devoted to characterize the support for the Opdam-Cherednik transform of the function in the Lebesgue space $L^p_A(\mathbb{R})$ for $p \in [1, 2)$, via the generalized potential function. In §6 we study the generalized tempered distributions with spectral gaps. Finally, in the last section we prove many versions of Roe's theorem for $T_{A,\varepsilon}$.

2 Preliminaries

This section gives an introduction to the harmonic analysis associated with the family of operators $T_{A,\varepsilon}$. The main reference is [5].

2.1 The eigenfunction of the operator $T_{A,\varepsilon}$

We consider the operators $T_{A,\varepsilon}$ given by the relation (1). To present the eigenfunctions $\Phi_{A,\varepsilon}(\lambda, .), \lambda \in \mathbb{C}$, of $T_{A,\varepsilon}$ satisfying the condition $\Phi_{A,\varepsilon}(\lambda, 0) = 1$, we consider first the eigenfunction $\varphi_\lambda, \lambda \in \mathbb{C}$, of the second order singular differential operator L on $(0, \infty)$

$$L = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

The function φ_λ, $\lambda \in \mathbb{C}$, is the unique analytic solution of the differential equation

$$\begin{cases} Lu(x) = -(\lambda^2 + \rho^2)u(x), \\ u(0) = 1, u'(0) = 0. \end{cases} \tag{7}$$

We denote also by φ_λ the even function on $\mathbb{R}$ equal to φ_λ on $[0, \infty)$.

For every $\lambda \in \mathbb{C}$, let us denote by $\Phi_{A,\varepsilon}(\lambda, .)$ the unique solution of the eigenvalue problem

$$\begin{cases} T_{A,\varepsilon} f(x) = i\lambda f(x), \\ f(0) = 1. \end{cases} \tag{8}$$

It is given for all $\lambda \in \mathbb{C}$, by

$$\forall\, x \in \mathbb{R}, \Phi_{A,\varepsilon}(\lambda, x) = \begin{cases} \varphi_{\mu_\varepsilon}(x) + \frac{1}{i\lambda - \varepsilon\rho} \frac{d}{dx}\varphi_{\mu_\varepsilon}(x), & \text{if } i\lambda \neq \varepsilon\rho, \\ 1 + \frac{2\varepsilon\rho\, \mathrm{sgn}(x)}{A(x)} \displaystyle\int_0^{|x|} A(t)dt, & \text{if } \lambda = i\varepsilon\rho, \end{cases}$$

where $\mu_\varepsilon^2 = \lambda^2 + (\varepsilon^2 - 1)\varrho^2$.

For $\lambda \neq -i\varepsilon\rho$, we can write it in the form

$$\forall\, x \in \mathbb{R}, \Phi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_\varepsilon}(x) + \mathrm{sgn}(x)\, \frac{i\lambda + \varepsilon\rho}{A(x)} \int_0^{|x|} \varphi_{\mu_\varepsilon}(z)A(z)dz.$$

It possesses the following properties:

i) For every $x \in \mathbb{R}$, the function $\lambda \to \Phi_{A,\varepsilon}(\lambda, x)$ is entire on $\mathbb{C}$.

ii) We assume that $\varepsilon \in [-1, 1]$. There exists a positive constant M such that for all $x \in \mathbb{R}$ and for all $\lambda \in \mathbb{R}$, with $|\lambda| \geq \sqrt{1 - \varepsilon^2}\varrho$

$$|\Phi_{A,\varepsilon}(\lambda, x)| \leq M(1 + |x|)(1 + \sqrt{\lambda^2 + \rho^2})e^{-\rho|x|}.$$

iii) For all $x \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}$, the function $\Phi_{A,\varepsilon}(\lambda, x)$ admits the Laplace type integral representation

$$\Phi_{A,\varepsilon}(\lambda, x) = \int_{-|x|}^{|x|} K(x, y)e^{i\lambda y}dy, \tag{9}$$

where $K(x, .)$ is a continuous function on $(-|x|, |x|)$, with support in $[-|x|, |x|]$.

We proceed as [24], we prove the following:

Proposition 1 *We assume that $\varepsilon \in [-1, 1]$. Let p be polynomial of degree m. Then there exists a positive constant C such that for all $\lambda \in \mathbb{R}$, with $|\lambda| \geq \sqrt{1 - \varepsilon^2}\varrho$ and for all $x \in \mathbb{R}$, we have*

$$|p(\frac{\partial}{\partial\lambda})\Phi_{A,\varepsilon}(\lambda, x)| \leq C(1 + |\lambda|)(1 + |x|)^{m+2}e^{-\varrho|x|}. \tag{10}$$

We finish this subsection by giving another version of Leibnitz formula.

Proposition 2 *([5]). We assume that $\varepsilon \in [-1, 1]$. Let p be polynomial of degree m. Then there exists a positive constant C such that for all $\lambda \in \mathbb{R}$, and for all $x \in \mathbb{R}$, we have*

$$|p(\frac{\partial}{\partial\lambda})\Phi_{A,\varepsilon}(\lambda, x)| \leq C(1 + |\lambda|)(1 + |x|)^{m+2}e^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|}. \tag{11}$$

2.2 Generalized Fourier transform

We denote by

$L^p_A(\mathbb{R})$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|_{L^p_A(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p A(x)dx\right)^{1/p} < \infty, \quad \text{if } 1 \le p < \infty$$
$$\|f\|_{L^\infty_A(\mathbb{R})} = ess \sup_{x\in\mathbb{R}} |f(x)| < \infty.$$

$\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

$\mathcal{S}_e(\mathbb{R})$ (resp. $\mathcal{S}_o(\mathbb{R})$) the subspace of $\mathcal{S}(\mathbb{R})$ consisting of even (resp. odd) functions.
$D(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ which are of compact support.
$\mathcal{S}^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1,1]$, the space of C^∞-functions on $\mathbb{R}$ such that for all $m,n \in \mathbb{N}$

$$q_{n,m}(f) := \sup_{x\in\mathbb{R}} e^{\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^m \left|\frac{d^n}{dx^n} f(x)\right| < \infty.$$

The topology of $\mathcal{S}^2_\varepsilon(\mathbb{R})$ is defined by the semi-norms $q_{n,m}$, $m,n \in \mathbb{N}$.

$\mathcal{S}^2_{\varepsilon,e}(\mathbb{R})$ (resp. $\mathcal{S}^2_{\varepsilon,o}(\mathbb{R})$) the subspace of $\mathcal{S}^2_\varepsilon(\mathbb{R})$ consisting of even (resp. odd) functions.

For $f \in L^1_A(\mathbb{R})$, the generalized Fourier transform is defined by

$$\mathcal{F}_{T_{A,\varepsilon}}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{A,\varepsilon}(\lambda,-x)A(x)dx, \quad \text{for all } \lambda \in \mathbb{C}. \tag{12}$$

Proposition 3 *([5]). For $\lambda \in \mathbb{C}$ and $g \in L^1_A(\mathbb{R})$, we have*

$$\mathcal{F}_{T_{A,\varepsilon}}(g)(\lambda) = 2\mathcal{F}_L(g_e)(\mu_\varepsilon) + 2(\varepsilon\varrho + i\lambda)\mathcal{F}_L(Jg_o)(\mu_\varepsilon), \tag{13}$$

where J is the integral operator defined by

$$Jf(x) = \int_{-\infty}^x f(t)dt, \quad x \in \mathbb{R}, \tag{14}$$

g_e (resp. g_o) denotes the even (resp. odd) part of g, and $\mathcal{F}_L$ stands for the Fourier transform related to the differential operator L, defined on $\mathcal{S}^2_{\varepsilon,e}(\mathbb{R})$ by

$$\mathcal{F}_L(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)A(x)dx, \quad \lambda \in \mathbb{R},$$

φ_λ being the eigenfunction of L as defined by (7).

Proposition 4 *([5]). For all $f \in D(\mathbb{R})$,*

$$\mathcal{F}^{-1}_{T_{A,\varepsilon}}(f)(x) = \int_{\mathbb{R}} f(\lambda)\Phi_{A,\varepsilon}(\lambda,x)d\sigma_\varepsilon(\lambda), \tag{15}$$

where

$$d\sigma_\varepsilon(\lambda) = (1-\frac{\varepsilon\varrho}{i\lambda})\frac{|\lambda|}{\sqrt{\lambda^2-(1-\varepsilon^2)\varrho^2}|c(\sqrt{\lambda^2-(1-\varepsilon^2)\varrho^2})|^2} 1_{\mathbb{R}\setminus(-\sqrt{1-\varepsilon^2}\varrho,\sqrt{1-\varepsilon^2}\varrho)}(\lambda)d\lambda, \tag{16}$$

with c is a continuous function on $(0,\infty)$ *such that*

$$|c(s)|^{-2} \sim \begin{cases} C_1 s^{2k} & as \quad s \to \infty \\ C_2 s^2 & as \quad s \to 0, \end{cases} \tag{17}$$

for some $C_1, C_2 \in \mathbb{R}_+$.

Remark 1 *1) For* $\varepsilon = 1$, $A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}$, $k \geq k' \geq 0$ *and* $k \neq 0$, *we have*

$$d\sigma_1(\lambda) = (1 - \frac{\varrho}{i\lambda})\frac{d\lambda}{|c(\lambda)|^2}$$

where

$$c\,(\lambda) := \frac{2^{\rho - i\lambda}\Gamma(k+\frac{1}{2})\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(k-k'+1+i\lambda))}, \quad \lambda \in \mathbb{C}\backslash i\mathbb{N}.$$

2) For $\varepsilon = 0$, $A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}$, $k \geq k' \geq 0$ *and* $k \neq 0$, *we have*

$$d\sigma_0(\lambda) = \frac{d\lambda}{\sqrt{\lambda^2 - \varrho^2}|c(\sqrt{\lambda^2 - \varrho^2})|^2} 1_{\mathbb{R}\backslash[-\varrho,\varrho]}(\lambda)d\lambda,$$

where

$$c\,(\lambda) := \frac{2^{\rho - i\lambda}\Gamma(k+\frac{1}{2})\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(k-k'+1+i\lambda))}, \quad \lambda \in \mathbb{C}\backslash i\mathbb{N}.$$

Proposition 5 *([5]).* i*)* **Plancherel formula for** $\mathcal{F}_{T_{A,\varepsilon}}$. *For all* f, g *in* $\mathcal{S}^2_\varepsilon(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} f(x)g(-x)A\,(x)\;dx = \int_{\mathbb{R}} \mathcal{F}_{T_{A,\varepsilon}}(f)(\xi)\mathcal{F}_{T_{A,\varepsilon}}(g)(\xi)d\sigma_\varepsilon(\xi). \tag{18}$$

3 Spectrum theorems of functions for the generalized Fourier transform

We begin by the following definition.

Definition 1 *Let u be a distribution on* $\mathbb{R}$ *and P a polynomial on* $\mathbb{R}$ *with complex coefficients. Then we let*

$$R(P,u) = \sup\Big\{|P(y)| : \quad y \in \operatorname{supp} u\Big\} \in [0,\infty],$$

where by convention $R(P,u) = 0$ *if* $u = 0$.

Theorem 1 *Let P be a non-constant polynomial. For any function* $f \in \mathcal{S}^2_\varepsilon(\mathbb{R})$ *the following relation holds*

$$\lim_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} = R(P, \mathcal{F}_{T_{A,\varepsilon}}(f)). \tag{19}$$

For prove this theorem we need the following key propositions.

Proposition 6 *([5]) The generalized Fourier transform* $\mathcal{F}_{T_{A,\varepsilon}}$ *is a bijection from* $\mathcal{S}^2_\varepsilon(\mathbb{R})$ *to* $\mathcal{S}(\mathbb{R})$.

Proposition 7 *(i) Let* $f\in\mathcal{S}^2_\varepsilon(\mathbb{R})$ *and* g *a nice function. Then*

$$\int_{\mathbb{R}} T_{A,\varepsilon}f(x)g(-x)A(x)dx=\int_{\mathbb{R}} f(x)T_{A,\varepsilon}g(-x)A(x)dx. \tag{20}$$

(ii) For $f\in\mathcal{S}^2_\varepsilon(\mathbb{R})$

$$\mathcal{F}_{T_{A,\varepsilon}}\left(T_{A,\varepsilon}f\right)(y)=iy\mathcal{F}_{T_{A,\varepsilon}}f(y),\quad \text{for all } y\in\mathbb{R}. \tag{21}$$

(iii) For $f\in\mathcal{S}^2_\varepsilon(\mathbb{R})$

$$\mathcal{F}_{T_{A,\varepsilon}}(\triangle_{A,\varepsilon}f)(y)=-y^2\mathcal{F}_{T_{A,\varepsilon}}(f)(y),\quad \text{for all } y\in\mathbb{R}, \tag{22}$$

where $\triangle_{A,\varepsilon}$ *is the generalized Laplace operator on* $\mathbb{R}$ *given by*

$$\forall\, x\in\mathbb{R},\quad \triangle_{A,\varepsilon}f(x):=T^2_{A,\varepsilon}f(x). \tag{23}$$

Proof. Let $f\in\mathcal{S}^2_\varepsilon(\mathbb{R})$ and g a nice function, and consider the bracket

$$\langle f,g\rangle=\int_{\mathbb{R}} f(x)g(-x)A(x)dx.$$

First, we have

$$\begin{aligned}\langle f',g\rangle &= \int_{\mathbb{R}} f'(x)g(-x)A(x)dx=-\int_{\mathbb{R}} f(x)\frac{d}{dx}[g(-x)A(x)]dx\\ &= -\int_{\mathbb{R}} f(x)g(-x)A'(x)dx+\int_{\mathbb{R}} f(x)g'(-x)A(x)dx\\ &= \langle f,g'\rangle+\langle f,g\tfrac{A'}{A}\rangle\end{aligned}$$

since $\frac{A'(x)}{A(x)}$ is odd.

Second, we have

$$\begin{aligned}\langle \tfrac12\tfrac{A'}{A}(f-\check f),g\rangle &= \int_{\mathbb{R}}\frac{A'(x)}{A(x)}\Big(\frac{f(x)-f(-x)}{2}\Big)g(-x)A(x)dx=\int_{\mathbb{R}}\Big(\frac{f(x)-f(-x)}{2}\Big)g(-x)A'(x)dx\\ &= \tfrac12\int_{\mathbb{R}}\Big(A'(x)f(x)g(-x)-A'(x)f(-x)g(-x)\Big)dx\\ &= \tfrac12\int_{\mathbb{R}}\Big(A'(x)f(x)g(-x)-A'(-x)f(x)g(x)\Big)dx\\ &= \tfrac12\int_{\mathbb{R}}\Big(A'(x)f(x)g(-x)+A'(x)f(x)g(x)\Big)dx\\ &= \tfrac12\int_{\mathbb{R}}A'(x)f(x)\Big(g(-x)+g(x)\Big)dx\\ &= \int_{\mathbb{R}}\frac{A'(x)}{A(x)}f(x)\Big(\frac{g(x)+g(-x)}{2}\Big)A(x)dx\\ &= -\langle f,\tfrac12\tfrac{A'}{A}(g+\check g)\rangle\end{aligned}$$

again using that $\frac{A'(x)}{A(x)}$ is odd.

Finally,

$$\begin{aligned}\langle(-\varepsilon\varrho\breve{f}),g\rangle &= -\varepsilon\varrho\int_{\mathbb{R}} f(-x)g(-x)A(x)dx\\ &= -\varepsilon\varrho\int_{\mathbb{R}} f(x)g(x)A(x)dx\\ &= \langle f,(-\varepsilon\varrho\breve{g})\rangle.\end{aligned}$$

All together, this gives

$$\begin{aligned}\langle T_{A,\varepsilon}f,g\rangle &= \langle f'+\tfrac{1}{2}\tfrac{A'}{A}(f-\breve{f})-\varepsilon\varrho\breve{f},g\rangle\\ &= \langle f,g'+g\tfrac{A'}{A}-\tfrac{1}{2}\tfrac{A'}{A}(g+\breve{g})-\varepsilon\varrho\breve{g}\rangle\\ &= \langle f,g'+\tfrac{1}{2}\tfrac{A'}{A}(g-\breve{g})-\varepsilon\varrho\breve{g}\rangle\\ &= \langle f,T_{A,\varepsilon}g\rangle.\end{aligned}$$

Assertion (ii) follows by substituting in (20) g by $\Phi_{A,\varepsilon}(\lambda,.)$. Assertion (iii) is immediately from (ii).

Remark 2 *The results in this proposition improve [[5], Lemma 3.1 and Lemma 8.10].*

Proposition 8 *Let P be a polynomial and $f\in\mathcal{S}^2_\varepsilon(\mathbb{R})$. Then in the extended positive real numbers*

$$\limsup_{n\to\infty}||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})}\leq R(P,\mathcal{F}_{T_{A,\varepsilon}}(f)). \tag{24}$$

Proof. Suppose firstly that $R(P,\mathcal{F}_{T_{A,\varepsilon}}(f))=0$. Then $\mathcal{F}_{T_{A,\varepsilon}}(f)=0$, and hence from Proposition 6, $f=0$. Thus (24) is immediately.

Moreover, the inequality (24), is clear when $R(P,\mathcal{F}_{T_{A,\varepsilon}}(f))=\infty$. So we can assume that

$$0<R(P,\mathcal{F}_{T_{A,\varepsilon}}(f))<\infty.$$

Hölder's inequality gives

$$||f||^2_{L^2_A(\mathbb{R})}=\int_{\mathbb{R}}(1+x^2)^{-1}(1+x^2)|f(x)|^2A(x)dx\leq C\sup_{x\in\mathbb{R}}e^{2\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^{2m}|f(x)|^2, \tag{25}$$

for $m\geq 1$. Thus

$$||f||_{L^2_A(\mathbb{R})}\leq C\sup_{x\in\mathbb{R}}e^{\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^m|f(x)|.$$

Consequently for all $n\in\mathbb{N}$, we deduce that

$$\begin{aligned}||P^n(-iT_{A,\varepsilon})f||_{L^2_A(\mathbb{R})} &\leq C\sup\nolimits_{x\in\mathbb{R}}e^{\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^m|P^n(-iT_{A,\varepsilon})f(x)|\\ &\leq C\sup\nolimits_{x\in\mathbb{R}}e^{\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^m\Big|\Big[\mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big(P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(f)(x)\Big)\Big]\Big|.\end{aligned}$$

Using the continuity of $\mathcal{F}^{-1}_{T_{A,\varepsilon}}$ we can show that

$$||P^n(-iT_{A,\varepsilon})f||_{L^2_A(\mathbb{R})} \leq C \sup_{\xi\in\mathbb{R}} \sum_{0\leq l,j\leq M} (1+\xi^2)^j \Big|\frac{d^l}{d\xi^l}\Big[P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(f)(\xi)\Big]\Big|, \tag{26}$$

with positive constant C and integer M, independent of n. Using Leibniz's rule we deduce that

$$||P^n(-iT_{A,\varepsilon})f||_{L^2_A(\mathbb{R})} \leq Cn^M \sup_{y\in \mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(f)} |P(y)|^{n-M},$$

with C is a constant independent of n. Hence, from the previous inequalities we obtain

$$\limsup_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \leq \sup_{y\in \mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(f)} |P(y)| = R(P,\mathcal{F}_{T_{A,\varepsilon}}(f)).$$

Proposition 9 *Let P be a polynomial. Suppose that $P^n(-iT_{A,\varepsilon})f \in L^2_A(\mathbb{R})$ for all $n\in\mathbb{N}_0$. Then in the extended positive real numbers*

$$\liminf_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \geq R(P,\mathcal{F}_{T_{A,\varepsilon}}(f)). \tag{27}$$

Proof. Fix $\xi_0 \in \mathrm{supp}\,\mathcal{F}_{T_{A,\varepsilon}}(f)$. We can assume that $|P(\xi_0)| \neq 0$. We will show that

$$\liminf_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \geq |P(\xi_0)| - \varepsilon,$$

for any fixed $\varepsilon > 0$ such that $0 < 2\varepsilon < |P(\xi_0)|$.
To this end, choose and fix $\chi \in D(\mathbb{R})$ such that $\langle \mathcal{F}_{T_{A,\varepsilon}}(f), \chi\rangle \neq 0$, and

$$\mathrm{supp}\,\chi \subset \Big\{\xi\in\mathbb{R}:\ |P(\xi_0)|-\varepsilon < |P(\xi)| < |P(\xi_0)|+\varepsilon\Big\}.$$

For $n\in\mathbb{N}$, let $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$. On the follow we want to estimate $||\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^2_A(\mathbb{R})}$. Indeed as the above we have

$$\begin{aligned} ||\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^2_A(\mathbb{R})} &\leq C\sup_{x\in\mathbb{R}} e^{\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^m|\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)(x)| \\ &\leq C\sup_{x\in\mathbb{R}} e^{\varrho(1+\sqrt{1-\varepsilon^2})|x|}(1+x^2)^m\Big|\Big[\mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big(P^{-n}(\xi)\chi\Big)(x)\Big]\Big|, \end{aligned}$$

with $m\geq 1$. Using the continuity of $\mathcal{F}^{-1}_{T_{A,\varepsilon}}$ we can show that

$$||\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^2_A(\mathbb{R})} \leq C \sup_{\xi\in\mathbb{R}} \sum_{0\leq l,j\leq M} (1+\xi^2)^j \Big|\frac{d^l}{d\xi^l}\Big[P^{-n}(\xi)\chi(\xi)\Big]\Big|, \tag{28}$$

with positive constant C and integer M, independent of n. Using Leibniz's rule we deduce that

$$||\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^2_A(\mathbb{R})} \leq Cn^M(|P(\xi_0)|-\varepsilon)^{-n}.$$

As

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(f), \chi\rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(f), P^n(\xi)\chi_n\rangle = \langle P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(f), \chi_n\rangle = \langle (P^n(-iT_{A,\varepsilon})f), \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)\rangle.$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{F}_{T_{A,\varepsilon}}(f), \chi\rangle| \leq C||P^n(-iT_{A,\varepsilon})f||_{L^2_A(\mathbb{R})}||\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^2_A(\mathbb{R})} \leq Cn^M(|P(\xi_0)|-\varepsilon)^{-n}||P^n(-iT_{A,\varepsilon})f||_{L^2_A(\mathbb{R})}.$$

Since $|\langle \mathcal{F}_{T_{A,\varepsilon}}(f), \chi\rangle| > 0$, we deduce that

$$\liminf_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \geq |P(\xi_0)| - \varepsilon.$$

Thus

$$\liminf_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \geq \sup_{y\in \operatorname{supp} \mathcal{F}_{T_{A,\varepsilon}}(f)} |P(y)| = R(P, \mathcal{F}_{T_{A,\varepsilon}}(f)).$$

Proof of Theorem 1. Putting Proposition 8 and Proposition 9 together, we get the result.

Definition 2 *Let P be a non-constant polynomial, we define the polynomial domain U_p by*

$$U_p := \Big\{x \in \mathbb{R} : |P(x)| \leq 1\Big\}.$$

We have the following result.

Corollary 1 *Let $f \in \mathcal{S}^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1, 1]$. The generalized Fourier transform $\mathcal{F}_{T_{A,\varepsilon}}(f)$ vanishes outside a domain U_P, if and only if,*

$$\limsup_{n\to\infty} ||P^n(-iT_{A,\varepsilon})f||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \leq 1. \tag{29}$$

Remark 3 *If we take $P(y) = -y^2$, then $P(-iT_{A,\varepsilon}) = \triangle_{A,\varepsilon}$, and Theorem 1 and Corollary 1 characterize functions such that the support of their generalized Fourier transform is $[-1, 1]$.*

4 Characterization of the functions which their generalized Fourier transform has the support inside or outside intervals

Notations. We denote by

$\mathcal{S}'^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1, 1]$, the space of generalized temperate distributions on $\mathbb{R}$, it is the dual space of $\mathcal{S}^2_\varepsilon(\mathbb{R})$.

$\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support.

Definition 3 *i) The generalized Fourier transform of a distribution τ in $\mathcal{S}'^2_\varepsilon(\mathbb{R})$ is defined by*

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(\tau), \phi\rangle = \langle \tau, \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\phi)\rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}). \tag{30}$$

ii) The inverse of the generalized Fourier transform of a distribution τ in $\mathcal{E}'(\mathbb{R})$ is defined by

$$\forall\, x \in \mathbb{R},\ \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\tau)(x) = \langle \tau_\lambda, \Phi_{\lambda,\varepsilon}(x)\rangle. \tag{31}$$

From the Proposition 7 it is easy to obtain the following:

Corollary 2 *The generalized Fourier transform $\mathcal{F}_{T_{A,\varepsilon}}$ is a topological isomorphism from $\mathcal{S}'^2(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$. Moreover, for all $\tau \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$, we have*

$$\mathcal{F}_{T_{A,\varepsilon}}(T_{A,\varepsilon}\,\tau) \;=\; iy\mathcal{F}_{T_{A,\varepsilon}}(\tau) \tag{32}$$

and

$$\mathcal{F}_{T_{A,\varepsilon}}(\triangle_{A,\varepsilon}\,\tau) \;=\; -y^2\mathcal{F}_{T_{A,\varepsilon}}(\tau). \tag{33}$$

Theorem 2 *Let $u \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1,1]$. Then the support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is contained in the compact $V_{r,\varepsilon} := \Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \leq r\Big\}$ for a polynomial P and a constant $r \geq 0$, if, and only if, for each $R > r$, there exist $N_R \in \mathbb{N}_0$ and a positive constant $C(R)$ such that*

$$|P^n(-iT_{A,\varepsilon})(u)(x)| \leq C(R)R^n(1+|x|)^{N_R}e^{-\varrho|x|}, \tag{34}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof. Assume that support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is contained in the compact $V_{r,\varepsilon}$. Let $R > r$ and let $\eta \in (0, R-r)$. We choose $\chi \in D(\mathbb{R})$ such that $\chi \equiv 1$ on an open neighborhood of support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$, and $\chi \equiv 0$ outside $V_{R-\frac{\eta}{3},\varepsilon}$. As $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is of order N, there exists a positive constant C such that for all $x \in \mathbb{R}$

$$\begin{aligned}
|P^n(-iT_{A,\varepsilon})(u)(x)| &= \Big|\mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big(P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u)\Big)(x)\Big| \\
&= \Big|\mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big(\chi(\xi)P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u)\Big)(x)\Big| \\
&= |\langle \chi(\xi)P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u)(\xi), \Phi_{A,\varepsilon}(\xi,x)\rangle| \\
&= |\langle \mathcal{F}_{T_{A,\varepsilon}}(u)(\xi), \chi(\xi)P^n(\xi)\Phi_{A,\varepsilon}(\xi,x)\rangle| \\
&\leq C\sup_{|\xi|\geq\sqrt{1-\varepsilon^2}\varrho}\sum_{0\leq j\leq N}\Big|D^j\Big(\chi(\xi)P^n(\xi)\Phi_{A,\varepsilon}(\xi,x)\Big)\Big|.
\end{aligned}$$

Thus from the Leibniz formula (10) we obtain that

$$\forall\, n \in \mathbb{N}_0, \quad |P^n(-iT_{A,\varepsilon})(u)(x)| \leq C_1(R)n^N(R-\frac{\eta}{3})^n(1+|x|)^{N+2}e^{-\varrho|x|} \leq C_2(R)R^n(1+|x|)^{N+2}e^{-\varrho|x|}.$$

Conversely we assume that we have (34).

Suppose $\xi_0 \in \mathbb{R}$ is fixed and such that $|\xi_0| \geq \sqrt{1-\varepsilon^2}\varrho$, and $|P(\xi_0)| \geq R+\eta$, for some $\eta > 0$. Choose and fix $\chi \in D(\mathbb{R})$ such that

$$\text{supp}\chi \subset \Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \geq R + \frac{\eta}{3}\Big\}.$$

For $n \in \mathbb{N}$, we introduce the function χ_n defined by $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$. We have

$$\begin{aligned}\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle &= \langle \mathcal{F}_{T_{A,\varepsilon}}(u), P^n(\xi)\chi_n\rangle = \langle P^n(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u), \chi_n\rangle \\ &= \langle \mathcal{F}_{T_{A,\varepsilon}}(P^n(-iT_{A,\varepsilon})u), \chi_n\rangle \\ &= \langle \Big(e^{\varrho|x|}(1+|x|)^{-N}P^n(-iT_{A,\varepsilon})u\Big), e^{-\varrho|x|}(1+|x|)^N\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)\rangle.\end{aligned}$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle| \leq ||e^{\varrho|x|}(1+|x|)^{-N}P^n(-iT_{A,\varepsilon})u||_{L^\infty_A(\mathbb{R})}||e^{-\varrho|x|}(1+|x|)^N\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^1_A(\mathbb{R})}.$$

We proceed as in Proposition 9, we prove that

$$||e^{-\varrho|x|}(1+|x|)^N\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^1_A(\mathbb{R})} \leq Cn^M(R+\frac{\eta}{3})^{-n}.$$

Thus

$$|\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle| \leq C(R)n^M\Big(\frac{R}{R+\frac{\eta}{3}}\Big)^n.$$

Hence we deduce $\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle = 0$, which implies that $\xi_0 \notin$ supp $\mathcal{F}_{T_{A,\varepsilon}}(u)$. Thus support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is contained in the compact $V_{r,\varepsilon}$.

We proceed as the above theorem, we use the same ideas and steps and the Leibnitz formula (11), we prove the following result.

Theorem 3 *Let $u \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1,1]$. Then the support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is contained in the compact $V_{r,1} := \Big\{\xi \in \mathbb{R} : |P(\xi)| \leq r\Big\}$ for a polynomial P and a constant $r \geq 0$, if, and only if, for each $R > r$, there exist $N_R \in \mathbb{N}_0$ and a positive constant $C(R)$ such that*

$$|P^n(-iT_{A,\varepsilon})(u)(x)| \leq C(R)R^n(1+|x|)^{N_R}e^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|}, \tag{35}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Notations. Let $\varepsilon \in [-1,1]$ and $r > 0$, we denote by

$$B_{r,\varepsilon} := \Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| < r\Big\}, \quad S_{r,\varepsilon} := \Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| = r\Big\}.$$

Theorem 4 *Let $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^2_\varepsilon(\mathbb{R})$, and consider the infinite series $\{u_{-n}\}_{n\in\mathbb{N}}$ of generalized tempered distributions defined as $u_{-n+1} = P(-iT_{A,\varepsilon})u_n$, for a polynomial P and for all $n \in \mathbb{N}$. Let $r > 0$. Assume, for all $R \in (0,r)$ there exist constants $N_R \in \mathbb{N}_0$ and $C(R) > 0$, such that*

$$\forall\, x \in \mathbb{R}, \quad |u_{-n}(x)| \leq C(R)R^{-n}(1+|x|)^{N_R}e^{-\varrho|x|}, \tag{36}$$

for all $n \in \mathbb{N}$. Then $\text{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \cap B_{r,\varepsilon} = \emptyset$.
On the other hand, if $\text{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \cap B_{r,\varepsilon} = \emptyset$ *and* $\text{supp}\mathcal{F}_{T_{A,\varepsilon}}(u)$ *is compact, then (36) holds, for all $R \in (0,r)$.*

Proof. Assume that we have (36). For a fixed $R \in (0, r)$ let $\eta > 0$. Choose and fix $\chi \in D(\mathbb{R})$ such that

$$\operatorname{supp} \chi \subset \Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \leq R - \frac{\eta}{3}\Big\},$$

and put $\chi_n = P^n(\xi)\chi$. We have

$$\begin{aligned} \langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle &= \langle \mathcal{F}_{T_{A,\varepsilon}}(u), P^{-n}(\xi)\chi_n\rangle = \langle P^{-n}(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u), \chi_n\rangle \\ &= \langle \mathcal{F}_{T_{A,\varepsilon}}(u_{-n}), \chi_n\rangle \\ &= \langle \Big(e^{\varrho|x|}(1+|x|)^{-N} u_{-n}\Big), e^{-\varrho|x|}(1+|x|)^N \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)\rangle. \end{aligned}$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle| \leq ||e^{\varrho|x|}(1+|x|)^{-N} u_{-n}||_{L^\infty_A(\mathbb{R})} ||e^{-\varrho|x|}(1+|x|)^N \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^1_A(\mathbb{R})}.$$

We proceed as in Proposition 9, we prove that

$$||e^{-\varrho|x|}(1+|x|)^N \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi_n)||_{L^1_A(\mathbb{R})} \leq C n^M (R - \frac{\eta}{3})^n.$$

Thus

$$\forall\, n \in \mathbb{N}, \quad |\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle| \leq C(R) n^M \Big(\frac{R - \frac{\eta}{3}}{R}\Big)^n.$$

Thus we deduce $\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \chi\rangle = 0$, which implies that $\operatorname{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \cap B_{r,\varepsilon} = \emptyset$.

Assume that $\operatorname{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \cap B_{r,\varepsilon} = \emptyset$ and $\operatorname{supp}\mathcal{F}_{T_{A,\varepsilon}}(u)$ is compact. Let $R \in (0, r)$ and let $\eta \in (0, r - R)$. Choose $\chi \in D(\mathbb{R})$ such that $\chi \equiv 1$ on an open neighborhood of support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$, and $\chi \equiv 0$ on $V_{R+\frac{\eta}{3},\varepsilon}$. As $u = P^n(-iT_{A,\varepsilon})u_{-n}$, we have

$$\begin{aligned} |u_{-n}(x)| &= \Big|\mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big(P^{-n}(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u)\Big)(x)\Big| \\ &= \Big|\mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big(\chi(\xi)P^{-n}(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u)\Big)(x)\Big| \\ &= |\langle \chi(\xi)P^{-n}(\xi)\mathcal{F}_{T_{A,\varepsilon}}(u)(\xi), \Phi_{A,\varepsilon}(\xi, x)\rangle| \\ &= |\langle \mathcal{F}_{T_{A,\varepsilon}}(u)(\xi), \chi(\xi)P^{-n}(\xi)\Phi_{A,\varepsilon}(\xi, x)\rangle| \\ &\leq C \sup_{|\xi| \geq \sqrt{1-\varepsilon^2}\varrho} \sum_{0 \leq j \leq N} \Big|D^j\Big(\chi(\xi)P^{-n}(\xi)\Phi_{A,\varepsilon}(\xi, x)\Big)\Big|. \end{aligned}$$

Thus from the Leibniz formula (10) we obtain that

$$\forall\, n \in \mathbb{N}_0, \quad |u_{-n}(x)| \leq C_1(R) n^N (R + \frac{\eta}{3})^{-n}(1+|x|)^{N+2} e^{-\varrho|x|} \leq C_2(R) R^{-n}(1+|x|)^{N+2} e^{-\varrho|x|}.$$

We proceed as the above theorem, we use the same ideas and steps and the Leibnitz formula (11), we prove the following result.

Theorem 5 *Let $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'^2_\varepsilon(\mathbb{R})$, and consider the infinite series $\{u_{-n}\}_{n\in\mathbb{N}}$ of generalized tempered distributions defined as $u_{-n+1} = P(-iT_{A,\varepsilon})u_n$, for a polynomial P and for all $n \in \mathbb{N}$. Let $r > 0$. Assume, for all $R \in (0, r)$ there exist constants $N_R \in \mathbb{N}_0$ and $C(R) > 0$, such that*

$$\forall\, x \in \mathbb{R}, \quad |u_{-n}(x)| \leq C(R) R^{-n}(1+|x|)^{N_R} e^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|}, \tag{37}$$

for all $n \in \mathbb{N}$. *Then* $\mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \cap B_{r,1} = \emptyset$.
On the other hand, if $\mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \cap B_{r,1} = \emptyset$ *and* $\mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(u)$ *is compact, then (37) holds, for all* $R \in (0, r)$.

Combining Theorem 2 and Theorem 4 together, we get

Corollary 3 *Let* $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap {\mathcal{S}'}^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1, 1]$, *and consider the infinite series* $\{u_n\}_{n\in\mathbb{Z}}$ *of generalized tempered distributions defined as* $u_{n+1} = P(-iT_{A,\varepsilon})u_n$, *for a polynomial* P *and for all* $n \in \mathbb{Z}$. *Let* $R > 0$. *Then* $\mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(u)$ *is contained in* $S_{R,\varepsilon}$, *if and only if for all* $\eta > 0$, *there exist constants* $N_\eta \in \mathbb{N}_0$ *and* $C_\eta > 0$, *such that*

$$\forall\, x \in \mathbb{R}, \quad |u_n(x)| \leq C_\eta R^n (1+\eta)^{|n|}(1+|x|)^{N_\eta} e^{-\varrho|x|} \tag{38}$$

for all $n \in \mathbb{Z}$.

Combining Theorem 3 and Theorem 5 together, we get

Corollary 4 *Let* $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap {\mathcal{S}'}^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1, 1]$, *and consider the infinite series* $\{u_n\}_{n\in\mathbb{Z}}$ *of generalized tempered distributions defined as* $u_{n+1} = P(-iT_{A,\varepsilon})u_n$, *for a polynomial* P *and for all* $n \in \mathbb{Z}$. *Let* $R > 0$. *Then* $\mathrm{supp}\mathcal{F}_{T_{A,\varepsilon}}(u)$ *is contained in* $S_{R,1}$, *if and only if for all* $\eta > 0$, *there exist constants* $N_\eta \in \mathbb{N}_0$ *and* $C_\eta > 0$, *such that*

$$\forall\, x \in \mathbb{R}, \quad |u_n(x)| \leq C_\eta R^n (1+\eta)^{|n|}(1+|x|)^{N_\eta} e^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|} \tag{39}$$

for all $n \in \mathbb{Z}$.

Remark 4 *Theorem 4 and Corollary 3 are the analogue of the new real Paley-Wiener theorems for the Fourier transform, proved by Andersen (see [2]).*

5 Characterisation for the spectrum of the Opdam-Cherednik transform on $L^p_A(\mathbb{R})$ via the generalized potential function

In this section, we assume that $\varepsilon = 1$, and $A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}$, $k \geq k' \geq 0$ and $k \neq 0$. In this case the generalized Fourier transform is the Opdam-Cherednik transform on $\mathbb{R}$.

Definition 4 *Let* $f \in {\mathcal{S}'}^2_\varepsilon(\mathbb{R})$. *The tempered generalized function* $R_0 f$ *is termed the generalized potential of* f *if* $-\triangle_{A,\varepsilon}(R_0 f) = f$, *that is*

$$\langle R_0 f, \triangle_{A,\varepsilon}\varphi\rangle = -\langle f, \varphi\rangle, \quad \text{for all } \varphi \in \mathcal{S}^2_\varepsilon(\mathbb{R}).$$

Theorem 6 *Let* $1 \leq p < 2$ *and* $R_0^n f \in L^p_A(\mathbb{R})$ *for all* $n \in \mathbb{N}_0$. *If* $0 \notin supp\, \mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big)$ *for all* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} = \frac{1}{\sigma_0^2}, \tag{40}$$

where

$$\sigma_0 = \inf\Big\{|\xi| \;:\; \xi \in supp\mathcal{F}_{T_{A,\varepsilon}}(f)\Big\}.$$

For prove this theorem we need the following lemmas.

Lemma 1 *If $\sigma_0 > 0$, then*

$$supp\,\mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big) = supp\,\mathcal{F}_{T_{A,\varepsilon}}(f), \quad n = 1, ... \tag{41}$$

Proof. As

$$(-\triangle_{A,\varepsilon})^n(R_0^n f) = f$$

we deduce that

$$\mathcal{F}_{T_{A,\varepsilon}}(f) = \xi^{2n}\mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big).$$

Therefore,

$$supp\,\mathcal{F}_{T_{A,\varepsilon}}(f) \subset supp\,\mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big) \subset \mathcal{F}_{T_{A,\varepsilon}}(f) \cup \Big\{0\Big\}.$$

So, to obtain (41), it is enough to use the hypothesis $0 \notin supp\,\mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big)$.

Lemma 2 *If $\sigma_0 > 0$, then*

$$\limsup_{n\to\infty} ||R_0^n f||_{L_A^p(\mathbb{R})}^{\frac{1}{n}} \leq \frac{1}{\sigma_0^2}. \tag{42}$$

Proof. From (41) we have

$$supp\,\mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big) \subset \mathbb{R}\backslash(-\sigma_0, \sigma_0). \tag{43}$$

For any $\eta > 0$, $\eta < \frac{\sigma_0}{2}$ we choose an even function $h \in \mathcal{E}(\mathbb{R})$ satisfying

$$h(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \geq \sigma_0 - \eta \\ 0 & \text{if} \quad |\xi| < \sigma_0 - 2\eta. \end{cases}$$

Let χ be an arbitrary element in $\mathcal{S}_\varepsilon^2(\mathbb{R})$. Then it follow from (43) that

$$\begin{aligned} \langle R_0^n f, \chi\rangle &= \langle \mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big), \mathcal{F}_{T_{A,\varepsilon}}(\chi)\rangle \\ &= \langle \mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big), h\mathcal{F}_{T_{A,\varepsilon}}(\chi)\rangle \\ &= \langle R_0^n f, (\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(h\mathcal{F}_{T_{A,\varepsilon}}(\chi)\Big)\rangle. \end{aligned}$$

Therefore,

$$\langle R_0^n f, \chi\rangle = \langle R_0^n f, \varphi\rangle, \tag{44}$$

where

$$\varphi = (\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(h\mathcal{F}_{T_{A,\varepsilon}}(\chi)\Big).$$

We put

$$\varphi_n = (\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\mathcal{F}_{T_{A,\varepsilon}}(\chi)\Big).$$

Then $\varphi_n \in \mathcal{S}^2_\varepsilon(\mathbb{R})$ and

$$\begin{aligned} |\langle f, \varphi_n\rangle| &= |\langle (-\triangle_{A,\varepsilon})^n R_0^n f, \varphi_n\rangle| \\ &= |\langle R_0^n f, (-\triangle_{A,\varepsilon})^n \varphi_n\rangle| \\ &= |\langle R_0^n f, \varphi\rangle|. \end{aligned} \tag{45}$$

Combining (44) and (45), we get

$$|\langle R_0^n f, \chi\rangle| = |\langle f, \varphi_n\rangle| = |\langle f, \chi *_k (\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\Big)\rangle|, \tag{46}$$

where $*_k$ is the generalized convolution associated with the Jacobi-Cherednik operator. Therefore, we have

$$\begin{aligned} ||R_0^n f||_{L^p_A(\mathbb{R})} &= \sup_{\left\{\chi\in\mathcal{S}^2_\varepsilon(\mathbb{R}):\quad ||\chi||_{L^2_A(\mathbb{R})}\le 1\right\}} \Big|\langle f, \chi *_k (\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\Big)\rangle\Big| \\ &\le \sup_{\left\{\chi\in\mathcal{S}^2_\varepsilon(\mathbb{R}):\quad ||\chi||_{L^2_A(\mathbb{R})}\le 1\right\}} ||f||_{L^p_A(\mathbb{R})} ||\chi *_k (\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\Big)||_{L^q_A(\mathbb{R})} \\ &\le ||f||_{L^p_A(\mathbb{R})} ||(\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\Big)||_{L^2_A(\mathbb{R})}. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} \le \limsup_{n\to\infty} ||(\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\Big)||^{\frac{1}{n}}_{L^2_A(\mathbb{R})}. \tag{47}$$

We put

$$\mu = \sigma_0 - 2\eta.$$

Using the Parseval identity we can show that

$$\limsup_{n\to\infty} ||(\mathcal{F}_{T_{A,\varepsilon}})^{-1}\Big(\frac{h(\xi)}{\xi^{2n}}\Big)||^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \le \frac{1}{\mu^2}. \tag{48}$$

Combining (47) and (48), we get

$$\limsup_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} \le \frac{1}{(\sigma_0 - 2\eta)^2}$$

and then (42) by letting $\eta \to 0$.

Lemma 3 *If $\sigma_0 > 0$, then*

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} \ge \frac{1}{\sigma_0^2}. \tag{49}$$

Proof. Without loss of generality we may assume that

$$\sigma_0 = \inf\Big\{\xi \in \mathbb{R}_+ : \xi \in supp\mathcal{F}_{T_{A,\varepsilon}}(f)\Big\}.$$

Hence, there exists a function $\chi \in D(\mathbb{R})$ such that

$$supp\chi \subset \Big\{\xi : \sigma_0 - \eta < |\xi| < \sigma_0 + \eta\Big\} \quad \text{and} \quad \langle \mathcal{F}_{T_{A,\varepsilon}}(f), \chi\rangle \neq 0.$$

Therefore,

$$\begin{aligned} 0 \neq |\langle f, \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)\rangle| &= |\langle (-\triangle_{A,\varepsilon})^n R_0^n f, \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)\rangle| \\ &= |\langle R_0^n f, (-\triangle_{A,\varepsilon})^n \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)\rangle| \\ &\leq ||R_0^n f||_{L^p_A(\mathbb{R})} ||(-\triangle_{A,\varepsilon})^n \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)||_{L^q_A(\mathbb{R})}. \end{aligned} \tag{50}$$

So

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} \geq \frac{1}{\limsup\limits_{n\to\infty} ||(-\triangle_{A,\varepsilon})^n \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)||_{L^q_A(\mathbb{R})}}. \tag{51}$$

We proceed as [17], we prove that

$$\limsup_{n\to\infty} ||(-\triangle_{A,\varepsilon})^n \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)||^{\frac{1}{n}}_{L^q_A(\mathbb{R})} \leq (\sigma_0 + \eta)^2.$$

So by (51) we obtain

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} \geq \frac{1}{(\sigma_0+\eta)^2}, \quad \eta > 0,$$

and then (49).

Proof of Theorem 6. We divide our proof into two cases.

Case 1. $\sigma_0 = 0$. We have $\xi_0 \in supp\mathcal{F}_{T_{A,\varepsilon}}(f)$. Hence, for any $\eta > 0$ there is a function $\chi \in D(\mathbb{R})$ such that $supp\,\chi \subset (-\eta, \eta)$ such that $\langle \mathcal{F}_{T_{A,\varepsilon}}(f), \chi\rangle \neq 0$. Arguing as above we obtain

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} \geq \frac{1}{\limsup\limits_{n\to\infty} ||(-\triangle_{A,\varepsilon})^n \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\chi)||^{\frac{1}{n}}_{L^q_A(\mathbb{R})}} \geq \frac{1}{\eta^2}.$$

Therefore

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} = \infty.$$

So we always have

$$\lim_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_A(\mathbb{R})} = \frac{1}{\sigma_0^2}.$$

Case 2. If $\sigma_0 > 0$. Combining (42) and (49), we arrive to (40).

6 Real Paley-Wiener theorems for the generalized Fourier transform on $\mathcal{S}'^2_\varepsilon(\mathbb{R})$

Let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$, $\varepsilon \in [-1, 1]$. We put

$$\Gamma_u := \inf\Big\{r \in (0, \infty] : \quad \mathrm{supp}(\mathcal{F}_{T_{A,\varepsilon}}(u)) \subset [-r, r]\Big\}.$$

Theorem 7 *Let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$. Then the support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is included in $[-M, M]$, $M > 0$, if and only if for all $R > M$ we have*

$$\lim_{n\to\infty} R^{-2n} \triangle^n_{A,\varepsilon} u = 0, \quad \text{in} \quad \mathcal{S}'^2_\varepsilon(\mathbb{R}).$$

Proof. Let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$ and $M > 0$ such that

$$\lim_{n\to\infty} R^{-2n} \triangle^n_{A,\varepsilon} u = 0, \quad \text{for all} \quad R > M.$$

Let $\varphi \in D(\mathbb{R})$ satisfy $\text{supp}(\varphi) \subset [-M, M]^c$. We have to prove that

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi \rangle = 0.$$

Let $r > M$ satisfy $\varphi(x) = 0$ for all $x \in [-r, r]$ and $R \in (M, r)$. Then for all $n \in \mathbb{N}$ the function $x^{-2n}\varphi$ is in $D(\mathbb{R})$ and we can write

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi \rangle = \langle (-x^2)^n R^{-2n} \mathcal{F}_{T_{A,\varepsilon}}(u), (-x^2)^{-n} R^{2n} \varphi \rangle,$$

and by formula (33), we have

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi \rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(R^{-2n} \triangle^n_{A,\varepsilon}(u)), (-x^2)^{-n} R^{2n} \varphi \rangle.$$

The hypothesis implies that $\mathcal{F}_{T_{A,\varepsilon}}(R^{-2n} \triangle^n_{A,\varepsilon}(u)) \to 0$ in $\mathcal{S}'(\mathbb{R})$. Moreover from the Leibniz formula we deduce that $(-x^2)^{-n} R^{2n} \varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi \rangle = 0.$$

Conversely, let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$ and $M > 0$ such that $\text{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \subset [-M, M]$. We are going to prove that for all $R > M$

$$\lim_{n\to\infty} R^{-2n} \triangle^n_{A,\varepsilon} u = 0, \quad \text{in} \quad \mathcal{S}'^2_\varepsilon(\mathbb{R}).$$

Let $M < R$ and choose $\varrho \in (M, R)$ and $\psi \in D(R)$ satisfying $\psi \equiv 1$ on a neighborhood of $[-M, M]$ and $\psi(x) = 0$ for all $x \notin [-\varrho, \varrho]$. Then for all $\varphi \in D(\mathbb{R})$ we have

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi \rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(u), \psi\varphi \rangle,$$

and then

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(R^{-2n} \triangle^n_{A,\varepsilon}(u)), \varphi \rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(u), (-x^2)^n R^{-2n} \psi\varphi \rangle.$$

Finally we deduce the result by using the fact that $(-x^2)^n R^{-2n} \psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$.

Corollary 5 *From the previous theorem we obtain*

$$\Gamma_u = \inf \Big\{ R > 0 : \quad \lim_{n\to\infty} R^{-2n} \triangle^n_{A,\varepsilon} u = 0, \quad \text{in} \quad \mathcal{S}'^2_\varepsilon(\mathbb{R}) \Big\}.$$

Let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$. We put $\gamma_u := \sup \Big\{ r \in [0, \infty) : \quad \text{supp}(\mathcal{F}_{T_{A,\varepsilon}}(u)) \subset (-r, r)^c \Big\}$.

Theorem 8 *Let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$ such that $(-x^2)^{-n}\mathcal{F}_{T_{A,\varepsilon}}(u) \in \mathcal{S}'(\mathbb{R})$ for all $n \in \mathbb{N}$.*

Let $u_n = \mathcal{F}^{-1}_{T_{A,\varepsilon}}\Big((-x^2)^{-n}\mathcal{F}_{T_{A,\varepsilon}}(u)\Big)$. Then the support of $\mathcal{F}_{T_{A,\varepsilon}}(u)$ is included in $(-M,M)^c$, $M > 0$, if and only if for all $R < M$ we have

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in} \quad \mathcal{S}'^2_\varepsilon(\mathbb{R}).$$

Proof. Let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$ and $M > 0$ such that

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{for all} \quad R < M.$$

Let $\varphi \in D(\mathbb{R})$ satisfy $\text{supp}(\varphi) \subset (-M,M)$. We want to prove that

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi\rangle = 0.$$

Let $r \in (0,M)$ such that $\text{supp}\,\varphi \subset (-r,r)$ and $R \in (r,M)$. Then for all $n \in \mathbb{N}$ the function $x^{2n}\varphi$ is in $D(\mathbb{R})$ and we can write

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi\rangle = \big\langle (-x^2)^{-n}R^{2n}\mathcal{F}_{T_{A,\varepsilon}}(u), (-x^2)^{n}R^{-2n}\varphi\big\rangle = \big\langle \mathcal{F}_{T_{A,\varepsilon}}(R^{2n}u_n), (-x^2)^{n}R^{-2n}\varphi\big\rangle\,.$$

The hypothesis implies that $\mathcal{F}_{T_{A,\varepsilon}}(R^{2n}u_n) \to 0$ in $\mathcal{S}'(\mathbb{R})$. Moreover from the Leibniz formula we deduce that $(-x^2)^{n}R^{-2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi\rangle = 0.$$

Conversely, let $u \in \mathcal{S}'^2_\varepsilon(\mathbb{R})$ and $M > 0$ such that $\text{supp}\mathcal{F}_{T_{A,\varepsilon}}(u) \subset (-M,M)^c$. We are going to prove that for all $R < M$

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in} \quad \mathcal{S}'^2_\varepsilon(\mathbb{R}).$$

Let $M > R$ and choose $\varrho \in (R,M)$ and $\psi \in D(R)$ satisfying $\psi(x) \equiv 1$ for $|x| \geq \frac{M+\varrho}{2}$ and $\psi(x) = 0$ for all $|x| \leq \varrho$. Then for all $\varphi \in D(\mathbb{R})$ we have

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(u), \varphi\rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(u), \psi\varphi\rangle,$$

and then

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(R^{n}u_n), \varphi\rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(u), (-x^2)^{-n}R^{2n}\psi\varphi\rangle.$$

Finally we deduce the result by using the fact that $(-x^2)^{-n}R^{2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$.

Corollary 6 *From the previous theorem we obtain*

$$\gamma_u = \sup\Big\{R > 0, \lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in} \quad \mathcal{S}'^2_\varepsilon(\mathbb{R})\Big\}.$$

7 Roe's theorem associated with type family of operators $T_{A,\varepsilon}$

In [21] Roe proved that if a doubly-infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies $\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \le M$ for all $j = 0, \pm1, \pm2, ...$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x + b)$ where a and b are real constants.

The purpose of this section is to generalize this theorem for the operators $T_{A,\varepsilon}$.

Theorem 9 *Suppose* $P(\xi) = \sum_n a_n\xi^n$ *is real-valued and let* $\{f_j\}_{-\infty}^{\infty}$ *be a sequence of complex-valued functions on* $\mathbb{R}$ *so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-iT_{A,\varepsilon})f_j.$$

(i) Let $a \ge 0$, $R > 0$, *and assume that* $\{f_j\}_{-\infty}^{\infty}$ *satisfies*

$$|f_j(x)| \le M_jR^j(1 + |x|)^a e^{-\varrho|x|}, \tag{52}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{53}$$

Then $f = f_+ + f_-$ *where* $P(-iT_{A,\varepsilon})f_+ = Rf_+$ *and* $P(-iT_{A,\varepsilon})f_- = -Rf_-$.
If R *(or* $-R$*) is not in the range of* P *then* $f_+ = 0$ *(or* $f_- = 0$*).*

(ii) If we replace (53) with

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0, \tag{54}$$

for all $j > 0$, *then the span of* $(f_j)_j$ *is finite dimensional. Moreover,* $f_0 = f_+ + f_-$, *where, for some integer* N , $(P(-iT_{A,\varepsilon}) - R)^N f_+ = 0$ *and* $(P(-iT_{A,\varepsilon}) + R)^N f_- = 0$. *Thus* f_+ *(or* f_- *) is a generalized eigenfunction of* $P(-iT_{A,\varepsilon})$ *with eigenvalue* R *(or* $-R$*).*

In order to prove Theorem 9 we need the following lemmas:

Lemma 4 *Let* $(f_j)_{j\in\mathbb{Z}}$ *is be a sequence of functions on* $\mathbb{R}$ *satisfying*

$$f_{j+1} = P(-iT_{A,\varepsilon})f_j, \tag{55}$$

$$|f_j(x)| \le M_jR^j(1 + |x|)^a e^{-\varrho|x|}, \tag{56}$$

and

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0, \tag{57}$$

for all $\varepsilon > 0$, *then*

$$\mathrm{supp}(\mathcal{F}_{T_{A,\varepsilon}}(f_0)) \subset S_{R,\varepsilon} := \Big\{\xi \in \mathbb{R} : |\xi| \ge \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| = R\Big\}.$$

Proof. First we show that $\mathcal{F}_{T_{A,\varepsilon}}(f_0)$ is supported in

$$\Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \leq R\Big\}.$$

To do this we need to show that

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), \phi\rangle = 0$$

if $\phi \in D(\mathbb{R})$ and $\text{supp}(\phi) \cap \Big\{\xi :\ |P(\xi)| \leq R\Big\} = \emptyset$. Since $\text{supp}(\phi)$ is compact, there is some $r < \frac{1}{R}$ so that $\frac{1}{|P(\xi)|} \leq r$, for all $\xi \in \text{supp}(\phi)$. Then

$$\begin{aligned}
\langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), \phi\rangle &= \langle P^j \mathcal{F}_{T_{A,\varepsilon}}(f_0), \tfrac{\phi}{P^j}\rangle \\
&= \langle \mathcal{F}_{T_{A,\varepsilon}}\Big(P^j(-iT_{A,\varepsilon})f_0\Big), \tfrac{\phi}{P^j}\rangle \\
&= \langle P^j(-iT_{A,\varepsilon})f_0, \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\tfrac{\phi}{P^j})\rangle.
\end{aligned}$$

Choose an integer m with $2m \geq 2a+2$. A calculation, using the hypothesis of the lemma and Cauchy-Schwartz inequality, implies

$$\begin{aligned}
|\langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), \phi\rangle| &\leq \int_{\mathbb{R}} |P^j(-iT_{A,\varepsilon})f_0(x)||\mathcal{F}^{-1}_{T_{A,\varepsilon}}(\frac{\phi}{P^j})(x)|A(x)dx \\
&\leq CM_j R^j \sup_{x\in\mathbb{R}} |e^{\varrho|x|}(1+x^2)^m \mathcal{F}^{-1}_{T_{A,\varepsilon}}(\tfrac{\phi}{P^j})(x)]|.
\end{aligned}$$

Using the continuity of $\mathcal{F}^{-1}_{T_{A,\varepsilon}}$ and the fact that ϕ is supported in $\Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \geq R+\eta\Big\}$ for some fixed $\eta > 0$, it is not hard to prove that the right-hand side of this goes to zero as $j \to \infty$ and so $\langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), \phi\rangle = 0$. To complete the proof we need to show that $\mathcal{F}_{T_{A,\varepsilon}}(f_0)$ is also supported in $\Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \geq R\Big\}$, which means $\langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), \phi\rangle = 0$ if ϕ is supported in $\Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } |P(\xi)| \leq R\Big\}$. Here we use (55) to obtain

$$\langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), \phi\rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(f_{-j}), P^j\phi\rangle$$

and the argument proceeds as before.

Lemma 5 *We assume that $-R$ is not a value of $P(\xi)$. There exists an integer N such that*

$$(P(\xi) - R)^{N+1}\mathcal{F}_{T_{A,\varepsilon}}(f_0) = 0. \tag{58}$$

Proof. Using Lemma 4 and proceeding as in [14], we prove the result.

Lemma 6 *([8]). Let X be a finite dimensional complex vector space, and let $\Lambda : X \to X$ be a linear map with eigenvalues $\lambda_1, ..., \lambda_p$. Then $X = X_1 \oplus ... \oplus X_p$, where $X_j = ker((\Lambda - \lambda_j)^N)$ and $dimX = N$.*

Proof of Theorem 9

We want to prove (i). Inverting the generalized Fourier transform in (58) yields that
$$(P(-iT_{A,\varepsilon}) - R)^{N+1} f_0 = 0. \tag{59}$$
This equation implies
$$\operatorname{span}\Big\{f_0, f_1, f_2, ...\Big\} = \operatorname{span}\Big\{f_0, P(-iT_{A,\varepsilon})f_0, ..., P^N(-iT_{A,\varepsilon})f_0\Big\}.$$
We shall now show that we can take $N = 0$ in (59). If not then $(P(-iT_{A,\varepsilon}) - R)f_0 \neq 0$. Let p be the largest positive integer so that $(P(-iT_{A,\varepsilon}) - R)^p f_0 \neq 0$. Clearly $p \leq N$. Thus
$$f := (P(-iT_{A,\varepsilon}) - R)^{p-1} f_0 \in \operatorname{span}\Big\{f_0, f_1, ..., f_N\Big\}$$
will satisfy
$$(P(-iT_{A,\varepsilon}) - R)^2 f = 0 \quad \text{and} \quad (P(-iT_{A,\varepsilon}) - R)f \neq 0. \tag{60}$$
Write
$$f = a_0 f_0 + ... + a_N f_N,$$
for constants $a_0, ..., a_N$. Then
$$P^j(-iT_{A,\varepsilon})f = a_0 f_j + ... + a_N f_{N+j}.$$
If
$$C_j = |a_0| R^0 M_j + ... + |a_N| R^N M_{j+N},$$
then this and (52) imply
$$|P^j(-iT_{A,\varepsilon})f(x)| \leq C_j R^j (1 + |x|)^a e^{-\varrho|x|}. \tag{61}$$
By (53) these satisfy the sublinear growth condition
$$\lim_{j\to\infty} \frac{C_j}{j} = 0. \tag{62}$$
An induction using (60) implies for $j \geq 2$ that
$$P^j(-iT_{A,\varepsilon})f = R^{j-1} j P(-iT_{A,\varepsilon})f - R^j (j-1) f = R^{j-1} j (P(-iT_{A,\varepsilon}) - R) f + R^j f.$$
Thus
$$|(P(-iT_{A,\varepsilon}) - R)f(x)| \leq \frac{1}{jR^{j-1}} |P^j(-iT_{A,\varepsilon})f(x)| + \frac{R|f(x)|}{j} \leq \frac{C_j R}{j} (1+|x|)^a e^{-\varrho|x|} + \frac{R|f(x)|}{j}.$$
Letting $j \to \infty$ and using (62) implies $(P(-iT_{A,\varepsilon}) - R)f = 0$. But this contradicts (60). Consequently, $N = 0$ in (59). This completes the proof in the case that $-R$ is not in the range of P.

In the case that R is not in the range of P we apply the same argument to $-P(-iT_{A,\varepsilon})$ to conclude $P(-iT_{A,\varepsilon})f_0 = -Rf_0$.
In the general case, let $\mathfrak{L} = P^2(-iT_{A,\varepsilon})$. Then $\mathcal{F}_{T_{A,\varepsilon}}(\mathfrak{L}f)(\xi) = P^2(\xi)\mathcal{F}_{T_{A,\varepsilon}}(f)(\xi)$. $\mathfrak{L}f_{2p} = f_{2(p+1)}$ and $P^2(\xi) \neq -R$. Thus we can (as before) conclude, for the sequence $(f_{2p})_{p\in\mathbb{Z}}$ that

$$\mathfrak{L}f_0 = P^2(-iT_{A,\varepsilon})f_0 = R^2 f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + \frac{1}{R}P(-iT_{A,\varepsilon})f_0)$ and $f_- = \frac{1}{2}(f_0 - \frac{1}{R}P(-iT_{A,\varepsilon})f_0)$.
Then $f = f_+ + f_-$, $P(-iT_{A,\varepsilon})f_+ = Rf_+$ and $P(-iT_{A,\varepsilon})f_- = -Rf_-$. This completes the proof of (i).

Now we want to prove (ii).
We first prove (ii) under the assumption that $P(\xi) \neq -R$. Using the growth condition (54) and Lemma 6, we may still conclude that

$$\mathrm{supp}(\mathcal{F}_{T_{A,\varepsilon}}(f_0)) \subset S_{R,\varepsilon} := \Big\{\xi \in \mathbb{R} : |\xi| \geq \sqrt{1-\varepsilon^2}\varrho \text{ and } P(\xi) = R\Big\}.$$

But then, as before, we can conclude that (59) holds. But this is enough to complete the proof in this case. A similar argument shows that if $P(\xi) \neq R$, then $(P(-iT_{A,\varepsilon})+R)^N f_0 = 0$.
In the general case we again let $\mathfrak{L} = P^2(-iT_{A,\varepsilon})$ and $P_0 = P^2$. Then $P_0(\xi) \neq -R$ and the span of $(f_{2j})_j$ is finite dimensional. The map $P(-iT_{A,\varepsilon})$ takes the span of $(f_{2j})_j$ onto the span of $(f_{2j+1})_j$. Thus X is finite dimensional. Any $f \in X$ will have $\mathrm{supp}(f)$ inside the set defined by $P(\xi) = \pm R$. From this it is not hard to show the only possible eigenvalues of $P(-iT_{A,\varepsilon})$ restricted to X are R and $-R$. The result now follows from the last lemma.

Remark 5 *(i) If we take $P(y) = -y^2$, then $P(-iT_{A,\varepsilon}) = \triangle_{A,\varepsilon}$ and Theorem 9 give $\triangle_{A,\varepsilon}f_0 = -Rf_0$. This characterizes eigenfunctions f of generalized Laplace operator $\triangle_{A,\varepsilon}$ with polynomial growth in terms of the size of the powers $\triangle^j_{A,\varepsilon}f$, $-\infty < j < \infty$.*

(ii) The previous theorem generalizes and improves the version presented in [4, 16, 17].

Theorem 10 *Let $\varepsilon \in [-1,1]$. Suppose $P(\xi) = \sum_n a_n\xi^n$ is a non-constant polynomial with complex coefficients. Let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-iT_{A,\varepsilon})f_j.$$

1) Let $a \geq 0$ and let $R > 0$. Assume that for all $\eta > 0$, there exist constants $N \in \mathbb{N}_0$ and $C > 0$, such that

$$\forall\, x \in \mathbb{R}, \quad |f_n(x)| \leq CR^n(1+\eta)^{|n|}(1+|x|)^N e^{-\varrho|x|} \tag{63}$$

is satisfied for all $n \in \mathbb{Z}$. Then

$$f_0 = \sum_{\lambda\in S_{R,\varepsilon}} \sum_{j=0}^{N} c(\lambda,j)\frac{d^j}{d\xi^j}_{|\xi=\lambda}\Phi_{A,\varepsilon}(\xi,.), \tag{64}$$

for constants $c(\lambda, j) \in \mathbb{C}$ and $N \in \mathbb{N}$.

2) Let $a \geq 0$ and let $R > 0$ and assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies

$$|f_j(x)| \leq M_j R^j (1 + |x|)^a e^{-\varrho|x|}, \tag{65}$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the subpotential growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j^m} = 0, \tag{66}$$

for some $m \geq 0$.

We have

(i) If $P'(\lambda_p) \neq 0$, for all $\lambda_p \in S_{R,\varepsilon}$, then $N < m$ in (64).

In particular, if $m = 1$, then

$$f_0 = \sum_{\lambda_p \in S_{R,\varepsilon}} f_{\lambda_p}, \quad \text{where } f_{\lambda_p} = c(\lambda_p)\Phi_{A,\varepsilon}(\lambda_p, .)$$

(ii) If $S_{R,\varepsilon}$ consists of one point λ_0 and $m = 1$ in (66), then

$$P(-iT_{A,\varepsilon})f_0 = P(\lambda_0)f_0.$$

Proof. 1) Assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies (63). Then Corollary 3 implies that the support of $\mathcal{F}_{T_{A,\varepsilon}}(f_0)$ is contained in the finite set $S_{R,\varepsilon}$. A standard result in distribution theory, see e.g., [[22], Theorem 6.25], infers that

$$\mathcal{F}_{T_{A,\varepsilon}}(f_0) = \sum_{\lambda \in S_{R,\varepsilon}} \sum_{0\leq j\leq N} c(\lambda, j)\delta_\lambda^{(j)}$$

for constants $c(\lambda, j) \in \mathbb{C}$, and some integer N. Here δ_ξ^j denotes the jth distributional derivative of the delta function δ_ξ at ξ.

The result follows with $f_0 = \mathcal{F}_{T_{A,\varepsilon}}^{-1}\left(\sum_{\lambda\in S_{R,\varepsilon}} \sum_{0\leq j\leq N} c(\lambda, j)\delta_\lambda^{(j)}\right)$.

We want to prove 2) (i). For $n \geq 0$, we have

$$\langle f_n, \chi\rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), P^n(\lambda)\mathcal{F}_{T_{A,\varepsilon}}(\chi)\rangle,$$

for any $\chi \in \mathcal{S}_\varepsilon^2(\mathbb{R})$. Fix $\lambda_p \in S_{R,\varepsilon}$ such that $P'(\lambda_p) \neq 0$ and let N_p be the order of $\mathcal{F}_{T_{A,\varepsilon}}(f)$ at λ_p. Choose $\chi \in \mathcal{S}_\varepsilon^2(\mathbb{R})$ such that $\mathcal{F}_{T_{A,\varepsilon}}(\chi) = 1$ in a small neighborhood of λ_p, and $\mathcal{F}_{T_{A,\varepsilon}}(\chi) = 0$ around the points $V_{R,\varepsilon}\backslash\{\lambda_p\}$. Then, for $n > N_p$

$$\begin{aligned}\langle f_n, \chi\rangle = \langle \mathcal{F}_{T_{A,\varepsilon}}(f_0), P^n(\lambda)\mathcal{F}_{T_{A,\varepsilon}}(\chi)\rangle &= \langle \textstyle\sum_{0\leq j\leq N_p}\left(c(\lambda_p, j)\delta_{\lambda_p}^{(j)}\right), P^n(\lambda)\mathcal{F}_{T_{A,\varepsilon}}(\chi)\rangle \\ &= c(\lambda_p, N_p)n^{N_p}P^{n-N_p}(\lambda_p)(P'(\lambda_p))^{N_p} + ...\end{aligned}$$

plus lower order terms in n. Since $|\langle f_n, \chi\rangle| \leq CM_nR^n$ for a constant $C > 0$, by (65), we have $c(\lambda_p, N_p) = 0$ for $N_p \geq m$ by(66).

If we assume that $m = 1$, then $N_p = 0$ and condition (66) implies that the condition (39) is satisfied. Thus from the above, Eq. (64) becomes

$$f_0 = \sum_{\lambda_p \in S_{R,\varepsilon}} f_{\lambda_p}, \quad \text{where } f_{\lambda_p} = c(\lambda_p)\Phi_{A,\varepsilon}(\lambda_p, .)$$

for a constant $c(\lambda_p) \in \mathbb{C}$.

We want to prove 2) (ii). Indeed, as in the above and from the assumptions on $\{f_j\}_{-\infty}^{\infty}$ we prove that

$$(P(-iT_{A,\varepsilon}) - P(\lambda_0))^{N+1} f_0 = 0. \tag{67}$$

This equation implies

$$\mathrm{span}\Big\{f_0, f_1, f_2, ...\Big\} = \mathrm{span}\Big\{f_0, P(-iT_{A,\varepsilon})f_0, ..., P^N(-iT_{A,\varepsilon})f_0\Big\}.$$

We shall now show that we can take $N = 0$ in (67).
If not then $(P(-iT_{A,\varepsilon}) - P(\lambda_0))f_0 \neq 0$. Let p be the largest positive integer so that $(P(-iT_{A,\varepsilon}) - P(\lambda_0))^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := (P(-iT_{A,\varepsilon}) - P(\lambda_0))^{p-1} f_0 \in \mathrm{span}\Big\{f_0, f_1, ..., f_N\Big\}$$

will satisfy

$$(P(-iT_{A,\varepsilon}) - P(\lambda_0))^2 f = 0 \quad \text{and} \quad (P(-iT_{A,\varepsilon}) - P(\lambda_0))f \neq 0. \tag{68}$$

Write

$$f = a_0 f_0 + ... + a_N f_N,$$

for constants $a_0, ..., a_N$. Then

$$P^j(-iT_{A,\varepsilon})f = a_0 f_j + ... + a_N f_{N+j}.$$

If we put

$$C_j := |a_0| R^0 M_j + ... + |a_N| R^N M_{j+N},$$

then by (65) we obtain

$$|P^j(-iT_{A,\varepsilon})f(x)| \leq C_j R^j (1 + |x|)^a e^{-\varrho|x|}. \tag{69}$$

By (66) C_j satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{C_j}{j} = 0. \tag{70}$$

An induction using (68) implies for $j \geq 2$ that

$$P^j(-iT_{A,\varepsilon})f = jP(\lambda_0)^{j-1}P(-iT_{A,\varepsilon})f - (j-1)P(\lambda_0)^j f = jP(\lambda_0)^{j-1}(P(-iT_{A,\varepsilon}) - P(\lambda_0))f + P(\lambda_0)^j f.$$

Thus

$$|(P(-iT_{A,\varepsilon}) - P(\lambda_0))f(x)| \leq \frac{1}{jR^{j-1}}|P^j(-iT_{A,\varepsilon})f(x)| + \frac{R|f(x)|}{j} \leq \frac{C_j R}{j}(1+|x|)^a e^{-\varrho|x|} + \frac{R|f(x)|}{j}.$$

Letting $j \to \infty$ and using (70) implies $(P(-iT_{A,\varepsilon}) - P(\lambda_0))f = 0$. But this contradicts (68). Consequently, $N = 0$ in (67). This completes the proof.

We proceed as the above theorem, we use the same ideas and steps and the Corollary 4, we prove the following result

Theorem 11 *Let* $\varepsilon \in [-1,1]$. *Suppose* $P(\xi) = \sum_n a_n \xi^n$ *is a non-constant polynomial with complex coefficients. Let* $\{f_j\}_{-\infty}^{\infty}$ *be a sequence of complex-valued functions on* $\mathbb{R}$ *so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-iT_{A,\varepsilon}) f_j.$$

1) Let $a \geq 0$ *and let* $R > 0$. *Assume that for all* $\eta > 0$, *there exist constants* $N \in \mathbb{N}_0$ *and* $C > 0$, *such that*

$$\forall\, x \in \mathbb{R}, \quad |f_n(x)| \leq C R^n (1+\eta)^{|n|} (1+|x|)^N e^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|} \tag{71}$$

is satisfied for all $n \in \mathbb{Z}$. *Then*

$$f_0 = \sum_{\lambda \in S_{R,1}} \sum_{j=0}^{N} c(\lambda, j) \frac{d^j}{d\xi^j}_{|\xi=\lambda} \Phi_{A,\varepsilon}(\xi, .), \tag{72}$$

for constants $c(\lambda, j) \in \mathbb{C}$ *and* $N \in \mathbb{N}$.

2) Let $a \geq 0$ *and let* $R > 0$ *and assume that* $\{f_j\}_{-\infty}^{\infty}$ *satisfies*

$$|f_j(x)| \leq M_j R^j (1+|x|)^a e^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|}, \tag{73}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the subpotential growth condition*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j^m} = 0, \tag{74}$$

for some $m \geq 0$.

We have

(i) If $P'(\lambda_p) \neq 0$, *for all* $\lambda_p \in S_{R,1}$, *then* $N < m$ *in (72).*

In particular, if $m = 1$, *then*

$$f_0 = \sum_{\lambda_p \in S_{R,1}} f_{\lambda_p}, \quad \text{where } f_{\lambda_p} = c(\lambda_p)\Phi_{A,\varepsilon}(\lambda_p, .)$$

(ii) If $S_{R,1}$ *consists of one point* λ_0 *and* $m = 1$ *in (74), then*

$$P(-iT_{A,\varepsilon}) f_0 = P(\lambda_0) f_0.$$

Remark 6 *(i) I studied the analogue of the results presented in this paper in the cadre of the Dunkl transform, Jacobi-Dunkl transform and the Opdam-Cherednik transform.*

(ii) In a forthcoming paper, we study the characterisation for the spectrum of other generalized Fourier transforms via the generalized potential function.

(iii) The previous theorem is the analogue for the Theorems 1 and 6 of [2].

8 Open Problem

I conjecture that the condition $0 \notin supp\,\mathcal{F}_{T_{A,\varepsilon}}\Big(R_0^n f\Big)$ for all $n \in \mathbb{N}$ in the theorem 6 is not necessary.

Acknowledgements. The author is deeply indebted to the referees for providing constructive comments and help in improving the contents of this paper. Thanks also to Professors K. Trimèche and N-B. Andersen for their helps.

References

[1] **N.B. Andersen,** *On the range of the Chébli-Trimèche transform. Monatsh. Math.* **144***, (2005), 193-201.*

[2] **N.B. Andersen,** *Roe's theorem revisited,* Integral Transf. and Special Functions V. **26**, Issue 3, (2015), 165-172.

[3] **H.H. Bang,** *A property of infinitely differentiable functions,* Proc. Amer. Math. Soc. **108**, N^o.1, (1990), 73-76.

[4] **N. Barhoumi and M. Mili,** *On the range of the generalized Fourier transform associated with a Cherednick type operator on the real line,* Arab J. Math. Sci. (2013). doi:10.1016/j.ajmsc.2013.11.001.

[5] **S. Ben Said, A. Boussen and M. Sifi,** *On a family of differential-reflection operators: interwining operators, and Fourier transform of rapidly decreasing functions,* Arxiv: 1507.00936v1.

[6] **N. Ben Salem and A. Ould Ahmed Salem,** *Convolution structure associated with the Jacobi-Dunkl operator on* $\mathbb{R}$, Ramanujan J. 12 (2006), no. 3, 359-378.

[7] **J.J. Betancor, J.D. Betancor and J.M.R. Mendez,** *Paley-Wiener type theorems for Chébli-Trimèche transforms,* Publ. Math. Debrecen **60**, 3-4 (2002), 347-358.

[8] **G. Birkoff and S. MacLane,** *A Survey of Modern Algebra, MacMillan, New York, 1965.*

[9] **C. Chettaoui, Y. Othmani and K. Trimèche,** *On the range of the Dunkl transform on* $\mathbb{R}^d$*. Anal. and Appl. Vol.2, N^{o}3, (2004), 177-192.*

[10] **F. Chouchene, M. Mili and K. Trimèche,** *Positivity of the intertwining operator and harmonic analysis associated with the Jacobi-Dunkl operator on* $\mathbb{R}$, Anal. and Appl. Vol. **1** (2003), 387-412.

[11] **C. F. Dunkl,** Differential-difference operators associated with reflections groups, *Trans. Amer. Math. Soc.* **311** (1989), 167-183.

[12] **J.-P. Gabardo,** *Tempered distributions with spectral gaps, Math. Proc. Camb. Phil. Soc.* **106***, (1989), 143-162.*

[13] **L. Gallardo and K. Trimèche,** *Positivity of the Jacobi-Cherednik intertwining operator and its dual,* Adv. Pure Appl. Math. **1** (2010), no.2, 163-194.

[14] **R. Howard and M. Reese,** *Characterization of eigenfunctions by boundedness conditions,* Canad. Math. Bull. **35** (1992), 204-213.

[15] **H. Mejjaoli and K. Trimèche,** *Spectrum of functions for the Dunkl transform on* $\mathbb{R}^d$*. Fract. Calc. Appl. Anal. 10 (2007), no. 1, 19–38.*

[16] **H. Mejjaoli,** *Spectral theorems associated with the Dunkl type operator on the real line,* Int. J. Open Problems Complex Analysis, Vol. 7, No. 2, June (2015) , 17-42.

[17] **H. Mejjaoli,** *Paley-Wiener theorems of generalized Fourier transform associated with a Cherednik type operator on the real line,* Complex Analysis and Operator Theory, (2015), Doi: 10.1007/s11785-015-0456-9.

[18] **M.A. Mourou and K. Trimèche,** *Transmutation operators and Paley-Wiener associated with a singular differential-difference operator on the real line* , Analysis and Applications, Vol. 1, No. 1 (2003), 43-70.

[19] **M.A. Mourou,** *Transmutation operators and Paley-Wiener associated with a Cherednik type operator on the real line,* Anal. Appl. **8** (2010), 387-408.

[20] **Y. Othmani and K. Trimèche,** *Real Paley-Wiener theorems associated with the Weinstein operator,* Mediterranean Journal of Mathematics (2006), Volume 3, Issue 1, 105-118.

[21] **J. Roe,** *A characterization of the sine function,* Math. Proc. Comb. Phil. Soc. **87** (1980), 69-73.

[22] **W. Rudin,** *Functional Analysis,* McGraw-Hill Book Co., 1973.

[23] **K. Trimèche,** *Inversion of the J.L. Lions transmutation operators using generalized wavelets*, Applied and Computational Harmonic Analysis, 4 (1997), 97-112

[24] **K. Trimèche,** *The transmutation operators relating to a Dunkl Type operator on* $\mathbb{R}$ *and their positivity,* Mediterr. J. Math. Vol. 12, Issue 2, (2015), 349-369.

[25] **V.K. Tuan,** *Paley-Wiener theorems for a class of integral transforms,* J. Math. Anal. Appl. **266** (2002), 200-226.