Int. J. Open Problems Complex Analysis, Vol. 8, No. 2, July 2016
ISSN 2074-2827;
www.i-csrs.org

Boas and Roe Theorems for the Dunkl Transform on the Real Line

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Abstract

In this paper we prove the Boas theorem for the Dunkl transform on the spaces $L^p_\alpha(\mathbb{R})$, $p \in [1,\infty]$, and we state many versions for the Roe's theorem associated for the Dunkl operator on the real line.

Keywords: *Dunkl operator, Dunkl transform, Dunkl potential function, Boas theorem, Paley-Wiener theorems, Roe's theorem*
2000 Mathematical Subject Classification: 35L05, 22E30
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1 Introduction

For $\alpha > -1$, let J_α denote the Bessel function of order α:

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,\Gamma(\alpha+n+1)}$$

(a classical reference on Bessel functions is [24]).

It is well known that when we consider radial functions in $\mathbb{R}^d$, $f(x) = g(\|x\|)$, the Fourier transform becomes the Hankel transform of order $d/2-1$. Indeed, taking the Hankel transform of order α, with $\alpha \geq -1/2$, as

$$\mathcal{H}_\alpha g(s) = \int_0^\infty g(r) j_\alpha(sr)\, d\omega_\alpha(r), \qquad s > 0,$$

where $d\omega_\alpha(r) = (2^\alpha \Gamma(\alpha+1))^{-1} r^{2\alpha+1}\, dr$ and $j_\alpha(z) = \Gamma(\alpha+1)(z/2)^{-\alpha} J_\alpha(z)$, it is verified that $\widehat{f}(\xi) = \mathcal{H}_{\frac{d}{2}-1} g(\|\xi\|)$.

The Dunkl transform on the real line is both an extension of the Hankel transform to the whole real line and a generalization of the Fourier transform. It is defined by the identity

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-iyx)\, d\mu_\alpha(x), \qquad y \in \mathbb{R},$$

where

$$E_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz)$$

and $d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}\, dx$. The Fourier transform corresponds with the case $\alpha = -1/2$ because $E_{-1/2}(z) = e^z$ and $d\mu_{-1/2}$ is, up to a multiplicative factor, the Lebesgue measure on $\mathbb{R}$. This transform is related to the Dunkl operator on the real line. The Dunkl operators on $\mathbb{R}^d$ are differential-difference operators associated with some finite reflection groups (cf. [7]). We consider the Dunkl operator Λ_α, $\alpha \geq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x) - f(-x)}{2}\right).$$

The Dunkl kernel E_α is, for $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the unique solution of the initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), & x \in \mathbb{R}, \\ f(0) = 1. \end{cases}$$

Very recently, many authors have been investigating the behavior of the Dunkl transform with respect to several problems already studied for the Fourier transform; for instance, multipliers [4], Paley-Wiener theorems [6], Cowling-Price's theorem [11], transplantation [17], Riesz transforms [18], uncertainty [21], and so on.

In the last few years there has been a great interest to the study of the spectrum of functions i.e. the support of the transform of these functions relative to certain integral transforms. The Paley-Wiener and Boas theorems give a characterization of two classes of functions in terms of the behavior of their Fourier transforms. See [5, 22] for an overview of references and details for this question.

More precisely, let $f \in L^2(\mathbb{R})$ and $\hat{f}$ be its Fourier transform. The Paley-Wiener theorem stated that $\hat{f} \in L^2(\mathbb{R})$ has compact support if and only if $f \in L^2(\mathbb{R})$ is analytically extendable into the complex plane as an entire function of exponential type. In [2] Bang proved another version of the Paley-Wiener theorem as follow:

Theorem 1 *$\hat{f}$ has compact support $[-\sigma, \sigma]$ if and only if f is infinitely differentiable, $D^n f \in L^2(\mathbb{R})$ for any n, and*

$$\lim_{n\to\infty} \|D^n f\|_{L^2(\mathbb{R})}^{1/n} \leq \sigma.$$

It is a real-valued version of the Paley-Wiener theorem since no complexification of f was required.

Since any function $f \in L^2(\mathbb{R})$ can be written $f = f_1 + f_2$, where $f_1 \in L^2[-\sigma, \sigma]$ and $f_2 \in L^2(I)$, where $I = \mathbb{R}\backslash[-\sigma, \sigma]$, it is natural to ask if there is any characterization of the space of all functions of the latter type (Boas problem).

The original Boas theorem asserts that if $f \in L^2(\mathbb{R})$, then a necessary and sufficient condition that f vanishes almost everywhere on $(-1, 1)$ is that

$$(B(B\widehat{f}))(\xi) = -\widehat{f}(\xi),$$

where $B(g)$ is the Boas transform of a function g defined by

$$(Bg)(x) = \frac{1}{\pi}\int_0^1 \frac{g(x+t) - g(x-t)}{t^2}\sin t\, dt,$$

whenever the integral exists.

Recently in [12] by using the spectral theory associated with the Dunkl Laplace operator we have characterize the class of square integrable functions vanishing in a neighborhood of a point ξ_0 under the Dunkl transform.

In this paper, we are interested in completing our work by obtaining a Boas-type theorem for the Dunkl transform on $\mathbb{R}$, on the spaces $L^p_\alpha(\mathbb{R})$, $p \in [1, \infty]$. More precisely, we prove a new characterisation for the support of the Dunkl transform under the behaviour of L^p_α-norms of iterated Dunkl potentials.

The structure of the paper is as follows. In §2 we state the precise notations and give some preliminaries related to the Dunkl operator on the real line. The §3 is devoted to characterize the support for the Dunkl transform of the function in the Lebesgue space $L^p_\alpha(\mathbb{R})$ for $p \in [1, \infty]$, via the Dunkl potentials. Finally, in the last section we state many versions of Roe's theorem for Λ_α.

2 Preliminaries

This section gives an introduction to the harmonic analysis associated with the Dunkl operator. Main references are [7, 8, 9, 20]. In the following we denote by

$\mathcal{P}(\mathbb{R})$	the set of polynomials on $\mathbb{R}$.
$C(\mathbb{R})$	the space of continuous functions on $\mathbb{R}$.
$C_c(\mathbb{R})$	the space of continuous functions on $\mathbb{R}$ with compact support.
$C^p(\mathbb{R})$	the space of functions of class C^p on $\mathbb{R}$.
$C^p_b(\mathbb{R})$	the space of bounded functions of class C^p.
$\mathcal{E}(\mathbb{R})$	the space of C^∞-functions on $\mathbb{R}$.
$\mathcal{S}(\mathbb{R})$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
$D(\mathbb{R})$	the space of C^∞-functions on $\mathbb{R}$ which are of compact support.
$\mathcal{S}'(\mathbb{R})$	the space of tempered distributions on $\mathbb{R}$.
$PW(\mathbb{C})$	the space of entire functions on $\mathbb{C}$, rapidly decreasing and of exponential type.
$\mathcal{PW}(\mathbb{C})$	the space of entire functions on $\mathbb{C}$, slowly increasing and of exponential type.
$\mathcal{E}'(\mathbb{R})$	the space of distributions on $\mathbb{R}$ with compact support.

Some properties of the Λ_α, are given in the following :

For all f and g in $C^1(\mathbb{R})$ with at least one of them is even, we have

$$\Lambda_\alpha(fg) = (\Lambda_\alpha f)g + f\Lambda_\alpha g. \tag{1}$$

For f of class C^1 on $\mathbb{R}$ with compact support and g of class C^1 on $\mathbb{R}$, we have :

$$\int_{\mathbb{R}} \Lambda_\alpha f(x) g(x) |x|^{2\alpha+1} dx = -\int_{\mathbb{R}} f(x) \Lambda_\alpha g(x) |x|^{2\alpha+1} dx. \tag{2}$$

For every $\lambda \in \mathbb{C}$, let us denote by $E_\alpha(\lambda.)$ the unique solution of the eigenvalue problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), \\ f(0) = 1. \end{cases} \tag{3}$$

Proposition 1 *For all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we have*

$$|\frac{d^n}{d\lambda^n} E_\alpha(\lambda x)| \leq |x|^n e^{|Re\lambda||x|}. \tag{4}$$

Notations. We denote by
$L^p_\alpha(\mathbb{R})$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\begin{aligned} \|f\|_{L^p_\alpha(\mathbb{R})} &= \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty \\ \|f\|_{L^\infty_\alpha(\mathbb{R})} &= ess \sup_{x \in \mathbb{R}} |f(x)| < \infty. \end{aligned}$$

For $\alpha \geq \frac{-1}{2}$, and $f \in C_c(\mathbb{R})$, the Dunkl transform is defined by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x) E_\alpha(-i\lambda x) d\mu_\alpha(x), \quad \text{for all } \lambda \in \mathbb{C}. \tag{5}$$

The inverse Fourier transform of a suitable function g on $\mathbb{R}$ is given by:

$$\mathcal{F}_\alpha^{-1}(g)(x) = \int_{\mathbb{R}} g(\lambda) E_\alpha(i\lambda x) d\mu_\alpha(\lambda). \tag{6}$$

Next, we give some properties of this transform.
i) For f in $L^1_\alpha(\mathbb{R})$ we have

$$||\mathcal{F}_\alpha(f)||_{L^\infty_\alpha(\mathbb{R})} \leq ||f||_{L^1_\alpha(\mathbb{R})}. \tag{7}$$

ii) For f in $\mathcal{S}(\mathbb{R})$ we have

$$\mathcal{F}_\alpha(\Lambda_\alpha f)(y) = iy \mathcal{F}_\alpha(f)(y), \quad \text{for all } y \in \mathbb{R}. \tag{8}$$

Proposition 2 i*)* **Plancherel formula for $\mathcal{F}_\alpha$.**
For all f in $\mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\xi)|^2 d\mu_\alpha(\xi). \tag{9}$$

ii*)* **Plancherel theorem for $\mathcal{F}_\alpha$.**
The Dunkl transform can be uniquely extended to an isomorphism from $L^2_\alpha(\mathbb{R})$ onto $L^2_\alpha(\mathbb{R})$.

Proposition 3 *The Dunkl transform $\mathcal{F}_\alpha$ is a topological isomorphism from*

i) $D(\mathbb{R})$ *onto* $PW(\mathbb{C})$.

ii) $\mathcal{S}(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$.

Definition 1 *Let $x \in \mathbb{R}$ and let $f \in C_b(\mathbb{R})$. For $\alpha \geq \frac{-1}{2}$, we define the generalized translation operator τ_x^α by*

$$\tau_x^\alpha f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^\alpha(z) \tag{10}$$

here

$$d\mu_{x,y}^\alpha(z) = \begin{cases} \mathcal{K}_\alpha(x,y,z)|z|^{2\alpha+1}dz & \text{if} \quad xy \neq 0 \\ d\delta_x(z) & \text{if} \quad y = 0 \\ d\delta_y(z) & \text{if} \quad x = 0 \end{cases}$$

where $\mathcal{K}_\alpha(x,y,z)$ is given explicitly in [4]. Moreover

$$supp(d\mu_{x,y}^\alpha) \subset \Big[-|x|-|y|, -\Big| |x|-|y| \Big| \Big] \bigcup \Big[\Big| |x|-|y| \Big|, |x|+|y| \Big].$$

Definition 2 *For suitable functions f and g, we define the convolution product $f *_\alpha g$ by*

$$f *_\alpha g(x) = \int_{\mathbb{R}} \tau_x^\alpha f(-y) g(y) |y|^{2\alpha+1} dy. \tag{11}$$

Remark 1 *It is clear that this convolution product is both commutative and associative:*

i) $f *_\alpha g = g *_\alpha f$.

ii) $(f *_\alpha g) *_\alpha h = f *_\alpha (g *_\alpha h)$.

Proposition 4 i*) Let f be in $L^2_\alpha(\mathbb{R})$ and g in $L^1_\alpha(\mathbb{R})$. Then the function $f *_\alpha g$ defined almost everywhere on $\mathbb{R}$ by*

$$f *_\alpha g(y) = \int_{\mathbb{R}} \tau_y^\alpha(f)(-x) g(x) |x|^{2\alpha+1} dx,$$

belongs to $L^2_\alpha(\mathbb{R})$ and we have

$$\|f *_\alpha g\|_{L^2_\alpha(\mathbb{R})} \leq C \|f\|_{L^2_\alpha(\mathbb{R})} \|g\|_{L^1_\alpha(\mathbb{R})}.$$

ii*) Assume that $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every $f \in L^p_\alpha(\mathbb{R})$ and $g \in L^q_\alpha(\mathbb{R})$, we have $f *_\alpha g \in L^r_\alpha(\mathbb{R})$, and*

$$\|f *_\alpha g\|_{L^r_\alpha(\mathbb{R})} \leq C \|f\|_{L^p_\alpha(\mathbb{R})} \|g\|_{L^q_\alpha(\mathbb{R})}. \tag{12}$$

Proposition 5 i*) Let $D_a(\mathbb{R})$ be the space of smooth functions on $\mathbb{R}$ supported in $[-a, a]$. For $f \in D_a(\mathbb{R})$ and $g \in D_b(\mathbb{R})$, we have $f *_\alpha g \in D_{a+b}(\mathbb{R})$ and*

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)(\lambda)\mathcal{F}_\alpha(f)(\lambda). \tag{13}$$

ii*) For $f \in L^2_\alpha(\mathbb{R})$ and $g \in L^1_\alpha(\mathbb{R})$ we have*

$$\mathcal{F}_\alpha(f *_\alpha g)(\lambda) = \mathcal{F}_\alpha(f)(\lambda)\mathcal{F}_\alpha(f)(\lambda). \tag{14}$$

Definition 3 *i) The Dunkl transform of a distribution τ in $\mathcal{S}'(\mathbb{R})$, is defined by*

$$\langle \mathcal{F}_\alpha(\tau), \varphi\rangle = \langle \tau, \mathcal{F}_\alpha^{-1}(\varphi)\rangle, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}). \tag{15}$$

ii) The Dunkl transform of a distribution S in $\mathcal{E}'(\mathbb{R})$ is defined by

$$\forall\, \lambda \in \mathbb{R}, \quad \mathcal{F}_\alpha(S)(\lambda) = \langle S, E_\alpha(-i\lambda.)\rangle. \tag{16}$$

Proposition 6 *The Dunkl transform $\mathcal{F}_\alpha$ is a topological isomorphism from*

i) $\mathcal{E}'(\mathbb{R})$ onto $\mathcal{PW}(\mathbb{C})$.

ii) $\mathcal{S}'(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$.

Let τ be in $\mathcal{S}'(\mathbb{R})$. We have

$$\mathcal{F}_\alpha(\Lambda_\alpha \tau) \quad = \quad iy\mathcal{F}_\alpha(\tau). \tag{17}$$

3 Boas theorem for the Dunkl transform

Definition 4 *Let $f \in \mathcal{S}'(\mathbb{R})$. The tempered generalized function $R_0 f$ is termed the Dunkl potential of f if $\Lambda_\alpha(R_0 f) = f$, that is*

$$\langle R_0 f, \Lambda_\alpha \varphi\rangle = -\langle f, \varphi\rangle, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

Remark 2 *We proceed as in [3], and using the potential theory we can characterize the Dunkl potential for tempered distributions.*

Theorem 2 *Let $1 \le p \le \infty$. If $R_0^n f \in L^p_\alpha(\mathbb{R})$ for all $n \in \mathbb{N}_0$, then*

$$\lim_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} = \frac{1}{\sigma_0}, \tag{18}$$

where

$$\sigma_0 = \inf\Big\{|\xi| \;:\; \xi \in supp\mathcal{F}_\alpha(f)\Big\}.$$

For prove this theorem we need the following lemmas.

Lemma 1 *If $\sigma_0 > 0$, then*

$$supp\,\mathcal{F}_\alpha\Big(R_0^n f\Big) = supp\,\mathcal{F}_\alpha(f), \quad n = 1, \ldots. \tag{19}$$

Proof. As

$$\Lambda_\alpha^n(R_0^n f) = f$$

we deduce that

$$\mathcal{F}_\alpha(f) = (i\xi)^n \mathcal{F}_\alpha\Big(R_0^n f\Big).$$

Therefore,

$$supp\,\mathcal{F}_\alpha(f) \subset supp\,\mathcal{F}_\alpha\Big(R_0^n f\Big) \subset \mathcal{F}_\alpha(f) \cup \Big\{0\Big\}.$$

So, to obtain (19), it is enough to show that $0 \notin supp\, \mathcal{F}_\alpha\Big(R_0^n f\Big)$.
We choose numbers $a, b : 0 < a < b < \sigma_0$ and a function $h \in D(\mathbb{R})$ such that $supp\, h \subset (-b, b)$ and $h(x) \equiv 1$ in $(-a, a)$. Then

$$supp\Big(h\mathcal{F}_\alpha(R_0^n f)\Big) \subset \Big\{0\Big\}.$$

Suppose that $supp\Big(h\mathcal{F}_\alpha(R_0^n f)\Big) = \Big\{0\Big\}$, then there is a numbers $N(n) \in \mathbb{N}$ such that

$$h\mathcal{F}_\alpha\Big(R_0^n f\Big) = \sum_{j=0}^{N(n)} C_j(N(n))\delta^j,$$

where δ^j denote the jth distributional derivative of the delta function δ at 0. Hence,

$$\mathcal{F}_\alpha^{-1}(h) *_\alpha R_0^n f(\xi) = \sum_{j=0}^{N(n)} C_j(N(n))(-i\xi)^j.$$

As $R_0^n f \in L^p_\alpha(\mathbb{R})$ and $\mathcal{F}_\alpha^{-1}(h) \in L^q_\alpha(\mathbb{R})$, we get $\mathcal{F}_\alpha^{-1}(h) *_\alpha R_0^n f \in L^\infty_\alpha(\mathbb{R})$.
Therefore

$$\mathcal{F}_\alpha^{-1}(h) *_\alpha R_0^n f(\xi) = C_0(N(n)),\ n \in \mathbb{N}.$$

Note that

$$\begin{aligned} C_0(N(n)) &= \mathcal{F}_\alpha^{-1}(h) *_\alpha R_0^n f(\xi) \\ &= \mathcal{F}_\alpha^{-1}(h) *_\alpha \Lambda_\alpha R_0^{n+1} f(\xi) \\ &= \Lambda_\alpha\Big(\mathcal{F}_\alpha^{-1}(h) *_\alpha R_0^{n+1} f(\xi)\Big) \\ &= \Lambda_\alpha(C_0(N(n+1))) = 0. \end{aligned}$$

Thus we deduce that $C_0(N(n)) = 0$. So $h\mathcal{F}_\alpha\Big(R_0^n f\Big) = 0$.
Assume now the contrary that

$$\Big\{0\Big\} \subset supp\ \mathcal{F}_\alpha\Big(R_0^n f\Big).$$

Then there is a function $\chi \in D(\mathbb{R})$, with $supp\, \chi \subset (-a, a)$ and such that

$$\langle \mathcal{F}_\alpha\Big(R_0^n f\Big), \chi\rangle \neq 0.$$

So, as $h(x) = 1$ for $|x| < a$, we get

$$0 \neq \langle \mathcal{F}_\alpha\Big(R_0^n f\Big), \chi\rangle = \langle \mathcal{F}_\alpha\Big(R_0^n f\Big), h\chi\rangle = \langle h\mathcal{F}_\alpha\Big(R_0^n f\Big), \chi\rangle = 0,$$

which is impossible. Thus we have proved (19).

Lemma 2 *If $\sigma_0 > 0$, then*

$$\limsup_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} \leq \frac{1}{\sigma_0}. \tag{20}$$

Proof. From (19) we have

$$supp\,\mathcal{F}_\alpha\Big(R_0^n f\Big) \subset \mathbb{R}\backslash(-\sigma_0, \sigma_0). \tag{21}$$

For any $\varepsilon > 0$, $\varepsilon < \frac{\sigma_0}{2}$ we choose a function $h \in \mathcal{E}(\mathbb{R})$ satisfying

$$h(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \geq \sigma_0 - \varepsilon \\ 0 & \text{if} \quad |\xi| < \sigma_0 - 2\varepsilon. \end{cases}$$

Let χ be an arbitrary element in $\mathcal{S}(\mathbb{R})$. Then it follow from (21) that

$$\begin{aligned} \langle R_0^n f, \chi\rangle &= \langle \mathcal{F}_\alpha\Big(R_0^n f\Big), \mathcal{F}_\alpha^{-1}(\chi)\rangle \\ &= \langle \mathcal{F}_\alpha\Big(R_0^n f\Big), h\mathcal{F}_\alpha^{-1}(\chi)\rangle \\ &= \langle R_0^n f, \mathcal{F}_\alpha\Big(h\mathcal{F}_\alpha^{-1}(\chi)\Big)\rangle. \end{aligned}$$

Therefore,

$$\langle R_0^n f, \chi\rangle = \langle R_0^n f, \varphi\rangle, \tag{22}$$

where

$$\varphi = \mathcal{F}_\alpha\Big(h\mathcal{F}_\alpha^{-1}(\chi)\Big).$$

We put

$$\varphi_n = \mathcal{F}_\alpha\Big(\frac{h(\xi)}{\xi^n}\mathcal{F}_\alpha^{-1}(\chi)\Big).$$

Then $\varphi_n \in \mathcal{S}(\mathbb{R})$ and

$$\begin{aligned} |\langle f, \varphi_n\rangle| &= |\langle \Lambda_\alpha^n R_0^n f, \varphi_n\rangle| \\ &= |\langle R_0^n f, \Lambda_\alpha^n \varphi_n\rangle| \\ &= |\langle R_0^n f, \varphi\rangle|. \end{aligned} \tag{23}$$

Combining (22) and (23), we get

$$|\langle R_0^n f, \chi\rangle| = |\langle f, \varphi_n\rangle| = |\langle f, \chi *_\alpha \mathcal{F}_\alpha(\frac{h(\xi)}{\xi^n})\rangle|. \tag{24}$$

Therefore, we have

$$\begin{aligned} ||R_0^n f||_{L_\alpha^p(\mathbb{R})} &= \sup_{\Big\{\chi\in\mathcal{S}(\mathbb{R}):\quad ||\chi||_{L_\alpha^q(\mathbb{R})}\leq 1\Big\}} \Big|\langle f, \chi *_\alpha \mathcal{F}_\alpha(\frac{h(\xi)}{\xi^n})\rangle\Big| \\ &\leq \sup_{\Big\{\chi\in\mathcal{S}(\mathbb{R}):\quad ||\chi||_{L_\alpha^q(\mathbb{R})}\leq 1\Big\}} ||f||_{L_\alpha^p(\mathbb{R})}||\chi *_\alpha \mathcal{F}_\alpha(\frac{h(\xi)}{\xi^n})||_{L_\alpha^q(\mathbb{R})} \\ &\leq C||f||_{L_\alpha^p(\mathbb{R})}||\mathcal{F}_\alpha(\tfrac{h(\xi)}{\xi^n})||_{L_\alpha^1(\mathbb{R})}. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} ||R_0^n f||_{L_\alpha^p(\mathbb{R})}^{\frac{1}{n}} \leq \limsup_{n\to\infty} ||\mathcal{F}_\alpha(\frac{h(\xi)}{\xi^n})||_{L_\alpha^1(\mathbb{R})}^{\frac{1}{n}}. \tag{25}$$

Using the continuity for the Dunkl transform on $\mathcal{S}(\mathbb{R})$, we prove that

$$\limsup_{n\to\infty} ||\mathcal{F}_\alpha(\frac{h(\xi)}{\xi^n})||^{\frac{1}{n}}_{L^1_\alpha(\mathbb{R})} \leq \frac{1}{\sigma_0 - 2\varepsilon}. \tag{26}$$

Combining (25) and (26), we get

$$\limsup_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} \leq \frac{1}{\sigma_0 - 2\varepsilon}$$

and then (20) by letting $\varepsilon \to 0$.

Lemma 3 *If $\sigma_0 > 0$, then*

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} \geq \frac{1}{\sigma_0}. \tag{27}$$

Proof. From the definition of σ_0, there exists a function $\chi \in D(\mathbb{R})$ such that

$$supp\chi \subset \Big\{\xi : \sigma_0 - \varepsilon < |\xi| < \sigma_0 + \varepsilon\Big\} \quad \text{and} \quad \langle \mathcal{F}_\alpha(f), \chi\rangle \neq 0.$$

Therefore,

$$\begin{aligned} 0 \neq |\langle f, \chi\rangle| &= |\langle \Lambda_\alpha^n R_0^n f, \chi\rangle| \\ &= |\langle R_0^n f, \Lambda_\alpha^n \chi\rangle| \\ &\leq ||R_0^n f||_{L^p_\alpha(\mathbb{R})} ||\Lambda_\alpha^n \chi||_{L^q_\alpha(\mathbb{R})}. \end{aligned} \tag{28}$$

So

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} \geq \frac{1}{\limsup\limits_{n\to\infty} ||\Lambda_\alpha^n \chi||_{L^q_\alpha(\mathbb{R})}}. \tag{29}$$

We proceed as above we prove that

$$\limsup_{n\to\infty} ||\Lambda_\alpha^n \chi||^{\frac{1}{n}}_{L^q_\alpha(\mathbb{R})} \leq \sigma_0 + \varepsilon.$$

So by (29) we obtain

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} \geq \frac{1}{(\sigma + \varepsilon)}, \quad \varepsilon > 0,$$

and then (27).

Proof. of Theorem 2.

We divide our proof into two cases.

Case 1. $\sigma_0 = 0$. We have $\xi_0 \in supp\mathcal{F}_\alpha(f)$. Hence, for any $\varepsilon > 0$ there is a function $\chi \in D(\mathbb{R})$ such that $supp\,\chi \subset (-\varepsilon, \varepsilon)$ such that $\langle \mathcal{F}_\alpha(f), \chi\rangle \neq 0$. Arguing as above we obtain

$$\liminf_{n\to\infty} ||R_0^n f||^{\frac{1}{n}}_{L^p_\alpha(\mathbb{R})} \geq \frac{1}{\limsup\limits_{n\to\infty} ||\Lambda_\alpha^n \chi||^{\frac{1}{n}}_{L^q_\alpha(\mathbb{R})}} \geq \frac{1}{\varepsilon}.$$

Therefore

$$\liminf_{n\to\infty} ||R_0^n f||_{L^p_\alpha(\mathbb{R})}^{\frac{1}{n}} = \infty.$$

So we always have

$$\lim_{n\to\infty} ||R_0^n f||_{L^p_\alpha(\mathbb{R})}^{\frac{1}{n}} = \frac{1}{\sigma_0}.$$

Case 2. $\sigma_0 > 0$. Combining (20) and (27), we arrive to (18).

Remark 3 *We have proved the analogues of the Theorem 2 for the hypergeometric Fourier transform and for the Opdam-Cherednik transform (see [15, 16]).*

4 Roe's theorem associated with the Dunkl operator Λ_α

In [19] Roe proved that if a doubly-infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies $\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \le M$ for all $j = 0, \pm1, \pm2, \ldots$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x+b)$ where a and b are real constants.

In this section we state many versions of the Roe's theorem associated for the Dunkl operator on the real line. These versions are proved in the context of the Dunkl-type operator, which is more general than the Dunkl operator in real line. (See [13]).

Theorem 3 *Suppose $P(\xi) = \sum_n a_n \xi^n$ is real-valued and let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-i\Lambda_\alpha) f_j.$$

(i) Let $a \ge 0$, $R > 0$, and assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies

$$|f_j(x)| \le M_j R^j (1+|x|)^a, \tag{30}$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{31}$$

Then $f = f_+ + f_-$ where $P(-i\Lambda_\alpha)f_+ = Rf_+$ and $P(-i\Lambda_\alpha)f_- = -Rf_-$. If R (or $-R$) is not in the range of P then $f_+ = 0$ (or $f_- = 0$).

(ii) If we replace (31) with

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0, \tag{32}$$

for all $j > 0$, then the span of $(f_j)_j$ is finite dimensional. Moreover, $f_0 = f_+ + f_-$, where, for some integer N , $(P(-i\Lambda_\alpha) - R)^N f_+ = 0$ and $(P(-i\Lambda_\alpha) + R)^N f_- = 0$. Thus f_+ (or f_-) is a generalized eigenfunction of $P(-i\Lambda_\alpha)$ with eigenvalue R (or $-R$).

Theorem 4 *Suppose* $P(\xi) = \sum_n a_n \xi^n$ *is a non-constant polynomial with complex coefficients. Let* $\{f_j\}_{-\infty}^{\infty}$ *be a sequence of complex-valued functions on* $\mathbb{R}$ *so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(-i\Lambda_\alpha) f_j.$$

1) Let $a \geq 0$ *and let* $R > 0$*. Assume that for all* $\varepsilon > 0$*, there exist constants* $N \in \mathbb{N}_0$ *and* $C > 0$*, such that*

$$\forall\, x \in \mathbb{R}, \quad |f_n(x)| \leq C R^n (1+\varepsilon)^{|n|} (1+|x|)^N \tag{33}$$

is satisfied for all $n \in \mathbb{Z}$*. Then*

$$\forall\, x \in \mathbb{R}, \quad f_0(x) = \sum_{\lambda \in S_R} \sum_{j=0}^{N} c(\lambda, j) \frac{d^j}{d\xi^j}\Big|_{\xi=\lambda} E_\alpha(i\xi x), \tag{34}$$

for constants $c(\lambda, j) \in \mathbb{C}$*,* $N \in \mathbb{N}$ *and* $S_R := \Big\{\xi \in \mathbb{R} : P(\xi) = R\Big\}$.

2) Let $a \geq 0$ *and let* $R > 0$ *and assume that* $\{f_j\}_{-\infty}^{\infty}$ *satisfies*

$$|f_j(x)| \leq M_j R^j (1+|x|)^a, \tag{35}$$

where $(M_j)_{j \in \mathbb{Z}}$ *satisfies the subpotential growth condition*

$$\lim_{j \to \infty} \frac{M_{|j|}}{j^m} = 0, \tag{36}$$

for some $m \geq 0$.

We have

(i) If $P'(\lambda_p) \neq 0$*, for all* $\lambda_p \in S_R$*, then* $N < m$ *in (34).*

In particular, if $m = 1$*, then*

$$\forall\, x \in \mathbb{R}, \quad f_0(x) = \sum_{\lambda_p \in S_R} f_{\lambda_p}(x), \quad \text{where } f_{\lambda_p}(x) = c(\lambda_p) E_\alpha(i\lambda_p x).$$

(ii) If S_R *consists of one point* λ_0 *and* $m = 1$ *in (36), then* $P(-i\Lambda_\alpha) f_0 = P(\lambda_0) f_0$.

Remark 4 *The previous theorem is the analogue for the Theorems 1 and 6 of [1].*

5 Open Problem

The first purpose of the future work is to characterize the Dunkl potential for tempered distributions.
The second purpose is to prove the analogous of Theorem 2 for the generalized Fourier transforms.

Acknowledgements. The author is deeply indebted to the referees for providing constructive comments and help in improving the contents of this paper. Thanks also to Professors K. Trimèche and N-B. Andersen for their helps.

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