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Convolution Properties for Some Subclasses of Meromorphic Bounded Functions of Complex Order

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Abstract

Making use of the operator $\mathcal{L}_{\upsilon,\alpha,\beta}$ for functions of the form $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1}$, which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$, we introduce two subclasses of meromorphic bounded functions of complex order and investigate convolution properties, coefficient estimates and containment properties for these subclasses.

Keywords and phrases: Univalent meromorphic functions, Hadamard product (or convolution), subordination, complex order.

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1 Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1}, \tag{1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^{k-1}, \tag{2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

We recall some definitions which will be used in our paper.
Definition 1.1 [9]. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let us consider the second order linear homogenous differential equation (see, Baricz [4, p. 7]):

$$z^2 w''(z) + \alpha z w'(z) + \left[\beta z^2 - \upsilon^2 + (1-\alpha)\right] w(z) = 0 \ (\alpha, \beta, \upsilon \in \mathbb{C}). \tag{4}$$

The function $w_{\upsilon,\alpha,\beta}(z)$, which is called the generalized Bessel function of the first kind of order υ where υ is an unrestricted (real or complex) number, is defined a particular solution of (4). The function $w_{\upsilon,\alpha,\beta}(z)$ has the representation

$$w_{\upsilon,\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{\Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)} \left(\frac{z}{2}\right)^{2k+\nu}.$$

Let us define

$$\begin{aligned} \mathcal{L}_{\upsilon,\alpha,\beta}(z) &= \frac{2^\upsilon\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{z^{\upsilon/2+1}} w_{\upsilon,\alpha,\beta}(z^{1/2}) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-\beta)^k\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)} z^{k-1}. \end{aligned}$$

The operator $\mathcal{L}_{\upsilon,\alpha,\beta}(z)$ is a modification of the of the operator introduced by Deniz [6] for analytic funtions.

By using the Hadamard product (or convolution), we define the operator $\mathcal{L}_{\upsilon,\alpha,\beta}$ as follows:

$$\begin{aligned}(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z) &= \mathcal{L}_{\upsilon,\alpha,\beta}(z) * f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-\beta)^k \ \Gamma\left(\upsilon + \frac{\alpha+1}{2}\right)}{4^k \ \Gamma(k+1) \ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)} a_k z^{k-1}. \end{aligned} \tag{5}$$

It is easy to verify that

$$z(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)'(z) = \left(\upsilon + \frac{\alpha+1}{2}\right)(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z) - \left(\upsilon + 1 + \frac{\alpha+1}{2}\right)(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)(z). \tag{6}$$

We note that: $(\mathcal{L}_{\upsilon,1,1}f)(z) = (\mathcal{L}_{\upsilon}f)(z)$ (see Aouf et al. [2]).

Definition 1.2 [1]. For $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, let $\mathcal{F}^*(b, M)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1) and satisfying the analytic criterion

$$-\frac{zf'(z)}{f(z)} \prec \frac{1 + [b(1+m) - m]\, z}{1 - mz} \quad \left(M \geq 1;\ m = 1 - \frac{1}{M};\ z \in \mathbb{U}^*\right), \tag{7}$$

or, equivalently,

$$\left|\frac{b - 1 - \frac{zf'(z)}{f(z)}}{b} - M\right| < M \quad \left(M \geq 1;\ m = 1 - \frac{1}{M};\ z \in \mathbb{U}^*\right). \tag{8}$$

Also, let $\mathcal{G}^*(b, M)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1) and satisfying the analytic criterion:

$$-\frac{zf''(z)}{f'(z)} \prec 2 + \frac{b(1+m)z}{1 - mz} \quad \left(M \geq 1;\ m = 1 - \frac{1}{M};\ z \in \mathbb{U}^*\right), \tag{9}$$

or, equivalently,

$$\left|\frac{b - 2 - \frac{zf''(z)}{f'(z)}}{b} - M\right| < M \quad \left(M \geq 1;\ m = 1 - \frac{1}{M};\ z \in \mathbb{U}^*\right). \tag{10}$$

It is easy to verify from (7) and (9) that,

$$f(z) \in \mathcal{G}^*(b, M) \Leftrightarrow -zf'(z) \in \mathcal{F}^*(b, M). \tag{11}$$

We note that:
(i) $\mathcal{F}^*(b,\infty) = \mathcal{F}^*(b)$ $(b \in \mathbb{C}^*)$ and $\mathcal{G}^*(b,\infty) = \mathcal{G}^*(b)$ $(b \in \mathbb{C}^*)$ (see Aouf [1]) where $\mathcal{F}^*(b)$ and $\mathcal{G}^*(b)$ are the classes of meromorphic starlike and convex functions of complex order b;

(ii) $\mathcal{F}^*(1-\alpha, M) = \mathcal{F}^*_M(\alpha)$ $(0 \le \alpha < 1)$ (see Kaczmarski [8]) and $\mathcal{G}^*(1-\alpha, M) = \mathcal{G}^*_M(\alpha)$ $(0 \le \alpha < 1)$ (see Aouf [1]) where $\mathcal{F}^*_M(\alpha)$ and $\mathcal{G}^*_M(\alpha)$ are the classes of meromorphic bounded starlike and convex functions of order α;

(iii) $\mathcal{F}^*(1,\infty) = \mathcal{F}^*(1)$ (see Clunie [5]) and $\mathcal{G}^*(1,\infty) = \mathcal{G}^*(1)$ (see Aouf [1]) where $\mathcal{F}^*(1)$ and $\mathcal{G}^*(1)$ are the classes of meromorphic starlike and convex functions;

(iv) $\mathcal{F}^*(1-\alpha,\infty) = \mathcal{F}^*(1-\alpha)$ $(0 \le \alpha < 1)$ (see Kaczmarski [8] and Pommeremke [10]) and $\mathcal{G}^*(1-\alpha,\infty) = \mathcal{G}^*(1-\alpha)$ $(0 \le \alpha < 1)$ (see Aouf [1]) where $\mathcal{F}^*(1-\alpha)$ and $\mathcal{G}^*(1-\alpha)$ are the classes of meromorphic starlike and convex functions of order α;

(v) $\mathcal{F}^*((1-\alpha)e^{-i\beta}\cos\beta,\infty) = \mathcal{F}^*(\alpha,\beta)$ $(0 \le \alpha < 1,\ |\beta| < \frac{\pi}{2})$ (see Kaczmarski [8]) and $\mathcal{G}^*((1-\alpha)e^{-i\beta}\cos\beta,\infty) = \mathcal{G}^*(\alpha,\beta)$ $(0 \le \alpha < 1,\ |\beta| < \frac{\pi}{2})$ (see Aouf [1]) where $\mathcal{F}^*(\alpha,\beta)$ and $\mathcal{G}^*(\alpha,\beta)$ are the classes of meromorphic $\beta-$spirallike and $\beta-$Robertson functions of order α.

Definition 1.3. For $M \ge 1,\ b \in \mathbb{C}^*$ and $\upsilon,\ \alpha,\ \beta \in \mathbb{C}$, let

$$\mathcal{F}^*_{\upsilon,\alpha,\beta}(b,M) = \left\{ f(z) \in \Sigma : (\mathcal{L}_{\upsilon,\alpha,\beta} f)(z) \in \mathcal{F}^*(b,M) \right\}, \tag{12}$$

and

$$\mathcal{G}^*_{\upsilon,\alpha,\beta}(b,M) = \left\{ f(z) \in \Sigma : (\mathcal{L}_{\upsilon,\alpha,\beta} f)(z) \in \mathcal{G}^*(b,M) \right\}. \tag{13}$$

It is easy to show that

$$f(z) \in \mathcal{G}^*_{\upsilon,\alpha,\beta}(b,M) \Leftrightarrow -zf'(z) \in \mathcal{F}^*_{\upsilon,\alpha,\beta}(b,M). \tag{14}$$

2 Main Result

Unless otherwise mentioned, we assume throughout this paper that $M \ge 1,\ b \in \mathbb{C}^*$ and $\upsilon,\ \alpha,\ \beta \in \mathbb{C}$.

Lemma 2.1 [3]. If $f(z) \in \Sigma$, then $f(z) \in \mathcal{F}^*(b,M)$ if and only if

$$z\left[f(z) * \frac{1+(C-1)z}{z(1-z)^2} \right] \ne 0 \ for\ z \in \mathbb{U}, \tag{15}$$

where $C = C_\theta = \frac{e^{-i\theta}-m}{b(1+m)},\ \theta \in [0, 2\pi)$.

Lemma 2.2 [3]. If $f(z) \in \Sigma$, then $f(z) \in \mathcal{G}^*(b, M)$ if and only if

$$z\left[f(z) * \frac{1-3z-2(C-1)z^2}{z(1-z)^3}\right] \neq 0\ for z \in \mathbb{U}, \tag{16}$$

where $C = C_\theta = \frac{e^{-i\theta}-m}{b(1+m)},\ \theta \in [0, 2\pi)$.

Lemma 2.3 [7]. Let $h(z)$ be convex (univalent) in $\mathbb{U}$ with $Re\,(\beta h(z) + \gamma) > 0$ for all $z \in \mathbb{U}$. If p is analytic in $\mathbb{U}$ with $p(0) = h(0)$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Theorem 2.1. If $f(z) \in \Sigma$, then $f(z) \in \mathcal{F}^*_{\upsilon,\alpha,\beta}(b, M)$ if and only if

$$1 + \sum_{k=1}^{\infty} \frac{\left[k(e^{-i\theta}-m) + b(1+m)\right]}{b(1+m)} \frac{(-\beta)^k\ \Gamma\left(\upsilon + \frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)} a_k z^k \neq 0, \tag{17}$$

for all $\theta \in [0, 2\pi)$.

Proof. If $f(z) \in \Sigma$, from Lemma 2.1, we have $f(z) \in \mathcal{F}^*_{\upsilon,\alpha,\beta}(b, M)$ if and only if

$$z\left[(\mathcal{L}_{\upsilon,\alpha,\beta} f)(z) * \frac{1+(C-1)z}{z(1-z)^2}\right] \neq 0\ for\ z \in \mathbb{U}, \tag{18}$$

where $C = C_\theta = \frac{e^{-i\theta}-m}{b(1+m)},\ \theta \in [0, 2\pi)$. Since

$$\frac{1+(C-1)z}{z(1-z)^2} = \frac{1}{z} + \sum_{k=1}^{\infty}(kC+1)z^{k-1}.$$

It is easy to show that (18) holds if and only if (17) holds. This completes the proof of Theorem 2.1.

Theorem 2.2. If $f(z) \in \Sigma$, then $f(z) \in \mathcal{G}^*_{\upsilon,\alpha,\beta}(b, M)$ if and only if

$$1 - \sum_{k=1}^{\infty} \frac{(k-1)\left[k(e^{-i\theta}-m) + b(1+m)\right]}{b(1+m)} \frac{(-\beta)^k\ \Gamma\left(\upsilon + \frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)} a_k z^k \neq 0, \tag{19}$$

for all $\theta \in [0, 2\pi)$.

Proof. If $f(z) \in \Sigma$, from Lemma 2.2, we have $f(z) \in \mathcal{G}^*_{\upsilon,\alpha,\beta}(b, M)$ if and only if

$$z\left[(\mathcal{L}_{\upsilon,\alpha,\beta} f)(z) * \frac{1-3z-2(C-1)z^2}{z(1-z)^3}\right] \neq 0\ for\ z \in \mathbb{U}, \tag{20}$$

where $C = C_\theta = \frac{e^{-i\theta}-m}{b(1+m)},\ \theta \in [0, 2\pi)$. Since

$$\frac{1-3z-2(C-1)z^2}{z(1-z)^3} = \frac{1}{z} - \sum_{k=1}^{\infty}(k-1)(kC+1)z^{k-1}.$$

It is easy to show that (20) holds if and only if (19) holds. This completes the proof of Theorem 2.2.

Unless otherwise mentioned, we assume throughout the remainder of this section that $\upsilon > -\frac{\alpha+1}{2}$ and $\alpha,\ \beta > 0$.

Theorem 2.3. If $f(z) \in \Sigma$ satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{(k+|b|)\left(|\beta|\right)^k\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)}\left|a_k\right| < |b|, \tag{21}$$

then $f(z) \in \mathcal{F}^*_{\upsilon,\alpha,\beta}(b, M)$.
Proof. Since

$$\begin{aligned} &\left|1+\sum_{k=1}^{\infty}\frac{\left[k(e^{-i\theta}-m)+b(1+m)\right]}{b(1+m)}\frac{(-\beta)^k\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)}a_k z^k\right| \\ \geq\ & 1-\sum_{k=1}^{\infty}\left|\frac{\left[k(e^{-i\theta}-m)+b(1+m)\right]}{b(1+m)}\right|\frac{\left(|\beta|\right)^k\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)}\left|a_k\right|\left|z^k\right| \\ \geq\ & 1-\sum_{k=1}^{\infty}\frac{(k+|b|)\left(|\beta|\right)^k\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^k\ |b|\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)}\left|a_k\right| > 0, \end{aligned}$$

which implies that inequality (21). Thus the proof of Theorem 2.3 is completed.

Using similar arguments to those in the proof of Theorem 2.3, we obtain the following theorem.
Theorem 2.4. If $f(z) \in \Sigma$ satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{(k-1)(k+|b|)\left(|\beta|\right)^k\ \Gamma\left(\upsilon+\frac{\alpha+1}{2}\right)}{4^k\ \Gamma(k+1)\ \Gamma\left(k+\upsilon+\frac{\alpha+1}{2}\right)}\left|a_k\right| < |b|, \tag{22}$$

then $f(z) \in \mathcal{G}^*_{\upsilon,\alpha,\beta}(b, M)$.

Theorem 2.5. If

$$\frac{\cos\left(\theta+\arg b\right)-m\cos\left(\arg b\right)}{1+m^2-2m\cos\theta} < \frac{1}{|b|\,(m+1)}\left(\upsilon+\frac{\alpha+1}{2}\right), \tag{23}$$

then $\mathcal{F}^*_{\upsilon,\alpha,\beta}(b,M) \subset \mathcal{F}^*_{\upsilon+1,\alpha,\beta}(b,M)$ with $(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)(z) \neq 0 \ \ (z \in \mathbb{U}^*)$.
Proof. Let $f(z) \in \mathcal{F}^*_{\upsilon,\alpha,\beta}(b,M)$ and

$$p(z) = -\frac{z(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)'(z)}{(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)(z)}, \tag{24}$$

we see that $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (6) in (24), we have

$$\frac{(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z)}{(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)(z)} = -\frac{1}{\left(\upsilon + \frac{\alpha+1}{2}\right)}p(z) + \left(\frac{\upsilon + 1 + \frac{\alpha+1}{2}}{\upsilon + \frac{\alpha+1}{2}}\right), \tag{25}$$

Differentiating (25) logarithmically and using (24), we have

$$-\frac{z(\mathcal{L}_{\upsilon,\alpha,\beta}f)'(z)}{(\mathcal{L}_{\upsilon,\alpha,\beta}f)(z)} = p(z) + \frac{zp'(z)}{-p(z) + \upsilon + 1 + \frac{\alpha+1}{2}} \prec \frac{1 + [b(1+m) - m]\, z}{1 - mz} = h(z). \tag{26}$$

A simple computation shows that $\operatorname{Re}\left(-h(z) + \upsilon + 1 + \frac{\alpha+1}{2}\right) > 0$ is equivalent to

$$Re\left(\frac{bz}{1-mz}\right) < \frac{1}{m+1}\left(\upsilon + \frac{\alpha+1}{2}\right),$$

which implies to (23). Since the function $h(z)$ is convex, then by using Lemma 2.3, we have $p(z) \prec h(z)$. This completes the proof of Theorem 2.5.

Using the same arguments as in the proof of Theorem 2.5, we obtain the following theorem.
Theorem 2.6. If

$$\frac{\cos(\theta + \arg b) - m\cos(\arg b)}{1 + m^2 - 2m\cos\theta} < \frac{1}{|b|\,(m+1)}\left(\upsilon + \frac{\alpha+1}{2}\right), \tag{27}$$

then $\mathcal{G}^*_{\upsilon,\alpha,\beta}(b,M) \subset \mathcal{G}^*_{\upsilon+1,\alpha,\beta}(b,M)$ with $(\mathcal{L}_{\upsilon+1,\alpha,\beta}f)'(z) \neq 0 \ \ (z \in \mathbb{U}^*)$.

3 Open Problem

The authors suggest to study the idea of this paper on the class Σ_p of $p-$valent meromorphic functions, where Σ_p denotes the class of analytic and $p-$valent meromorphic functions in the punctured disc $\mathbb{U}^*$ of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \ (p \in \mathbb{N}). \tag{28}$$

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