Int. J. Open Problems Complex Analysis, Vol. 8, No. 2, July 2016
ISSN 2074-2827;
www.i-csrs.org

Some Properties of a Subclass of Harmonic Univalent Functions Defined By Salagean Operator

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Received 1 February 2016; Accepted 12 May 2016

Abstract

In this paper, we introduced the classes $M_H^l(m, n, \phi, \psi; \gamma)$ ***and*** $V_H^l(m, n, \phi, \psi; \gamma)$, $l = \{0, 1\}$ ***consisting of harmonic univalent functions*** $f = h + \overline{g}$. ***We studied the coefficients estimate, distortion theorem, extreme points, convex combination and family of integral operators. Also, we established some results concerning the convolution. In proving our results certain conditions on the coefficients of*** ϕ ***and*** ψ ***are considered which lead various well-known results proved earlier.***

Keywords: *Harmonic functions, univalent functions, Salagean operator, extreme points, convolution.*
2000 Mathematical Subject Classification: 30C45, 30C50.

1 Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$, that is u, v satisfy, respectively, the Laplace equations. It follows that every analytic function is a complex-valued harmonic function. In any simply connected domain $\mathbb{D}$, we can write $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathbb{D}$ (see[5]).

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z'(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

We, note that S_H reduces to the class of normalized analytic functions if the co-analytic part of f is identically zero; that is $g = 0$, then

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

In 1984 Clunie and Sheil-Small [5] investigated the class S_H as well as its geometric subclasses and some coefficient bounds for functions in S_H were obtained. Also, various subclasses of S_H were investigated by several authors (see [1], [2], [3], [7] and [13]).

Definition 1.1 *A function $f(z) \in S_H$ is said to be in the class of harmonic starlike functions of order α denote by $S_H^*(\alpha)$ if it satisfies the following condition*

$$\frac{\partial}{\partial\theta}\left(arg(f(re^{i\theta}))\right) > \alpha$$

which is equivalent to the condition

$$Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h'(z) + \overline{g'(z)}}\right\} \geq \alpha, \quad (0 \leq \alpha < 1; z \in \mathbb{U}).$$

Definition 1.2 *A function $f(z) \in S_H$ is said to be in the class of harmonic convex functions of order α denote by $K_H(\alpha)$ if it satisfies the following condition*

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right) > \alpha$$

which is equivalent to the condition

$$Re\left\{1 + \frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right\} \geq \alpha, \quad (0 \leq \alpha < 1; z \in \mathbb{U}).$$

The classes $S_H^*(\alpha)$ and $K_H(\alpha)$ were introduced and studied by Jahangiri [9]. From the above conditions, we see that

$$f(z) \in K_H(\alpha) \Longleftrightarrow zf'(z) \in S_H^*(\alpha).$$

A necessary condition for a function $f(z) = h + \overline{g}$, where $h(z)$ and $g(z)$ are of the form (1) belong to the classes $S_H^*(\alpha)$ and $K_H(\alpha)$

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=1}^{\infty}(k+\alpha)|b_k| \leq 1-\alpha \Rightarrow f(z) \in S_H^*(\alpha),$$

and

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| + \sum_{k=1}^{\infty} k(k+\alpha)|b_k| \leq 1-\alpha \Rightarrow f(z) \in K_H(\alpha).$$

For $1 < \gamma \leq 4/3$ and $z \in \mathbb{U}$, Porwal and Dixit [14] introduced and studied the classes $M_H(\gamma)$ of harmonic functions $f = h + \overline{g}$ of the form (1) satisfying the condition

$$\frac{\partial}{\partial\theta}\left(arg f(z)\right) = Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \leq \gamma, \qquad (z \in \mathbb{U}),$$

and $L_H(\gamma)$ the class of harmonic functions of the form (1) satisfying the condition

$$\frac{\partial}{\partial\theta}\left\{arg\left(\frac{\partial}{\partial\theta}f(z)\right)\right\} = Re\left\{1 + \frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right\} \leq \gamma, \quad (z \in \mathbb{U}).$$

Further, let V_H and U_H be the subclasses of S_H consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k, \tag{3}$$

and

$$f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k, \tag{4}$$

respectively.

Let $V_H(\gamma) \equiv M_H(\gamma) \cap V_H$, and $U_H(\gamma) \equiv L_H(\gamma) \cap U_H$. A necessary and sufficient condition for a function $f(z)$ is of the form (3) belongs to the classes $V_H(\gamma)$ and

$$\sum_{k=2}^{\infty}(k-\gamma)|a_k| + \sum_{k=1}^{\infty}(k+\gamma)|b_k| \leq \gamma - 1, \tag{5}$$

where $1 < \gamma \leq \frac{4}{3}$ [14].

A necessary and sufficient condition for a function $f(z)$ is of the form (4) belongs to the classes $U_H(\gamma)$ and

$$\sum_{k=2}^{\infty} k(k-\gamma)|a_k| + \sum_{k=1}^{\infty} k(k+\gamma)|b_k| \leq \gamma - 1, \tag{6}$$

where $1 < \gamma \leq \frac{4}{3}$ [14]. We note that for $g = 0$ the classes $M_H(\gamma) \equiv M(\gamma)$, $L_H(\gamma) \equiv L(\gamma)$, $V_H(\gamma) \equiv V(\gamma)$ and $U_H(\gamma) \equiv U(\gamma)$ were studied in [18].

For $f = h + \overline{g}$ with h and g are of the form (1), the modified differential Salagean operator D^n for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is given by

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \tag{7}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \qquad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

Sharma et al. [15] define a generalized class $S_H^l(m, n, \phi, \psi; \alpha)$ of functions $f = h + \overline{g} \in S_H$ satisfying for $l \in \{0, 1\}$, the condition

$$\Re \left\{ \frac{D^m h(z) * \phi(z) + (-1)^{m+l} \overline{D^m g(z) * \psi(z)}}{D^n h(z) + (-1)^n \overline{D^n g(z)}} \right\} > \alpha,$$

where $m, n \in \mathbb{N}_0$, $m \geq n$, $0 \leq \alpha < 1$, $\phi(z) = z + \sum_{k=2}^{\infty} \beta_k z^k$ and $\psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$ are analytic in $\mathbb{U}$ with the conditions $\beta_k, \mu_k \geq 1$.

Further denote by $TS_H^l(m, n, \phi, \psi; \alpha)$, a subclass of $S_H^l(m, n, \phi, \psi; \alpha)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = (-1)^{m+l-1} \sum_{k=1}^{\infty} |b_k| z^k, \ \ |b_1| < 1. \tag{8}$$

It is interesting to note that by specializing the parameters m, n, l and the functions ϕ and ψ we obtain the following known subclasses of S_H studied earlier.

(i) $S_H^0(m, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = S_H(m, n; \alpha)$ and $\mathrm{T}S_H^0(m, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = \mathrm{T}S_H(m, n; \alpha)$ studied by [19].

(ii) $S_H^0(n+1, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = S_H(n; \alpha)$ and $\mathrm{T}S_H^0(n+1, n, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = \mathrm{T}S_H(n; \alpha)$ studied by [10].

(iii) $S_H^0(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = S_H^*(\alpha)$ and $\mathrm{T}S_H^0(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = \mathrm{T}S_H^*(\alpha)$ studied by [9] .

(iv) $S^0_H(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = K_H(\alpha)$ and $\mathrm{T}S^0_H(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha) = \mathrm{T}K_H(\alpha)$ studied by [9].

(v) $S^1_H(0, 0, \phi, \psi; \alpha) = S_H(\phi, \psi; \alpha)$ and $\mathrm{T}S^0_H(0, 0, \phi, \psi; \alpha) = \mathrm{T}S^1_H(0, 0, \phi, \psi; \alpha) = \mathrm{T}S_H(\phi, \psi, \alpha)$ studied by [8].

(vi) $S^0_H(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; 0) = K_H$, $\mathrm{T}S^0_H(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; 0) = \mathrm{T}K_H$, $S^0_H(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; 0) = S^*_H$, and $\mathrm{T}S^0_H(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; 0) = \mathrm{T}S^*_H$ studied by [16] and [17].

Using the operator D^n, we define a generalized class $M^l_H(m, n, \phi, \psi; \gamma)$ of functions $f = h + \overline{g} \in S_H$ satisfying for $l \in \{0, 1\}$, the condition

$$\Re\left\{\frac{D^m h(z) * \phi(z) + (-1)^{m+l}\overline{D^m g(z) * \psi(z)}}{D^n h(z) + (-1)^n \overline{D^n g(z)}}\right\} \leq \gamma, \tag{9}$$

where m, $n \in \mathbb{N}_0$, $m \geq n$, $1 < \gamma \leq \frac{4}{3}$, and $\phi(z) = z + \sum_{k=2}^{\infty} \beta_k z^k$ and $\psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$ are analytic in $\mathbb{U}$ with the conditions $\beta_k, \mu_k \geq 1$.

Further denote by $V^l_H(m, n, \phi, \psi; \gamma)$, a subclass of $M^l_H(m, n, \phi, \psi; \gamma)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = (-1)^{m+l} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{10}$$

It is interesting to note that by specializing the parameters m, n, l and the functions ϕ and ψ we obtain the following known subclasses of S_H studied earlier.

(i) $V^0_H(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; \gamma) = V_H(\gamma)$ and $M^0_H(1, 0, \frac{z}{1-z}, \frac{z}{1-z}; \gamma) = M_H(\gamma)$ studied by [14].

(ii) $V^0_H(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; \gamma) = V_H(\gamma)$ and $M^0_H(2, 1, \frac{z}{1-z}, \frac{z}{1-z}; \gamma) = M_H(\gamma)$ studied by [14].

In the present paper, we prove a number of coefficients estimate, distortion theorem, extreme points, a family of integral operators, convolution properties and convex combination for functions in $M^l_H(m, n, \phi, \psi; \gamma)$ and $V^l_H(m, n, \phi, \psi; \gamma)$ under certain conditions on the coefficients of ϕ and ψ.

2 Coefficients Estimate

In this section, we studies a sufficient coefficient condition for functions in $M^l_H(m, n, \phi, \psi; \gamma)$ under certain conditions on the coefficients of ϕ and ψ.

Theorem 2.1 *Let $f(z) = h(z) + \overline{g(z)}$, where h and g are given by (1) and satisfy the condition*

$$\sum_{k=2}^{\infty}(\beta_k k^m - \gamma k^n)|a_k| + \sum_{k=1}^{\infty}(\mu_k k^m - (-1)^{m+l-n}\gamma k^n)|b_k| \leq \gamma - 1, \tag{11}$$

where

$$k \leq \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}, \qquad k \leq \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} \qquad \textit{for } k \geq 2. \tag{12}$$

$l \in \{0,1\}$, $m \in \mathbb{N}, n \in \mathbb{N}_0$, $m > n$, $\beta_k, \mu_k \geq 1$, $k \geq 1, 1 \leq \gamma < 4/3$. *Then $f(z)$ is sense-preserving, harmonic univalent in $\mathbb{U}$ and $f(z) \in M_H^l(m,n,\phi,\psi;\gamma)$.*

Proof. To show that f is sense-preserving in $\mathbb{U}$.

$$\begin{aligned}
|h'(z)| &= \left|1 + \sum_{k=2}^{\infty} k|a_k|z^{k-1}\right| \geq 1 - \sum_{k=2}^{\infty} k|a_k|r^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\
&> 1 - \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k| > \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k| \\
&> \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k|r^{k-1} \geq |g'(z)|.
\end{aligned}$$

To show that f is univalent in $\mathbb{U}$, suppose $z_1, z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$, then we have

$$\begin{aligned}
\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &= 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| \\
&= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_2^k - z_1^k)}{(z_2 - z_1) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right| \\
&> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \\
&> 1 - \frac{\sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k|} > 0.
\end{aligned}$$

Now, we show that $f \in M_H^l(\phi,\psi,m,n;\gamma)$. We only need to show that if (11) holds then the condition (9) is satisfied, then we want to proof that

$$\left|\frac{\frac{D^m h(z)*\phi(z) + (-1)^{m+l}\overline{D^m g(z)*\psi(z)}}{D^n h(z) + (-1)^n D^n g(z)} - 1}{\frac{D^m h(z)*\phi(z) + (-1)^{m+l} D^m g(z)*\psi(z)}{D^n h(z) + (-1)^n D^n g(z)} - (2\gamma - 1)}\right| < 1, \qquad z \in U.$$

$$\frac{D^m h(z) * \phi(z) + \overline{(-1)^{m+l} D^m g(z) * \psi(z)}}{D^n h(z) + \overline{(-1)^n D^n g(z)}} - 1$$
$$= \frac{\sum_{k=2}^{\infty}(\beta_k k^m - k^n) a_k z^k + \sum_{k=1}^{\infty}(\mu_k k^m - (-1)^{m+l-n} k^n) b_k z^k}{z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n b_k z^k},$$

$$\frac{D^m h(z) * \phi(z) + \overline{(-1)^{m+l} D^m g(z) * \psi(z)}}{D^n h(z) + \overline{(-1)^n D^n g(z)}} - (2\gamma - 1)$$
$$= \frac{z + \sum_{k=2}^{\infty} \beta_k k^m a_k z^k + (-1)^{m+l} \sum_{k=1}^{\infty} \mu_k k^m b_k z^k}{z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n b_k z^k} - (2\gamma - 1)$$
$$= \frac{(2-2\gamma)z + \sum_{k=2}^{\infty}(\beta_k k^m - (2\gamma-1)k^n) a_k z^k + \sum_{k=1}^{\infty}((-1)^{m+l}\mu_k k^m - (-1)^n (2\gamma-1) k^n) b_k z^k}{z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n b_k z^k},$$

we have

$$\left| \frac{\frac{D^m h(z)*\phi(z) + (-1)^{m+l}\overline{D^m g(z)*\psi(z)}}{D^n h(z) + (-1)^n D^n g(z)} - 1}{\frac{D^m h(z)*\phi(z) + (-1)^{m+l} D^m g(z)*\psi(z)}{D^n h(z) + (-1)^n D^n g(z)} - (2\gamma - 1)} \right|$$
$$\leq \frac{\sum_{k=2}^{\infty} [\beta_k k^m - k^n] |a_k| + \sum_{k=1}^{\infty} \left|(-1)^{m+l}\mu_k k^m - (-1)^n k^n\right| |b_k|}{2(\gamma-1) - \sum_{k=2}^{\infty} [\beta_k k^m - (2\gamma-1)k^n] |a_k| - \sum_{k=1}^{\infty} |(-1)^{m+l}\mu_k k^m - (-1)^n (2\gamma-1) k^n| |b_k|}.$$

The last expression is bounded above by 1, if

$$\sum_{k=2}^{\infty}[\beta_k k^m - k^n]|a_k| + \sum_{k=1}^{\infty} |(-1)^{m+l}\mu_k k^m - (-1)^n k^n||b_k|$$
$$\leq 2(\gamma-1) - \sum_{k=2}^{\infty}[\beta_k k^m - (2\gamma-1)k^n]|a_k| - \sum_{k=1}^{\infty} |(-1)^{m+l}\mu_k k^m - (-1)^n(2\gamma-1)k^n||b_k|,$$

which is equivalent to

$$\sum_{k=2}^{\infty}[\beta_k k^m - \gamma k^n]|a_k| + \sum_{k=1}^{\infty}[\mu_k k^m - (-1)^{m+l-n}\gamma k^n]|b_k| \leq \gamma - 1. \tag{13}$$

But (13) is true by hypothesis and then Theorem 2.1 is proved.

Sharpness of the coefficient inequality (11) can be seen by the function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{\gamma - 1}{\beta_k k^m - \gamma k^n} x_k z^k + \sum_{k=1}^{\infty} \frac{\gamma - 1}{\mu_k k^m - (-1)^{m+l-n}\gamma k^n} \overline{y_k z^k},$$

where $l \in \{0,1\}$, $m \in \mathbb{N}, n \in \mathbb{N}_0$, $m \geq n, \beta_k, \mu_k \geq 1$, $k \geq 1, 1 \leq \gamma < 4/3$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$.

In the following theorem, it is shown that the condition (11) is also necessary for function $f(z)$ given by (10) belong to the class $V_H^l(m,n,\phi,\psi;\gamma)$.

Theorem 2.2 *let $f(z)$ be given by (10), then $f(z) \in V_H^l(m,n,\phi,\psi;\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}(\beta_k k^m - \gamma k^n)|a_k| + \sum_{k=1}^{\infty}(\mu_k k^m - (-1)^{m+l-n}\gamma k^n)|b_k| \leq \gamma - 1. \tag{14}$$

Proof. Since $V_H^l(m,n,\phi,\psi;\gamma) \subseteq M_H^l(m,n,\phi,\psi;\gamma)$, we only need to prove the only if part of the theorem. To prove the only if, let $f(z) \in V_H^l(m,n,\phi,\psi;\gamma)$ then we have

$$\Re\left\{\frac{D^m h(z) * \phi(z) + (-1)^{m+l}\overline{D^m g(z) * \psi(z)}}{D^n h(z) + (-1)^n \overline{D^n g(z)}}\right\} \leq \gamma.$$

$$\Re\left\{\frac{(z + \sum_{k=2}^{\infty} k^m a_k z^k) * (z + \sum_{k=2}^{\infty} \beta_k z^k) + (-1)^{2m+2l}\overline{(\sum_{k=1}^{\infty} k^m |b_k| z^k) * (z + \sum_{k=2}^{\infty} \mu_k z^k)}}{z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^{n+m+l}\overline{\sum_{k=1}^{\infty} k^n |b_k| z^k}}\right\} \leq \gamma.$$

This is equivalent to

$$\Re\left\{\frac{(\gamma-1)z - \sum_{k=2}^{\infty}[\beta_k k^m - \gamma k^n]|a_k| z^k - (-1)^{2m+2l}\sum_{k=1}^{\infty}[\mu_k k^m - (-1)^{m+l-n}\gamma k^n]|b_k|\overline{z}^k}{z + \sum_{k=2}^{\infty} k^n |a_k| z^k + (-1)^{n+m+l}\sum_{k=1}^{\infty} k^n |b_k| \overline{z}^k}\right\} \geq 0.$$

The above condition must hold for all values of $z \in \mathbb{U}$, so that on taking $z = r < 1$, the above inequality reduces to

$$\frac{(\gamma-1) - \sum_{k=2}^{\infty}[\beta_k k^m - \gamma k^n]|a_k| r^{k-1} - \sum_{k=1}^{\infty}[\mu_k k^m - (-1)^{m+l-n}\gamma k^n]|b_k| r^{k-1}}{1 + \sum_{k=2}^{\infty} k^n |a_k| r^{k-1} + (-1)^{n+m+l}\sum_{k=1}^{\infty} k^n |b_k| r^{k-1}} \geq 0. \tag{15}$$

If the condition (11) does not hold then the numerator of (15) is negative for r and sufficintly close to 1. This contradicts the required condition for $f(z) \in V_H^l(m,n,\phi,\psi;\gamma)$. This complete the proof of Theorem 2.2.

Remark 2.3

(i) Putting $\phi = \psi = \frac{z}{1-z}$, $l = 0, m = 1, n = 0$ in Theorem 2.1, we obtain the result obtained in [14 Theorem 2.1.];

(ii) Putting $\phi = \psi = \frac{z}{1-z}$, $l = 0, m = 2, n = 1$ in Theorem 2.1, we obtain the result obtained in [14 Theorem 2.4.].

3 Distortion Theorem

Now, we give the distortion theorem for functions in $V_H^l(m,n,\phi,\psi;\gamma)$, which yield a covering result for $V_H^l(m,n,\phi,\psi;\gamma)$.

Theorem 3.1 *Let $f = h+\overline{g}$ with h and g are of the form (10) belongs to the class $V_H^l(m, n, \phi, \psi; \gamma)$ for functions ϕ and ψ with non-decreasing sequences $\{\beta_k\}$, $\{\mu_k\}$ satisfying $\beta_k, \mu_k \geq \beta_2$, $k \geq 2$, and $A_k \leq \beta_k k^m - \gamma k^n$, $B_k \leq \mu_k k^m - (-1)^{m+l-n}\gamma k^n$ for $k \geq 2$, $C = \min\{A_2, B_2\}$. Then for $|z| = r < 1$, we have*

$$|f(z)| \leq (1 + |b_1|)r + \frac{\gamma - 1}{C}\left[1 - \frac{1 - (-1)^{m+l-n}\gamma}{\gamma - 1}|b_1|\right] r^2, \tag{16}$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{\gamma - 1}{C}\left[1 - \frac{1 - (-1)^{m+l-n}\gamma}{\gamma - 1}|b_1|\right] r^2. \tag{17}$$

The equalities in (16) and (17) are attained for the functions $f(z)$ given by

$$f(z) = (1 + |b_1|)\overline{z} + \frac{\gamma - 1}{C}\left[1 - \frac{1 - (-1)^{m+l-n}\gamma}{\gamma - 1}|b_1|\right] \overline{z}^2,$$

and

$$f(z) = (1 - |b_1|)\overline{z} - \frac{\gamma - 1}{C}\left[1 - \frac{1 - (-1)^{m+l-n}\gamma}{\gamma - 1}|b_1|\right] \overline{z}^2.$$

where $|b_1| \leq \frac{\gamma-1}{1-(-1)^{m+l-n}\gamma}$.

Proof. Let $f \in V_H^l(m, n, \phi, \psi; \gamma)$ then on taking the absolute value of f, we get for $|z| = r < 1$

$$\begin{aligned}
|f(z)| &\leq |h(z)| + |\overline{g}(z)| \leq r + \sum_{k=2}^{\infty}|a_k|r^k + \sum_{k=1}^{\infty}|b_k|r^k, \qquad |b_1| < 1 \\
&= (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\
&\leq (1 + |b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k| + |b_k|) \\
&\leq (1 + |b_1|)r + \frac{r^2(\gamma - 1)}{C}\sum_{k=2}^{\infty}\frac{C}{(\gamma - 1)}|a_k| + \frac{C}{(\gamma - 1)}|b_k| \\
&\leq (1 + |b_1|)r + \frac{r^2(\gamma - 1)}{C}\sum_{k=2}^{\infty}\frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k| + \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k| \\
&\leq (1 + |b_1|)r + \frac{(\gamma - 1)}{C}(1 - \frac{\mu_1 - (-1)^{m+l-n}\gamma}{\gamma - 1}|b_1|)r^2 \\
&\leq (1 + |b_1|)r + \frac{\gamma - 1}{C}\left[1 - \frac{1 - (-1)^{m+l-n}\gamma}{\gamma - 1}|b_1|\right] r^2,
\end{aligned}$$

which proves the assertion (16) of Theorem 3.1. The proof of the assertion (17) is similar, thus, we omit it.

Corollary 3.2 *Let the function $f(z)$ given by (10) be in the class $V_H^l(m,n,\phi,\psi;\gamma)$, where $|b_1| \leq \frac{\gamma-1}{1-(-1)^{m+l-n}\gamma}$ and $A_k \leq \beta_k k^m - \gamma k^n$, $B_k \leq \mu_k k^m - (-1)^{m+l-n}\gamma k^n$ for $k \geq 2$, $C = \min\{A_2, B_2\}$. Then for $|z| = r < 1$, we have*

$$\left\{ w : |w| < \left(1 - \frac{\gamma - 1}{C}\right) + \left(\frac{1-(-1)^{m+l-n}\gamma}{C} - 1\right)|b_1| \right\} \subset f(\mathbb{U}).$$

4 Extreme Points

Theorem 4.1 *Let $f(z)$ be given by (10). Then $f(z) \in clcoV_H^l(m,n,\phi,\psi;\gamma)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)), \tag{18}$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_k(z) &= z + \frac{\gamma-1}{\beta_k k^m - \gamma k^n} z^k, \qquad (k \geq 2), \\ g_k(z) &= z + \frac{(-1)^{m+l}(\gamma-1)}{\mu_k k^m - (-1)^{m+l-n}\gamma k^n} \overline{z}^k, \qquad (k \geq 1), \end{aligned}$$

where $x_k \geq 0$, $y_k \geq 0$ and $\sum_{k=1}^{\infty}(x_k + y_k) = 1$. In particular, the extreme points of $V_H^l(m,n,\phi,\psi;\gamma)$ are $\{h_k\}$ and $\{g_k\}$, respectively.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)).$$

Then

$$f(z) = z + \sum_{k=2}^{\infty} \frac{\gamma-1}{\beta_k k^m - \gamma k^n} x_k z^k + \sum_{k=1}^{\infty} \frac{(-1)^{m+l}(\gamma-1) y_k \overline{z}^k}{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}$$

Since

$$\sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1} \frac{\gamma - 1}{\beta_k k^m - \gamma k^n} x_k + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} \frac{\gamma - 1}{\mu_k k^m - (-1)^{m+l-n}\gamma k^n} y_k$$
$$= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1.$$

Thus, $f(z) \in clcoV_H^l(m,n,\phi,\psi;\gamma)$.

Conversely, assume that $f(z) \in clcoV_H^l(m,n,\phi,\psi;\gamma)$. Set

$$x_k = \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k|, \qquad (0 \leq x_k \leq 1;\ k \geq 2),$$

and

$$y_k = \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k|, \qquad (0 \leq y_k \leq 1;\ k \geq 1),$$

where $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$. Therefore,

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_k| \overline{z}^k \\ &= z + \sum_{k=2}^{\infty} \frac{\gamma - 1}{\beta_k k^m - \gamma k^n} x_k z^k + \sum_{k=1}^{\infty} \frac{(-1)^{m+l}(\gamma - 1) y_k \overline{z}^k}{\mu_k k^m - (-1)^{m+l-n}\gamma k^n} \\ &= z + \sum_{k=2}^{\infty} (h_k(z) - z) x_k + \sum_{k=1}^{\infty} (g_k(z) - z) y_k \\ &= z\left(1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k\right) + \sum_{k=2}^{\infty} h_k(z) x_k + \sum_{k=1}^{\infty} g_k(z) y_k \\ &= \sum_{k=1}^{\infty} \left(h_k(z) x_k + g_k(z) y_k\right). \end{aligned}$$

This completes the proof of Theorem 4.1.

5 A family of integral operators

In this section, we examine a closure property of the class $V_H^l(m, n, \phi, \psi; \gamma)$ under the generalized Bernardi-Libera Livingston integral operator (see [4], [11] and [12]). Bernardi defined the integral operator J_c by

$$J_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (f \in V_H^l(m, n, \phi, \psi; \gamma),\ c > -1).$$

Now, we define the Bernardi integral operator $J_c(f)$ on the class S_H of harmonic univalent functions of the form (1) as follows:[6]

$$\begin{aligned} J_c(f) &= J_c(h) + \overline{J_c(g)} \\ &= z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n + \overline{\sum_{n=1}^{\infty} \frac{c+1}{c+n} b_n z^n}. \end{aligned}$$

Theorem 5.1 *Let the function $f(z)$ defined by (10) be in the class $V_H^l(m, n, \phi, \psi; \gamma)$ and c be a real number such that $c > -1$, then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt + (-1)^{m+l} \overline{\frac{c+1}{z^c} \int_0^z t^c g(t) dt}, \quad (c > -1), \tag{19}$$

also belongs to the class $V_H^l(m, n, \phi, \psi; \gamma)$.

Proof. Let the function $f(z)$ be defined by (10), then from the representation (19) of $F(z)$, it follows that

$$F(z) = z + \sum_{k=2}^{\infty} |\zeta_k| z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |\eta_k| \overline{z^k},$$

where

$$|\zeta_k| = \left(\frac{c+1}{c+k}\right)|a_k| \text{ and } |\eta_k| = \left(\frac{c+1}{c+k}\right)|b_k|.$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1} |\zeta_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} |\eta_k| \\
&= \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1} \left(\frac{c+1}{c+k}\right)|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} \left(\frac{c+1}{c+k}\right)|b_k| \\
&\leq \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} |b_k| \leq 1,
\end{aligned}$$

since $f(z) \in V_H^l(m, n, \phi, \psi; \gamma)$. Hence $F(z) \in V_H^l(m, n, \phi, \psi; \gamma)$. This completes the proof of Theorem 5.1.

6 convex combinations

The convex combination properties of the class $V_H^l(m, n, \phi, \psi; \gamma)$ is given in the following theorem.

Theorem 6.1 *The class $V_H^l(m, n, \phi, \psi; \gamma)$ is closed under convex combination.*

Proof. For $j = 1, 2, ...$, suppose that $f_j \in V_H^l(m, n, \phi, \psi; \gamma)$ where $f_j(z)$ is given by

$$f_j(z) = z + \sum_{k=2}^{\infty} |a_{j,k}| z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_{j,k}| z^k.$$

Then, by Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1} |a_{j,k}| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} |b_{j,k}| \leq 1.$$

For $\sum_{j=1}^{\infty} t_j = 1$, $0 \leq t_j \leq 1$, the convex combination of $f_j(z)$ may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z + \sum_{k=2}^{\infty}\sum_{j=1}^{\infty} t_j |a_{j,k}| z^k + (-1)^{m+l} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} t_j |b_{j,k}| z^k.$$

Now

$$\sum_{k=1}^{\infty}\frac{\beta_k k^m-\gamma k^n}{\gamma-1}\sum_{j=1}^{\infty}t_j|a_{j,k}|+\sum_{k=1}^{\infty}\frac{\mu_k k^m-(-1)^{m+l-n}\gamma k^n}{\gamma-1}\sum_{j=1}^{\infty}t_j|b_{j,k}|$$
$$=\sum_{j=1}^{\infty}t_j\left(\sum_{k=2}^{\infty}\frac{\beta_k k^m-\gamma k^n}{\gamma-1}|a_{j,k}|+\sum_{k=1}^{\infty}\frac{\mu_k k^m-(-1)^{m+l-n}\gamma k^n}{\gamma-1}|b_{j,k}|\right)$$
$$\leq 1,$$

we have $\sum_{j=1}^{\infty}t_jf_j(z)\in V_H^l(m,n,\phi,\psi;\gamma)$.

Theorem 6.2 *If $f\in V_H^l(m,n,\phi,\psi;\gamma)$ then, f is convex in the disc*

$$|z|\leq\min_k\left\{\frac{(\gamma-1)(1-|b_1|)}{k[\gamma-1-(1-(-1)^{m+l-n}\gamma)|b_1|]}\right\}^{\frac{1}{k-1}},\quad k=2,3,4,\ldots.$$

Proof. If $f\in V_H^l(m,n,\phi,\psi;\gamma)$ and let $0<r<1$ be fixed. Then $r^{-1}f(rz)\in V_H^l(m,n,\phi,\psi;\gamma)$ and we have

$$\begin{aligned}\sum_{k=2}^{\infty}k^2(|a_k|+|b_k|)r^{k-1} &= \sum_{k=2}^{\infty}k(|a_k|+|b_k|)(kr^{k-1})\\ &\leq \sum_{k=2}^{\infty}\left(\frac{\beta_k k^m-\gamma k^n}{\gamma-1}|a_k|+\frac{\mu_k k^m-(-1)^{m+l-n}\gamma k^n}{\gamma-1}|b_k|\right)kr^{k-1}\\ &\leq 1-|b_1|\end{aligned}$$

provided

$$kr^{k-1}\leq\frac{1-|b_1|}{1-\frac{1-(-1)^{m+l-n}\gamma}{\gamma-1}|b_1|}$$

which is true if

$$r\leq\min_k\left\{\frac{(1-|b_1|)(\gamma-1)}{k[(\gamma-1)-(1-(-1)^{m+l-n}\gamma)|b_1|]}\right\}^{\frac{1}{k-1}}\quad k=2,3,\ldots.$$

7 Convolution properties

For harmonic functions

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^{m+l}\sum_{k=1}^{\infty}|b_k|z^k,$$

and

$$F(z)=z+\sum_{k=2}^{\infty}|A_k|z^k+(-1)^{m+l}\sum_{k=1}^{\infty}|B_k|z^k,$$

we define the convolution of f and F as

$$(f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} |a_k||A_k| + (-1)^{m+l} \sum_{k=1}^{\infty} |b_k||B_k|z^k.$$

In the following theorem we examine the convolution properties of the class $V_H^l(m, n, \phi, \psi; \gamma)$.

Theorem 7.1 *If* $f \in V_H^l(m, n, \phi, \psi; \gamma)$ *and* $F \in V_H^l(m, n, \phi, \psi; \gamma)$ *then* $f * F \in V_H^l(m, n, \phi, \psi; \gamma)$.

Proof. let

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} |a_k|z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_k|z^k, \\ F(z) &= z + \sum_{k=2}^{\infty} |A_k|z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |B_k|z^k. \end{aligned}$$

Then, by Theorem 2.1, we have

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k| &\leq 1 \\ \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|A_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|B_k| &\leq 1 \end{aligned}$$

so for $f * F$, we may write

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k A_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k B_k| \\ &\leq \sum_{k=2}^{\infty} \frac{\beta_k k^m - \gamma k^n}{\gamma - 1}|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1}|b_k| \leq 1. \end{aligned}$$

Thus $f * F \in V_H^l(m, n, \phi, \psi; \gamma)$.

Theorem 7.2 *Let* $f_j(z) = z + \sum_{k=2}^{\infty} |a_{k,j}|z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_{k,j}|z^k$ *be in the class* $V_H^l(m, n, \phi, \psi; \alpha_j)$ *for all* $(j = 1, 2, ..., p)$, *then* $(f_1 * f_2 * ... * f_p)(z) \in V_H^l(m, n, \phi, \psi; \delta)$ *where*

$$\delta = 1 + \frac{(2^m\beta_2 - 2^n) \prod_{j=1}^{p}(\alpha_j - 1)}{\prod_{j=1}^{p}(2^m\beta_2 - 2^n\alpha_j) + 2^n \prod_{j=1}^{p}(\alpha_j - 1)} \tag{20}$$

.

Proof. We use the principle of mathematical induction in our proof. Let $f_1(z) = z + \sum_{k=2}^{\infty} |a_{k,1}|z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_{k,1}|z^k$ and $f_2(z) = z + \sum_{k=2}^{\infty} |a_{k,2}|z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_{k,2}|z^k$

be in the class $V_H^l(m,n,\phi,\psi;\alpha_1)$ and $V_H^l(m,n,\phi,\psi;\alpha_2)$, respectively. Then by Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\frac{(\beta_k k^m-\alpha_1 k^n)|a_{k,1}|}{\alpha_1-1}+\sum_{k=1}^{\infty}\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)|b_{k,1}|}{\alpha_1-1}\le 1$$
$$\sum_{k=2}^{\infty}\frac{(\beta_k k^m-\alpha_2 k^n)|a_{k,2}|}{\alpha_2-1}+\sum_{k=1}^{\infty}\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)|b_{k,2}|}{\alpha_2-1}\le 1.$$

We need to find the largest δ such that

$$\sum_{k=2}^{\infty}\frac{\beta_k k^m-\delta k^n}{\delta-1}|a_{k,1}||a_{k,2}|+\sum_{k=1}^{\infty}\frac{\mu_k k^m-(-1)^{m+l-n}\delta k^n}{\delta-1}|b_{k,1}||b_{k,2}|\le 1. \tag{21}$$

Then

$$\left[\sum_{k=2}^{\infty}\left(\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)|a_{k,1}|}{\alpha_1-1}}\right)^2\cdot\sum_{k=2}^{\infty}\left(\sqrt{\frac{(\beta_k k^m-\alpha_2 k^n)|a_{k,2}|}{\alpha_2-1}}\right)^2\right]^{\frac{1}{2}}$$
$$+\left[\sum_{k=1}^{\infty}\left(\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)|b_{k,1}|}{\alpha_1-1}}\right)^2\cdot\sum_{k=1}^{\infty}\left(\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)|b_{k,2}|}{\alpha_2-1}}\right)^2\right]^{\frac{1}{2}}\le 1. \tag{22}$$

thus, by applying Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty}\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)|a_{k,1}||a_{k,2}|}{(\alpha_1-1)(\alpha_2-1)}}$$
$$\le\left[\sum_{k=2}^{\infty}\left(\sqrt{\frac{\beta_k k^m-\alpha_1 k^n)|a_{k,1}|}{\alpha_1-1}}\right)^2\cdot\sum_{k=2}^{\infty}\left(\sqrt{\frac{(\beta_k k^m-\alpha_2 k^n)|a_{k,2}|}{\alpha_2-1}}\right)^2\right]^{1/2}$$
$$\sum_{k=1}^{\infty}\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)|b_{k,1}||b_{k,2}|}{(\alpha_1-1)(\alpha_2-1)}}$$
$$\le\left[\sum_{k=1}^{\infty}\left(\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)|b_{k,1}|}{\alpha_1-1}}\right)^2\cdot\sum_{k=1}^{\infty}\left(\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)|b_{k,2}|}{\alpha_2-1}}\right)^2\right]^{1/2}. \tag{23}$$

Then, we get

$$\sum_{k=2}^{\infty}\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)|a_{k,1}||a_{k,2}|}{(\alpha_1-1)(\alpha_2-1)}}$$
$$+\sum_{k=1}^{\infty}\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)|b_{k,1}||b_{k,2}|}{(\alpha_1-1)(\alpha_2-1)}}\le 1, \tag{24}$$

then, we get

$$\sum_{k=2}^{\infty}\frac{(\beta_k k^m-\delta k^n)|a_{k,1}||a_{k,2}|}{\delta-1}\leq\sum_{k=2}^{\infty}\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)|a_{k,1}||a_{k,2}|}{(\alpha_1-1)(\alpha_2-1)}}$$

$$\sum_{k=1}^{\infty}\frac{(\mu_k k^m-(-1)^{m+l-n}\delta k^n)|b_{k,1}||b_{k,2}|}{\delta-1}$$

$$\leq\sum_{k=1}^{\infty}\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)|b_{k,1}||b_{k,2}|}{(\alpha_1-1)(\alpha_2-1)}},\tag{25}$$

that is, if

$$\frac{(\beta_k k^m-\delta k^n)}{\delta-1}|a_{k,1}a_{k,2}|\leq\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}|a_{k,1}a_{k,2}|}$$

$$\frac{(\mu_k k^m-(-1)^{m+l-n}\delta k^n)}{\delta-1}|b_{k,1}b_{k,2}|\leq\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}|b_{k,1}b_{k,2}|},\tag{26}$$

hence that,

$$\sqrt{|a_{k,1}a_{k,2}|}\leq\frac{\delta-1}{\beta_k k^m-\delta k^n}\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}}$$

$$\sqrt{|b_{k,1}b_{k,2}|}\leq\frac{\delta-1}{\mu_k k^m-(-1)^{m+l-n}\delta k^n}\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)}{(\alpha_1-1)(\alpha_1-1)}}.\tag{27}$$

We know that

$$\sum_{k=2}^{\infty}\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}|a_{k,1}a_{k,2}|}\leq 1$$

$$\sum_{k=1}^{\infty}\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}|b_{k,1}b_{k,2}|}\leq 1,$$

then

$$\begin{aligned}\sqrt{|a_{k,1}a_{k,2}|}&\leq\sqrt{\frac{(\alpha_1-1)(\alpha_2-1)}{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)}},\\ \sqrt{|b_{k,1}b_{k,2}|}&\leq\sqrt{\frac{(\alpha_1-1)(\alpha_1-1)}{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)}}.\end{aligned}\tag{28}$$

Consequently, from Eqs. (27) and (28) we obtain

$$\sqrt{\frac{(\alpha_1-1)(\alpha_2-1)}{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m \alpha_2 k^n)}} \leq \frac{(\delta-1)}{(\beta_k k^m-\delta k^n)}\sqrt{\frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}},$$

$$\sqrt{\frac{(\alpha_1-1)(\alpha_2-1)}{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)}}$$

$$\leq \frac{(\delta-1)}{(\mu_k k^m-(-1)^{m+l-n}\delta k^n)}\sqrt{\frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)}{(\alpha_1-1)(\alpha_2-1)}}.$$

Then we see that

$$\delta \geq \frac{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)+\beta_k k^m(\alpha_1-1)(\alpha_2-1)}{(\beta_k k^m-\alpha_1 k^n)(\beta_k k^m-\alpha_2 k^n)+k^n(\alpha_1-1)(\alpha_2-1)}$$

$$\delta \geq \frac{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)+\mu_k k^m(\alpha_1-1)(\alpha_2-1)}{(\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n)(\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n)+k^n(\alpha_1-1)(\alpha_2-1)}, \tag{29}$$

then

$$\delta \geq 1+\frac{(\alpha_1-1)(\alpha_2-1)(\beta_k k^m-k^n)}{[\beta_k k^m-\alpha_1 k^n][\beta_k k^m-\alpha_2 k^n]+k^n(\alpha_1-1)(\alpha_2-1)}=\zeta(k),$$

$$\delta \geq 1+\frac{(\mu_k k^m-k^n)(\alpha_1-1)(\alpha_2-1)}{[\mu_k k^m-(-1)^{m+l-n}\alpha_1 k^n][\mu_k k^m-(-1)^{m+l-n}\alpha_2 k^n]+(-1)^{m+l-n}k^n(\alpha_1-1)(\alpha_2-1)}=\eta(k), \tag{30}$$

since $\zeta(k)$ for $k \geq 2$ and $\eta(k)$ for $k \geq 1$ are increasing, then $(f_1 * f_2)(z) \in V_H^l(m,n,\phi,\psi;\delta)$ where

$$\delta = 1+\frac{(\alpha_1-1)(\alpha_2-1)(\beta_2 2^m-2^n)}{[\beta_2 2^m-\alpha_1 2^n][\beta_2 2^m-\alpha_2 2^n]+2^n(\alpha_1-1)(\alpha_2-1)}$$

Next, we suppose that $(f_1 * f_2 * ... * f_p)(z) \in V_H^l(m,n,\phi,\psi;\gamma)$ then

$$\gamma = 1+\frac{(\beta_2 2^m-2^n)\prod_{j=1}^{p}(\alpha_j-1)}{\prod_{j=1}^{p}[\beta_2 2^m-\alpha_j 2^n]+2^n\prod_{j=1}^{p}(\alpha_j-1)}$$

we show that $(f_1 * f_2 * ... * f_{p+1})(z) \in V_H^l(m,n,\phi,\psi;\delta)$ then

$$\delta = 1+\frac{(\beta_2 2^m-2^n)(\gamma-1)(\alpha_{p+1}-1)}{[\beta_2 2^m-\gamma 2^n][\beta_2 2^m-\alpha_{p+1}2^n]+2^n(\gamma-1)(\alpha_{p+1}-1)}$$

since

$$(\gamma-1)(\alpha_{p+1}-1) = \frac{(\beta_2 2^m-2^n)\prod_{j=1}^{p+1}(\alpha_j-1)}{\prod_{j=1}^{p}(\beta_2 2^m-\alpha_j 2^n)+2^n\prod_{j=1}^{p}(\alpha_j-1)}.$$

$$(\beta_2 2^m-\gamma 2^n)(\beta_2^m-\alpha_{p+1}2^n) = \frac{(\beta_2 2^m-2^n)\prod_{j=1}^{p+1}(\beta_2 2^m-\alpha_j 2^n)}{\prod_{j=1}^{p}(\beta_2 2^m-\alpha_j 2^n)+2^n\prod_{j=1}^{p}(\alpha_j-1)},$$

we have

$$\delta = 1+\frac{(\beta_2 2^m-2^n)\prod_{j=1}^{p+1}(\alpha_j-1)}{\prod_{j=1}^{p+1}(\beta_2 2^m-\alpha_j 2^n)+2^n\prod_{j=1}^{p+1}(\alpha_j-1)}.$$

Theorem 7.3 *Let* $f_j(z) = z + \sum_{k=2}^{\infty} |a_{k,j}|z^k + (-1)^{m+l} \sum_{k=1}^{\infty} |b_{k,j}|z^k$ *be in the class* $V_H^l(m, n, \phi, \psi; \alpha)$ *for all* $(j = 1, 2, ..., p)$ *then* $(f_1 * f_2 * ... * f_p)(z) \in V_H^l(m, n, \phi, \psi; \delta)$ *where*

$$\delta = 1 + \frac{(\beta_2 2^m - 2^n)(\alpha - 1)^p}{(\beta_2 2^m - \alpha 2^n)^p + 2^n(\alpha - 1)^p} \tag{31}$$

.

8 Inclusion Results

To prove our next theorem, we need to the following lemma.

Lemma 8.1 *[15] Let* $f = h + \overline{g}$ *be given by (8), then* $f \in TS_H^l(m, n, \phi, \psi; \alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}(\beta_k k^m - \alpha k^n)|a_k| + \sum_{k=1}^{\infty}(\mu_k k^m - (-1)^{m+l-n}\alpha k^n)|b_k| \leq 1 - \alpha,$$

where $l \in \{0, 1\}$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m \geq n$, $\beta_k, \mu_k \geq 1$, $k \geq 1$, $0 \leq \alpha < 1$.

Theorem 8.2 *Let* $f \in V_H^l(m, n, \phi, \psi; \gamma)$, *then* $f \in TS_H^l(m, n, \phi, \psi; (4 - 3\gamma)/(3 - 2\gamma))$.

Proof. Since $f \in V_H^l(m, n, \phi, \psi; \gamma)$, then by Theorem 2.1, we have

$$\sum_{k=2}^{\infty}(\beta_k k^m - \gamma k^n)|a_k| + \sum_{k=1}^{\infty}(\mu_k k^m - (-1)^{m+l-n}\gamma k^n)|b_k| \leq \gamma - 1.$$

To show that $f(z) \in TS_H^l(m, n, \phi, \psi; (4 - 3\gamma)/(3 - 2\gamma))$ by Lemma 8.1, we have to show that

$$\sum_{k=2}^{\infty}(\beta_k k^m - \frac{4 - 3\gamma}{3 - 2\gamma}k^n)|a_k| + \sum_{k=1}^{\infty}(\mu_k k^m - (-1)^{m+l-n}\frac{4 - 3\gamma}{3 - 2\gamma}k^n)|b_k| \leq 1 - \frac{4 - 3\gamma}{3 - 2\gamma},$$

where $0 \leq \frac{4-3\gamma}{3-2\gamma} < 1$. For this, it is sufficient to prove that

$$\frac{\beta_k k^m - \gamma k^n}{\gamma - 1} \geq \frac{\beta_k k^m - \frac{4-3\gamma}{3-2\gamma}k^n}{1 - \frac{4-3\gamma}{3-2\gamma}}, \ (k = 2, 3, 4, ...),$$

$$\frac{\mu_k k^m - (-1)^{m+l-n}\gamma k^n}{\gamma - 1} \geq \frac{\mu_k k^m - (-1)^{m+l-n}\frac{4-3\gamma}{3-2\gamma}k^n}{1 - \frac{4-3\gamma}{3-2\gamma}}, \ (k = 1, 2, 3, ...),$$

or equivalently $2(\gamma - 1)\beta_k k^m - 4(\gamma - 1)k^n \geq 0$, and $2(\gamma - 1)\mu_k k^m - 4(-1)^{m+l-n}(\gamma - 1)k^n \geq 0$, which is true and the theorem is proved.

9 Open problem

The authors suggest to study the properties of the same classes $M_H^l(m,n,\phi,\psi;\gamma)$ and $V_H^l(m,n,\phi,\psi;\gamma)$ by using different operators and discuss Theorem 7.2.

Acknowledgement: The authors express their sincerest thanks to the referee for his useful comments.

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