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Some Characterizations for a Certain Generalized Bessel Function of the First Kind to be in Certain Subclasses of Analytic Functions

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Abstract

The main purpose of this paper is to introduce some characterizations for a certain generalized Bessel function of the first kind to be in the subclasses $S_p(\lambda, \alpha, \beta)$ and $TS_p(\lambda, \alpha, \beta)$ of normalized analytic functions in the open unit disk U and followed by various corollaries related to the well-known subclasses of starlike and convex functions. Furthermore, we consider some operators related to the generalized Bessel function.

Keywords: *analytic function, univalent function, starlike function, convex function, generalized Bessel functions.*

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1 Introduction

Let $\mathcal{A}$ denotes the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, we denote by T the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.2}$$

Let $S^*(\alpha)$ and $K(\alpha)$ denote the well-known subclasses of $\mathcal{A}$ consisting of starlike and convex functions of order $\alpha\,(0 \leq \alpha < 1)$ (see, for example, [26] and [20]), respectively. Also, let $T^*(\alpha) = S^*(\alpha) \cap T$ and $C(\alpha) = K(\alpha) \cap T$. We note that $S^*(0) = S^*$ and $K(0) = K$ the subclasses of starlike and convex functions (see Robertson [25]).

We denote by $UST(\alpha, \beta)$, $-1 \leq \alpha < 1$ and $\beta \geq 0$, the class of β-uniformly starlike functions of order α consisting of functions $f \in \mathcal{A}$ that satisfy the following condition:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (-1 \leq \alpha < 1;\ \beta \geq 0;\ z \in U).$$

Also, we denote by $UCV(\alpha, \beta)$, $-1 \leq \alpha < 1$ and $\beta \geq 0$, the class of β-uniformly convex functions of order α consisting of functions $f \in \mathcal{A}$ that satisfy the following condition:

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (-1 \leq \alpha < 1;\ \beta \geq 0;\ z \in U).$$

The classes $UST(\alpha, \beta)$ and $UCV(\alpha, \beta)$ was studied by Shams et al. [27] and Owa et al. [22]. We note that $UST(\alpha, 0) = S^*(\alpha)$ and $UCV(\alpha, \beta) = K(\alpha)$.

In 2010, Aouf et al. [2] introduced the following subclass of $\mathcal{A}$.
Definition 1. For $-1 \leq \alpha < 1$, $0 \leq \lambda \leq 1$ and $\beta \geq 0$, let $S(\lambda, \alpha, \beta)$ denotes the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) satisfying the analytic criterion:

$$\mathrm{Re}\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - \alpha\right\} > \beta\left|\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right|;\ z \in U. \tag{1.3}$$

Moreover, we will consider the following subclass $TS(\lambda,\alpha,\beta)$ of T as following:

$$TS(\lambda,\alpha,\beta) = S(\lambda,\alpha,\beta) \cap T. \tag{1.4}$$

We note that:
(*i*) $TS(0,\alpha,\beta) = TS(\alpha,\beta)$ and $TS(1,\alpha,\beta) = UCT(\alpha,\beta)$ are the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively. Also, $TS(0,\alpha,1)$=$TS(\alpha)$ and $TS(1,\alpha,1)$
=$UCT(\alpha)$ are the subclasses of uniformly starlike and convex functions of order α (see Bharati et al. [15]);
(*ii*) $TS(0,1,1) = TS$ and $TS(1,1,1) = UCT$ are the subclasses of uniformly starlike and convex functions, respectively. (see Bharati et al. [15]);
(*iii*) $TS(\lambda,\alpha,0) = T^*(\lambda,\alpha)$ (see Altintaş [1]);
(*iv*) $TS(0,\alpha,0) = T^*(\alpha)$ and $TS(1,\alpha,0) = C(\alpha)$ (see Silverman [28]).

Recently, Baricz [3] defined a generalized Bessel function $w_{p,b,c}(z) \equiv w(z)$ as follows:

$$w_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{k!\Gamma\left(k+p+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2k+p} (p,b,c \in \mathbb{C}), \tag{1.5}$$

which is the particular solution of the following second-order linear homogeneous differential equation:

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = 0 \quad (p,b,c \in \mathbb{C}), \tag{1.6}$$

which, in turn, is a natural generalization of the classical Bessel's equation. Solutions of (1.6) are regarded as the generalized Bessel function of order p. The particular solution given by (1.5) is called the generalized Bessel function of the first kind of order p. We also note that the function $w_{p,b,c}(z)$ is generally not univalent in U, although the series defined above is convergent everywhere. Now, we consider the function $u_{p,b,c}(z)$ defined by the following transformation:

$$u_{p,b,c}(z) = 2^p\ \Gamma\left(p + (b+1)/2\right)\ z^{-\frac{p}{2}}\ w_{p,b,c}(\sqrt{z}).$$

By using the well-known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\mu = 0), \\ \lambda(\lambda+1)\ldots(\lambda+\mu-1), & (\mu \in \mathbb{N}), \end{cases}$$

we can express $u_{p,b,c}(z)$ as follows:

$$u_p(z) = u_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c/4)^k}{(m)_k} \frac{z^k}{k!}, \tag{1.7}$$

where for convenience we have chosen $m = p+\frac{b+1}{2} (m \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0,-1,-2,...\})$. Then the function $u_p(z)$ is analytic on $\mathbb{C}$ and satisfies the following second-order linear differential equation:

$$4z^2u''(z) + 2(2p + b + 1)zu'(z) + czu(z) = 0.$$

We will consider two functions; v_p and w_p which are defined in terms of u_p as following:

$$v_p(z) := zu_p(z), \tag{1.8}$$

and

$$w_p(z) := (1 - \mu)\, v_p(z) + \mu z v_p'(z) \quad (\mu \geq 0)\,. \tag{1.9}$$

Using the convolution (or Hadamard product) of two series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ which is defined as the power series

$$(f * g)\,(z) = \sum_{k=0}^{\infty} a_k b_k z^k,$$

we define two linear operators $\mathcal{M}, \mathcal{N} : \mathcal{A} \longrightarrow \mathcal{A}$ by using as following:

$$(\mathcal{M}f)\,(z) := v_p(z) * f(z) = z+\sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(1)_{k-1}} a_k z^k, \tag{1.10}$$

$$(\mathcal{N}f)\,(z) := w_p(z) * f(z) = z+\sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(1)_{k-1}} [1+\mu(k-1)]\, a_k z^k. \tag{1.11}$$

For further result on this transformation of the generalized Bessel function, we refer to the recent papers [4]-[9], where some interesting functional inequalities, integral representations, extensions of some known trigonometric inequalities, starlikeness, convexity and univalence, were established. Recently, Baricz and Frasin [7] and Deniz et al. [9] were interested in the univalence of some integral operators which involved the normalized form of the ordinary Bessel function of the first kind and the normalized form of the generalized Bessel functions of the first kind, respectively. Frasin [18] obtained various sufficient conditions for the convexity and strong convexity of the integral operators defined by the normalized form of the ordinary Bessel function of the first kind. Also, the problem of geometric properties (such as univalence, starlikeness and convexity) of some generalized integral operators has been discussed by many authors [10]-[14], [17], [19], [23] and [29].

Very recently, Cho et al. [16] and Murugusundaramoorthy and Janani [21] (see also Porwal and Dixit [24]) introduced some characterization of generalized Bessel functions of first kind to be in certain subclasses of uniformly starlike and uniformly convex functions. In the present paper, we determine sufficient conditions for certain analytic functions defined by using the generalized Bessel functions. Also, we introduce some necessary and sufficient conditions.

2 Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that, $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \lambda \leq 1$, $\mu \geq 0$, $p, b, c \in \mathbb{C}$, and $m = p + (b+1)/2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

To establish our main results, we shall require the following lemmas:

Lemma 1 ([2]). *The sufficient condition for a function f of the form (1.1) to be in the class $S(\lambda, \alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] [1 + \lambda (k-1)] |a_k| \leq 1 - \alpha. \tag{2.1}$$

Lemma 2 ([2]). *The necessary and sufficient condition for a function f of the form (1.2) to be in the $TS(\lambda, \alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] [1 + \lambda (k-1)] a_k \leq 1 - \alpha. \tag{2.2}$$

Lemma 3 ([6]). *If $p, b, c \in \mathbb{C}$ and $m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, then the function u_p satisfies the following recursive relations:*

$$u_p'(z) = \frac{-c/4}{m} u_{p+1}(z),\ u_p''(z) = \frac{(-c/4)^2}{m(m+1)} u_{p+2}(z),\ u_p'''(z) = \frac{(-c/4)^3}{m(m+1)(m+2)} u_{p+3}(z). \tag{2.3}$$

The first result is to obtain the sufficient condition for $v_p(z)$ to be in the class $S(\lambda, \alpha, \beta)$.

Theorem 1. *Let $c < 0$ and $m > 0$, then $v_p(z) \in S(\lambda, \alpha, \beta)$ if*

$$\lambda(1+\beta) u_p''(1) + [(1+\lambda)(1+\beta) + \lambda(1-\alpha)] u_p'(1) + (1-\alpha)(u_p(1) - 1) \leq 1 - \alpha. \tag{2.4}$$

Proof. Since

$$v_p(z) = z u_p(z) = z + \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} z^k, \tag{2.5}$$

by virtue of Lemma 1, it is suffices to show that

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] [1 + \lambda(k-1)] \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} \leq (1 - \alpha). \tag{2.6}$$

By simple computation and using Lemma 3, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)]\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}\\
=\;&\sum_{k=0}^{\infty}[(k+2)(1+\beta)-(\alpha+\beta)][1+\lambda(k+1)]\frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!}\\
=\;&\sum_{k=0}^{\infty}\left[\lambda(1+\beta)(k+1)^2+[(1+\beta)+\lambda(1-\alpha)](k+1)+(1-\alpha)\right]\frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!}\\
=\;&\lambda(1+\beta)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k-1)!}+[(1+\lambda)(1+\beta)+\lambda(1-\alpha)]\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k)!}\\
&+(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!}\\
=\;&\lambda(1+\beta)\frac{(-c/4)^2}{m(m+1)}\sum_{k=0}^{\infty}\frac{(-c/4)^{k-1}}{(m+2)_{k-1}(k-1)!}\\
&+((1+\lambda)(1+\beta)+\lambda(1-\alpha))\frac{(-c/4)}{m}\sum_{k=0}^{\infty}\frac{(-c/4)^{k}}{(m+1)_{k}(k)!}\\
&+(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!},\\
=\;&\lambda(1+\beta)\tfrac{(-c/4)^2}{m(m+1)}u_{p+2}(1)+[(1+\lambda)(1+\beta)+\lambda(1-\alpha)]\tfrac{(-c/4)}{m}u_{p+1}(1)\\
&+(1-\alpha)(u_p(1)-1)\\
=\;&\lambda(1+\beta)u_p''(1)+[(1+\lambda)(1+\beta)+\lambda(1-\alpha)]u_p'(1)+(1-\alpha)(u_p(1)-1)\,. \qquad (2.7)
\end{aligned}$$

Therefore, we see that the last expression (2.7) is bounded above by $(1-\alpha)$ if (2.4) is satisfied. This completes the proof of Theorem 1.

Taking $\lambda=0$ and $\lambda=1$ in Theorem 1, then we can obtain the sufficient conditions for $v_p(z)$ to be in the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively, as following:

Corollary 1. *Let $c<0$ and $m>0$, then $v_p(z)$ is a $\beta-$uniformly starlike function of order α if the following inequality is satisfied:*

$$(1+\beta)\,u_p'(1)+(1-\alpha)\,u_p(1)\leq 2\,(1-\alpha)\,. \qquad (2.4.a)$$

Remark 1. *The result of Corollary 1 coincides with the result of Cho et al. [16, Theorem 2.1].*

Corollary 2. *Let $c < 0$ and $m > 0$, then $v_p(z)$ is a $\beta-$uniformly convex function of order α if the following inequality is satisfied:*

$$(1+\beta)\, u_p''(1) + (3 + 2\beta - \alpha)\, u_p'(1) + (1-\alpha)\, u_p(1) \leq 2\,(1 - \alpha)\,. \qquad (2.4.b)$$

Moreover, taking $\lambda = \beta = 0$ and $\lambda = 1, \beta = 0$, then we can obtain the sufficient conditions for $v_p(z)$ to be in the subclasses of starlike and convex functions of order α, respectively, as following:

Corollary 3. *Let $c < 0$ and $m > 0$, then $v_p(z)$ is a starlike function of order α if the following inequality is satisfied:*

$$u_p'(1) + (1 - \alpha)\, u_p(1) \leq 2\,(1 - \alpha)\,. \qquad (2.4.c)$$

Corollary 4. *Let $c < 0$ and $m > 0$, then $v_p(z)$ is a convex function of order α if the following inequality is satisfied:*

$$u_p''(1) + (3 - \alpha)\, u_p'(1) + (1-\alpha)\, u_p(1) \leq 2\,(1 - \alpha)\,. \qquad (2.4.d)$$

Theorem 2. *Let $c < 0$ and $m > 0$. If*

$$\widetilde{v}_p(z) := z\left(2 - \frac{v_p(z)}{z}\right), \qquad (2.7)$$

then $\widetilde{v}_p(z) \in TS(\lambda, \alpha, \beta)$ *if and only if, the condition (2.4) is satisfied.*

Proof. Since

$$\widetilde{v}_p(z) = z - \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!} z^k, \qquad (2.8)$$

in view of Lemma 2 and using the same techniques given in the proof of Theorem 1, we have immediately Theorem 2.

Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 2, then we obtain the necessary and sufficient conditions for $\widetilde{v}_p(z)$ to be in the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively, as following:

Corollary 5. *Let $c < 0$ and $m > 0$, then $\widetilde{v}_p(z)$ is a $\beta-$uniformly starlike function of order α if and only if the inequality (2.4.a) is satisfied.*

Remark 2. *The result of Corollary 5 coincides with the result of Murugusundaramoorthy and Janani [21, Theorem 5, with $\lambda = 0$], also, in case of* $\lambda = \beta = 0$ we have the result of Porwal and Dixit [24, Theorem 5, with $\lambda = 0$] (see also Baricz [6, Theorem 2.14, page 51]).

Corollary 6. *Let $c < 0$ and $m > 0$, then $\widetilde{v}_p(z)$ is a $\beta-$uniformly convex function of order α if and only if the inequality (2.4.b) is satisfied.*

Moreover, taking $\lambda = \beta = 0$ and $\lambda = 1, \beta = 0$ in Theorem 2, then we get the necessary and sufficient conditions for $\widetilde{v}_p(z)$ to be in the subclasses of starlike and convex functions of order α, respectively, as following:

Corollary 7. *Let $c < 0$ and $m > 0$, then $\widetilde{v}_p(z)$ is a starlike function of order α if and only if the inequality (2.4.c) is satisfied.*
Corollary 8. *Let $c < 0$ and $m > 0$, then $\widetilde{v}_p(z)$ is a convex function of order α if and only if the inequality (2.4.d) is satisfied.*

Also, in the next theorem we obtain the sufficient condition for $w_p(z)$ to be in the class $S(\lambda, \alpha, \beta)$.

Theorem 3. *Let $c < 0$ and $m > 0$, then $w_p(z) \in S(\lambda, \alpha, \beta)$ if*

$$\begin{aligned} &\lambda\mu\,(1+\beta)\,u_p'''(1) + \Big[\lambda\,(3\mu+1)\,(1+\beta) + \mu\,(\lambda\,(1-\alpha) + (1-\beta))\Big]\,u_p''(1) \\ &+ \Big[\lambda\,(\mu{+}1)\,(2{+}\beta{-}\alpha) + \mu\,(2{-}\alpha{-}\beta) + (1{+}\beta)\Big]\,u_p'(1) \\ &+ (1{-}\alpha)\,(u_p(1){-}1) \leq 1{-}\alpha. \end{aligned} \tag{2.9}$$

Proof. By virtue of Lemma 1, it is suffices to show that

$$\sum_{k=2}^{\infty} [k\,(1+\beta) - (\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(1)_{k-1}}\,[1+\mu(k-1)] \leq (1-\alpha).$$

By simple computation and using Lemma 3, we obtain

$$\sum_{k=2}^{\infty} [k\,(1+\beta) - (\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!}\,[1+\mu(k-1)]$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} [(k+2)\,(1+\beta) - (\alpha+\beta)]\,[1+\lambda\,(k+1)]\,\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k+1)!}\,[1+\mu(k+1)] \\ &= \sum_{k=0}^{\infty} \Big[\Big(\lambda\mu\,(1+\beta)\Big)\,(k+1)^3 + \Big(\lambda\,(1+\beta) + \mu\,\Big(\lambda\,(1-\alpha) + (1-\beta)\Big)\Big)\,(k+1)^2 \\ &\qquad + \Big(\,(\lambda+\mu)\,(1-\alpha) + (1+\beta)\Big)\,(k+1) + (1-\alpha)\Big]\,\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k+1)!} \end{aligned}$$

$$
\begin{aligned}
= \; & \lambda\mu\,(1+\beta)\left[\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k-2)!}+3\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k-1)!}+\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k)!}\right]\\
& +\Big(\lambda\,(1+\beta)+\mu\,\Big(\lambda\,(1-\alpha)+(1-\beta)\Big)\Big)\left[\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k-1)!}+\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k)!}\right]\\
& +\Big(\,(\lambda+\mu)\,(1-\alpha)+(1+\beta)\Big)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k)!}+(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k+1)!}\\
= \; & \lambda\mu\,(1+\beta)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k-2)!}\\
& +\Big(\lambda\,(1+\beta)\,(3\mu+1)+\mu\,\Big(\lambda\,(1-\alpha)+(1-\beta)\Big)\Big)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k-1)!}\\
& +\Big(\lambda\,(1+\mu)\,(2+\beta-\alpha)+\mu\,(2-\alpha-\beta)+(1+\beta)\Big)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k)!}\\
& +(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k+1)!},\\
= \; & \lambda\mu\,(1+\beta)\,\frac{(-c/4)^{3}}{m\,(m+1)\,(m+2)}\sum_{k=0}^{\infty}\frac{(-c/4)^{k-2}}{(m+3)_{k-2}\,(k-2)!}\\
& +\Big(\lambda\,(1+\beta)\,(3\mu+1)+\mu\,\Big(\lambda\,(1-\alpha)+(1-\beta)\Big)\Big)\frac{(-c/4)^{2}}{m\,(m+1)}\sum_{k=0}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!}\\
& +\Big(\lambda\,(1+\mu)\,(2+\beta-\alpha)+\mu\,(2-\alpha-\beta)+(1+\beta)\Big)\frac{(-c/4)}{m}\sum_{k=0}^{\infty}\frac{(-c/4)^{k}}{(m+1)_{k}\,(k)!}\\
& +(1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k+1)!}\\
= \; & \lambda\mu\,(1+\beta)\,\frac{(-c/4)^{3}}{m\,(m+1)\,(m+2)}u_{p+3}(1)\\
& +\Big(\lambda\,(1+\beta)\,(3\mu+1)+\mu\,\Big(\lambda\,(1-\alpha)+(1-\beta)\Big)\Big)\frac{(-c/4)^{2}}{m\,(m+1)}u_{p+2}(1)\\
& +\Big(\lambda\,(1+\mu)\,(2+\beta-\alpha)+\mu\,(2-\alpha-\beta)+(1+\beta)\Big)\frac{(-c/4)}{m}u_{p+1}(1)+(1-\alpha)\,(u_p(1)-1)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \lambda\mu\,(1+\beta)\,u_p'''(1)+\Big(\lambda\,(1+\beta)\,(3\mu+1)+\mu\,\Big(\lambda\,(1-\alpha)+(1-\beta)\Big)\Big)\,u_p''(1)\\
& +\Big(\lambda\,(1+\mu)\,(2+\beta-\alpha)+\mu\,(2-\alpha-\beta)+(1+\beta)\Big)\,u_p'(1)\\
& +(1-\alpha)\,(u_p(1)-1)\,. \qquad (2.10)
\end{aligned}
$$

Therefore, we see that the last expression (2.10) is bounded above by $(1-\alpha)$ if (2.9) is satisfied. The proof of Theorem 3 is completed.

Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 3, then we obtain the sufficient conditions for $w_p(z)$ to be in the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively, as following:
Corollary 9. *Let $c < 0$ and $m > 0$, then $w_p(z)$ is a $\beta-$uniformly starlike function of order α if the following inequality is satisfied:*

$$\mu\,(1-\beta)\,u_p''(1) + \Big(\mu\,(2-\alpha-\beta) + (1+\beta)\Big)\,u_p'(1) + (1-\alpha)\,u_p(1) \le 2\,(1-\alpha)\,. \tag{2.9.a}$$

Corollary 10. *Let $c < 0$ and $m > 0$, then $w_p(z)$ is a $\beta-$uniformly convex function of order α if the following inequality is satisfied:*

$$\begin{aligned} &\mu\,(1+\beta)\,u_p'''(1) + \Big((3\mu+1)\,(1+\beta) + \mu\,(2-\alpha-\beta)\Big)\,u_p''(1) \\ &+\Big((\mu{+}1)\,(2{+}\beta{-}\alpha) + \mu\,(2{-}\alpha{-}\beta) + (1{+}\beta)\Big)\,u_p'(1) + (1{-}\alpha)\,u_p(1) \\ \le\;& 2\,(1{-}\alpha)\,. \end{aligned} \tag{2.9.b}$$

Also, taking $\lambda = \beta = 0$ and $\lambda = 1, \beta = 0$, then we can obtain the sufficient conditions for $w_p(z)$ to be in the subclasses of starlike and convex functions of order α, respectively, as following:
Corollary 11. *Let $c < 0$ and $m > 0$, then $w_p(z)$ is a starlike function of order α if the following inequality is satisfied:*

$$\mu u_p''(1) + \Big(\mu\,(2-\alpha) + 1\Big)\,u_p'(1) + (1-\alpha)\,u_p(1) \le 2\,(1-\alpha)\,. \tag{2.9.c}$$

Corollary 12. *Let $c < 0$ and $m > 0$, then $w_p(z)$ is a convex function of order α if the following inequality is satisfied:*

$$\begin{aligned} &\mu u_p'''(1) + \Big(\mu\,(5-\alpha) + 1\Big)\,u_p''(1) + \Big((2\mu{+}1)\,(2{-}\alpha) + 1\Big)\,u_p'(1) + (1{-}\alpha)\,u_p(1) \\ \le\;& 2\,(1{-}\alpha)\,. \end{aligned} \tag{2.9.d}$$

Also, considering the same technique of Theorem 2, we can obtain the following theorem and the proof is omitted.
Theorem 4. *Let $c < 0$ and $m > 0$. If*

$$\widetilde{w}_p(z) := z\left(2 - \frac{w_p(z)}{z}\right), \tag{2.11}$$

then $\widetilde{w}_p(z) \in TS(\lambda, \alpha, \beta)$, if and only if the condition (2.9) is satisfied.

Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 4, then we obtain the necessary and sufficient conditions for $\widetilde{w}_p(z)$ to be in the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively, as following:

Corollary 13. *Let $c < 0$ and $m > 0$, then $\widetilde{w}_p(z)$ is a $\beta-$uniformly starlike function of order α if and only if the inequality (2.9.a) is satisfied.*

Corollary 14. *Let $c < 0$ and $m > 0$, then $\widetilde{w}_p(z)$ is a $\beta-$uniformly convex function of order α if and only if the inequality (2.9.b) is satisfied.*

Moreover, taking $\lambda = \beta = 0$ and $\lambda = 1, \beta = 0$ in Theorem 2, then we get the necessary and sufficient conditions for $\widetilde{w}_p(z)$ to be in the subclasses of starlike and convex functions of order α, respectively, as following:

Corollary 15. *Let $c < 0$ and $m > 0$, then $\widetilde{w}_p(z)$ is a starlike function of order α if and only if the inequality (2.9.c) is satisfied.*

Corollary 16. *Let $c < 0$ and $m > 0$, then $\widetilde{w}_p(z)$ is a convex function of order α if and only if the inequality (2.9.d) is satisfied.*

Now, we introduce necessary and sufficient conditions of the functions $w_p(z)$ and $v_p(z)$ to be in the class $TS(\lambda, \alpha, \beta)$.

Theorem 5. *Let $c < 0$ and $-1 < m < 0$, then $v_p(z) \in TS(\lambda, \alpha, \beta)$ if and only if*

$$\lambda (1 + \beta) u_p''(1) + [(1 + \lambda)(1 + \beta) + \lambda (1 - \alpha)] u_p'(1) + (1 - \alpha) u_p(1) \geq 0. \quad (2.12)$$

Proof. Using (2.5) in case of $c < 0$ *and* $-1 < m < 0$, *then*

$$v_p(z) = z - \frac{1}{|m|} \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m+1)_{k-2} (k-1)!} z^k.$$

By virtue of Lemma 2, we have

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] [1 + \lambda (k-1)] \frac{(-c/4)^{k-1}}{(m+1)_{k-2} (k-1)!} \leq |m| (1-\alpha). \quad (2.13)$$

By simple computation and using Lemma 3, we have

$$\begin{aligned} & \sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] [1 + \lambda (k-1)] \frac{(-c/4)^{k-1}}{(m+1)_{k-2} (k-1)!} \\ = & \sum_{k=0}^{\infty} \left[\lambda (1+\beta)(k+1)^2 + [(1+\beta) + \lambda (1-\alpha)](k+1) + (1-\alpha)\right] \frac{(-c/4)^{k+1}}{(m+1)_k (k+1)!} \\ = & \lambda (1+\beta) \left[\sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m+1)_k (k-1)!} + \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m+1)_k k!}\right] \\ & + [(1+\beta) + \lambda (1-\alpha)] \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m+1)_k (k)!} + (1-\alpha) \sum_{k=0}^{\infty} \frac{(-c/4)^{k+1}}{(m+1)_k (k+1)!} \end{aligned}$$

$$\begin{aligned}
&= \lambda m\,(1+\beta)\,\frac{(-c/4)^2}{m\,(m+1)}\sum_{k=0}^{\infty}\frac{(-c/4)^{k-1}}{(m+2)_{k-1}\,(k-1)!} \\
&\quad +m\left[\,(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]\frac{(-c/4)}{m}\sum_{k=0}^{\infty}\frac{(-c/4)^{k}}{(m{+}1)_{k}\,(k)!} \\
&\quad +m\,(1{-}\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}\,(k{+}1)!} \\
&= \lambda m\,(1{+}\beta)\,\frac{(-c/4)^2}{m\,(m+1)}u_{p+2}(1)+m\left[(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]\frac{(-c/4)}{m}u_{p+1}(1) \\
&\quad +m\,(1{-}\alpha)\,(u_p(1){-}1) \\
&= m\left[\lambda\,(1{+}\beta)\,u_p''(1){+}\left[(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]u_p'(1){+}\,(1{-}\alpha)\,u_p(1)\right] \\
&\quad -m\,(1{-}\alpha)\,. \qquad (2.14)
\end{aligned}$$

Substituting from (2.14) into (2.13), then we have

$$\begin{aligned}
& m\left[\lambda\,(1{+}\beta)\,u_p''(1)+\left[(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]u_p'(1)+(1{-}\alpha)\,u_p(1)\right]-m\,(1{-}\alpha) \\
= \;& m\left[\lambda\,(1{+}\beta)\,u_p''(1)+\left[(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]u_p'(1)+(1{-}\alpha)\,u_p(1)\right]+|m|\,(1{-}\alpha) \\
\leq \;& |m|\,(1-\alpha)\,,
\end{aligned}$$

or

$$m\left[\lambda\,(1{+}\beta)\,u_p''(1)+\left[(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]u_p'(1)+(1{-}\alpha)\,u_p(1)\right]\leq 0,$$

then

$$\lambda\,(1{+}\beta)\,u_p''(1)+\left[(1{+}\lambda)\,(1{+}\beta)+\lambda\,(1{-}\alpha)\right]u_p'(1)+(1{-}\alpha)\,u_p(1)\geq 0.$$

This completes the proof of Theorem 5.

Putting $\lambda = 0$ and $\lambda = 1$ in Theorem 5, then we obtain the necessary and sufficient conditions for $v_p(z)$ to be in the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively, as following:

Corollary 17. *Let $c < 0$ and $-1 < m < 0$, then $v_p(z)$ is a $\beta-$uniformly starlike function of order α if and only if the following inequality is satisfied*

$$(1{+}\beta)\,u_p'(1)+(1{-}\alpha)\,u_p(1)\geq 0.$$

Corollary 18. *Let $c < 0$ and $-1 < m < 0$, then $v_p(z)$ is a $\beta-$uniformly convex function of order α if and only if the following inequality is satisfied*

$$(1{+}\beta)\,u_p''(1)+(3+2\beta-\alpha)\,u_p'(1)+(1{-}\alpha)\,u_p(1)\geq 0.$$

Similarly, taking $\lambda=\beta=0$ and $\lambda=1,\beta=0$ in Theorem 5, then we obtain the necessary and sufficient conditions for $v_p(z)$ to be in the subclasses of starlike and convex functions of order α, respectively, as following:

Corollary 19. *Let $c<0$ and $-1<m<0$, then $v_p(z)$ is a starlike function of order α if and only if the following inequality is satisfied.*

$$u_p'(1)+(1-\alpha)\,u_p(1)\geq 0.$$

Corollary 20. *Let $c<0$ and $-1<m<0$, then $v_p(z)$ is a convex function of order α if and only if the following inequality is satisfied.*

$$u_p''(1)+(3-\alpha)\,u_p'(1)+(1-\alpha)\,u_p(1)\geq 0.$$

Theorem 6. *Let $c<0$ and $-1<m<0$, then $w_p(z)\in TS(\lambda,\alpha,\beta)$ if and only if*

$$\lambda\mu\,(1+\beta)\,u_p'''(1)+\Big(\lambda\,(1+\beta)\,(3\mu+1)+\mu\,\Big(\lambda\,(1-\alpha)+(1-\beta)\Big)\Big)\,u_p''(1)$$

$$+\Big(\lambda\,(1+\mu)\,(2+\beta-\alpha)+\mu\,(2-\alpha-\beta)+(1+\beta)\Big)\,u_p'(1)+(1-\alpha)\,u_p(1)\geq 0. \tag{2.15}$$

Proof. Using (1.9) in case of $c<0$ *and* $-1<m<0$, *then*

$$w_p(z)=z-\frac{1}{|m|}\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m+1)_{k-2}\,(k-1)!}\,[1+\mu(k-1)]\,z^k.$$

By virtue of Lemma 2, we have

$$\sum_{k=2}^{\infty}[k\,(1+\beta)-(\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m+1)_{k-2}\,(k-1)!}\,[1+\mu(k-1)]\leq|m|\,(1-\alpha)\,. \tag{2.16}$$

A Simple computation, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\lambda\left(k-1\right)\right]\frac{\left(-c/4\right)^{k-1}}{\left(m+1\right)_{k-2}\left(k-1\right)!}\left[1+\mu(k-1)\right]\\
=\;&\sum_{k=0}^{\infty}\left[\left(k+2\right)\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\lambda\left(k+1\right)\right]\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k+1\right)!}\left[1+\mu(k+1)\right]\\
=\;&\sum_{k=0}^{\infty}\Bigg[\Big(\lambda\mu\left(1+\beta\right)\Big)\left(k+1\right)^{3}+\Big(\lambda\left(1+\beta\right)+\mu\Big(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\Big)\Big)\left(k+1\right)^{2}\\
&\qquad+\Big(\left(\lambda+\mu\right)\left(1-\alpha\right)+\left(1+\beta\right)\Big)\left(k+1\right)+\left(1-\alpha\right)\Bigg]\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k+1\right)!}\\
=\;&\lambda\mu\left(1+\beta\right)\left[\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k-2\right)!}+3\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k-1\right)!}+\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k\right)!}\right]\\
&+\Big(\lambda\left(1+\beta\right)+\mu\Big(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\Big)\Big)\left[\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k-1\right)!}+\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k\right)!}\right]\\
&+\Big(\left(\lambda+\mu\right)\left(1-\alpha\right)+\left(1+\beta\right)\Big)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k\right)!}+\left(1-\alpha\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k+1\right)!}\\
=\;&\lambda\mu\left(1+\beta\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k-2\right)!}\\
&+\Big(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\Big(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\Big)\Big)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k-1\right)!}\\
&+\Big(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\Big)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k\right)!}\\
&+\left(1-\alpha\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m+1\right)_{k}\left(k+1\right)!}\\
=\;&\lambda\mu\left(1+\beta\right)\frac{\left(-c/4\right)^{3}}{\left(m+1\right)\left(m+2\right)}\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k-2}}{\left(m+3\right)_{k-2}\left(k-2\right)!}\\
&+\Big(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\Big(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\Big)\Big)\frac{\left(-c/4\right)^{2}}{\left(m+1\right)}\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k-1}}{\left(m+2\right)_{k-1}\left(k-1\right)!}\\
&+\Big(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\Big)\frac{\left(-c/4\right)}{1}\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k}}{\left(m+1\right)_{k}\left(k\right)!}\\
&+m\left(1-\alpha\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k+1\right)!}
\end{aligned}$$

$$\begin{aligned}
&= \lambda\mu m\left(1+\beta\right)\frac{(-c/4)^3}{m\left(m+1\right)\left(m+2\right)}u_{p+3}(1)\\
&\quad +m\left(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\right)\right)\frac{(-c/4)^2}{m\left(m+1\right)}u_{p+2}(1)\\
&\quad +m\left(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)\frac{(-c/4)}{m}u_{p+1}(1)+m\left(1-\alpha\right)\left(u_p(1)-1\right)\\
&= \lambda\mu m\left(1+\beta\right)u_p'''(1)+m\left(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\right)\right)u_p''(1)\\
&\quad +m\left(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)u_p'(1)\\
&\quad +m\left(1-\alpha\right)\left(u_p(1)-1\right). \qquad (2.17)
\end{aligned}$$

Substituting from (2.17) into (2.16), then we have

$$\begin{aligned}
&\lambda\mu m\left(1+\beta\right)u_p'''(1)+m\left(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\right)\right)u_p''(1)\\
&\quad +m\left(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)u_p'(1)+m\left(1-\alpha\right)\left(u_p(1)-1\right).\\
&= m\Big[\lambda\mu\left(1+\beta\right)u_p'''(1)+\left(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\right)\right)u_p''(1)\\
&\quad +\left(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)u_p'(1)+\left(1-\alpha\right)u_p(1)\Big]+|m|\left(1-\alpha\right)\\
&\le |m|\left(1-\alpha\right),
\end{aligned}$$

or

$$\begin{aligned}
&m\Big[\lambda\mu\left(1+\beta\right)u_p'''(1)+\left(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\right)\right)u_p''(1)\\
&\quad +\left(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)u_p'(1)+\left(1-\alpha\right)u_p(1)\Big]\le 0,
\end{aligned}$$

then

$$\begin{aligned}
&\lambda\mu\left(1+\beta\right)u_p'''(1)+\left(\lambda\left(1+\beta\right)\left(3\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\right)\right)u_p''(1)\\
&\quad +\left(\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)u_p'(1)+\left(1-\alpha\right)u_p(1)\ge 0.
\end{aligned}$$

This completes the proof of Theorem 6.

Putting $\lambda = 0$ and $\lambda = 1$ in Theorem 6, then we obtain the necessary and sufficient conditions for $w_p(z)$ to be in the subclasses of $\beta-$uniformly starlike and convex functions of order α, respectively, as following:

Corollary 21. *Let $c < 0$ and $-1 < m < 0$, then $w_p(z)$ is a $\beta-$uniformly starlike function of order α if and only if the following inequality is satisfied*

$$\mu\left(1-\beta\right)u_p''(1)+\left(\mu\left(2-\alpha-\beta\right)+\left(1+\beta\right)\right)u_p'(1)+\left(1-\alpha\right)u_p(1)\ge 0.$$

Corollary 22. *Let $c < 0$ and $-1 < m < 0$, then $w_p(z)$ is a $\beta-$uniformly convex function of order α if and only if the following inequality is satisfied*

$$\mu(1+\beta)\,u_p'''(1) + \Big((3\mu+1)(1+\beta) + \mu(2-\alpha-\beta)\Big)\,u_p''(1)$$

$$+\Big((1+\mu)(2+\beta-\alpha) + \mu(2-\alpha-\beta) + (1+\beta)\Big)\,u_p'(1) + (1-\alpha)\,u_p(1) \geq 0.$$

Also, putting $\beta = \lambda = 0$ and $\beta = 0, \lambda = 1$ in Theorem 6, then we obtain the necessary and sufficient conditions for $w_p(z)$ to be in the subclasses of starlike and convex functions of order α, respectively, as following:

Corollary 23. *Let $c < 0$ and $-1 < m < 0$, then $w_p(z)$ is a starlike function of order α if and only if the following inequality is satisfied.*

$$\mu u_p''(1) + (1 + \mu(2-\alpha))\,u_p'(1) + (1-\alpha)\,u_p(1) \geq 0.$$

Corollary 24. *Let $c < 0$ and $-1 < m < 0$, then $w_p(z)$ is a convex function of order α if and only if the following inequality is satisfied.*

$$\mu u_p'''(1) + (1 + \mu(5-\alpha))\,u_p''(1) + \Big(1 + (1+2\mu)(2-\alpha)\Big)\,u_p'(1) + (1-\alpha)\,u_p(1) \geq 0.$$

3 Further Results

Using Bieberbach conjecture (if f is defined by (1.1), then $|a_k| \leq k$ for all $f \in S^*$ and $|a_k| \leq 1$ for all $f \in K$), we introduce further results due to the operators $\mathcal{M}$ and $\mathcal{N}$.

Theorem 7. *Let $c < 0$ and $m > 0$. If the following inequality is satisfied*

$$\begin{aligned}&\lambda(1+\beta)\,u_p'''(1) + \Big((1+4\lambda)(1+\beta) + \lambda(1-\alpha)\Big)\,u_p''(1)\\ &+\Big(2(1+\lambda)(1+\beta) + (1+2\lambda)(1-\alpha)\Big)\,u_p'(1) + (1-\alpha)(u_p(1) - 1)\\ \leq\ & 1-\alpha, \end{aligned} \tag{3.1}$$

then $(\mathcal{M}f)(z)$ maps the class S^* into the class $S(\lambda,\alpha,\beta)$.

Proof. Since

$$(\mathcal{M}f)(z) = z + \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!} a_k z^k,$$

by virtue of Lemma 1, it is suffices to show that

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)]\,[1 + \lambda(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}\,|a_k| \leq (1-\alpha), \quad (f \in S^*)\,. \tag{3.2}$$

By using Bieberbach conjecture ($f \in S^* \Longrightarrow |a_k| \leq k$) and some computations, we have

$$\Phi(\alpha,\beta,\lambda,c,m) = \sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)] \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}|a_k|$$

$$\begin{aligned}
&\leq \sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)][1+\lambda(k-1)] \frac{(-c/4)^{k-1}}{(m)_{k-1}(k-1)!}k \\
&= \sum_{k=0}^{\infty} [(k+2)(1+\beta)-(\alpha+\beta)][1+\lambda(k+1)] \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!}(k+2) \\
&= \sum_{k=0}^{\infty} \left[\lambda(1+\beta)(k+1)^2+[(1+\beta)+\lambda(1-\alpha)](k+1)+(1-\alpha)\right] \frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!}(k+2) \\
&= \lambda(1+\beta)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k-2)!} + \Big[(1+4\lambda)(1+\beta)+\lambda(1-\alpha)\Big]\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k-1)!} \\
&\quad + \Big(2(1+\lambda)(1+\beta)+(1+2\lambda)(1-\alpha)\Big)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}k!} + (1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!} \\
&= \lambda(1+\beta)\frac{(-c/4)^3}{m(m+1)(m+2)}\sum_{k=0}^{\infty}\frac{(-c/4)^{k-2}}{(m+3)_{k-2}(k-2)!} \\
&\quad + \Big[(1+4\lambda)(1+\beta)+\lambda(1-\alpha)\Big]\frac{(-c/4)^2}{m(m+1)}\sum_{k=0}^{\infty}\frac{(-c/4)^{k-1}}{(m+2)_{k-1}(k-1)!} \\
&\quad + \Big(2(1+\lambda)(1+\beta)+(1+2\lambda)(1-\alpha)\Big)\frac{(-c/4)}{m}\sum_{k=0}^{\infty}\frac{(-c/4)^k}{(m+1)_k k!} + (1-\alpha)\sum_{k=0}^{\infty}\frac{(-c/4)^{k+1}}{(m)_{k+1}(k+1)!}
\end{aligned}$$

$$\begin{aligned}
&= \lambda(1+\beta)\frac{(-c/4)^3}{m(m+1)(m+2)}u_{p+3}(1) + \Big[(1+4\lambda)(1+\beta)+\lambda(1-\alpha)\Big]\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1) \\
&\quad + \Big(2(1+\lambda)(1+\beta)+(1+2\lambda)(1-\alpha)\Big)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\alpha)(u_p(1)-1) \\
&= \lambda(1+\beta)u_p'''(1) + \Big((1+4\lambda)(1+\beta)+\lambda(1-\alpha)\Big)u_p''(1) \\
&\quad + \Big(2(1+\lambda)(1+\beta)+(1+2\lambda)(1-\alpha)\Big)u_p'(1) + (1-\alpha)(u_p(1)-1)
\end{aligned}$$

Therefore, we see that the expression $\Phi(\alpha,\beta,\lambda,c,m)$ is bounded above by $(1-\alpha)$ if (3.1) is satisfied. This completes the proof of Theorem 7.

Theorem 8. *Let $c < 0$ and $m > 0$. If the inequality* (2.4) *holds true, then* $(\mathcal{M}f)(z)$ *maps the class K into the class $S(\lambda,\alpha,\beta)$.*

Proof. Similarly, by virtue of Lemma 1, it is suffices to show that

$$\sum_{k=2}^{\infty} [k\,(1+\beta)-(\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!}\,|a_k| \leq (1-\alpha), \quad (f\in K)\,. \tag{3.3}$$

By using Bieberbach conjecture ($f \in K \Longrightarrow |a_k| \leq 1$) and some computations, we have

$$\Phi(\alpha,\beta,\lambda,c,m) = \sum_{k=2}^{\infty} [k\,(1+\beta)-(\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!}\,|a_k|$$
$$\leq \sum_{k=2}^{\infty} [k\,(1+\beta)-(\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!},$$

which is exactly the left hand side of (2.6). Therefore, we see that the expression $\Phi(\alpha,\beta,\lambda,c,m)$ is bounded above by $(1-\alpha)$ if (2.4) is satisfied and the proof of Theorem 8 is now completed.

Theorem 9. *Let $c<0$ and $m>0$. If the following inequality*

$$\begin{aligned}&\lambda\mu\,(1{+}\beta)\,u_p^{(4)}(1) + \Big(\lambda\,(1{+}\beta)\,(7\mu{+}1) + \mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)\Big)\,u_p'''(1)\\&+\Big(2\lambda\,(1{+}\beta)\,(5\mu{+}2) + 4\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta) + (\lambda{+}\mu)\,(1{-}\alpha)+1{+}\beta\Big)\,u_p''(1)\\&+\Big(\lambda\,(1{+}\beta)\,(\mu{+}1) + \mu\,(\lambda\,(1{-}\alpha)+1{-}\beta) + (\lambda{+}\mu{+}1)\,(1{-}\alpha) + \lambda\,(1{+}\mu)\,(2{+}\beta{-}\alpha)\\&\quad+2\,(1{+}\beta)\Big)\,u_p'(1) + (1{-}\alpha)\,\Big(u_p(1)-1\Big) \leq 1-\alpha,\end{aligned} \tag{3.4}$$

is satisfied, then $(\mathcal{N}f)\,(z)$ maps the class S^* to the class $S(\lambda,\alpha,\beta)$.
Proof. Since

$$(\mathcal{N}f)\,(z) = z + \sum_{k=2}^{\infty} \frac{(-c/4)^{k-1}}{(m)_{k-1}\,(1)_{k-1}}\,[1+\mu(k-1)]\,a_k z^k.$$

Also, by virtue of Lemma 1, it is suffices to show that

$$\sum_{k=2}^{\infty} [k\,(1{+}\beta)-(\alpha{+}\beta)]\,[1{+}\lambda\,(k{-}1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(1)_{k-1}}\,[1{+}\mu(k{-}1)]\,|a_k| \leq (1{-}\alpha);\, f\in S^*.$$

Using Bieberbach conjecture ($f \in S^* \Longrightarrow |a_k| \leq k$) and some computations, we obtain

$$\begin{aligned}
&\Psi(\alpha,\beta,\lambda,\mu,c,m)\\
= &\sum_{k=2}^{\infty}\left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\lambda\left(k-1\right)\right]\frac{\left(-c/4\right)^{k-1}}{\left(m\right)_{k-1}\left(1\right)_{k-1}}\left[1+\mu(k-1)\right]|a_k|\\
\leq &\sum_{k=2}^{\infty}\left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\lambda\left(k-1\right)\right]\frac{\left(-c/4\right)^{k-1}}{\left(m\right)_{k-1}\left(1\right)_{k-1}}\left[1+\mu(k-1)\right]k\\
= &\sum_{k=0}^{\infty}\left[\left(k+2\right)\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\lambda\left(k+1\right)\right]\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k+1\right)!}\left[1+\mu(k+1)\right]\left(k+2\right)\\
= &\sum_{k=0}^{\infty}\left[\Big(\lambda\mu\left(1+\beta\right)\Big)\left(k+1\right)^3+\Big(\lambda\left(1+\beta\right)+\mu\Big(\lambda\left(1-\alpha\right)+\left(1-\beta\right)\Big)\Big)\left(k+1\right)^2\right.\\
&\left.\quad+\Big(\left(\lambda+\mu\right)\left(1-\alpha\right)+\left(1+\beta\right)\Big)\left(k+1\right)+\left(1-\alpha\right)\right]\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k+1\right)!}\left(k+2\right)\\
= &\lambda\mu\left(1+\beta\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k-3\right)!}\\
&+\Big(\lambda\left(1+\beta\right)\left(7\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+1-\beta\right)\Big)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k-2\right)!}\\
&+\Big(2\lambda\left(1+\beta\right)\left(5\mu+2\right)+4\mu\left(\lambda\left(1-\alpha\right)+1-\beta\right)+\left(\lambda+\mu\right)\left(1-\alpha\right)+1+\beta\Big)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k-1\right)!}\\
&+\Big(\lambda\left(1+\beta\right)\left(\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+1-\beta\right)+\left(\lambda+\mu+1\right)\left(1-\alpha\right)+\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)\\
&+2\left(1+\beta\right)\Big)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}k!}+\left(1-\alpha\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k+1\right)!}\\
= &\lambda\mu\left(1+\beta\right)\frac{\left(-c/4\right)^{4}}{m\left(m+1\right)\left(m+2\right)\left(m+3\right)}\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k-3}}{\left(m+4\right)_{k-3}\left(k-3\right)!}\\
&+\Big(\lambda\left(1+\beta\right)\left(7\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+1-\beta\right)\Big)\frac{\left(-c/4\right)^{3}}{m\left(m+1\right)\left(m+2\right)}\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k-2}}{\left(m+3\right)_{k-2}\left(k-2\right)!}\\
&+\Big(2\lambda\left(1+\beta\right)\left(5\mu+2\right)+4\mu\left(\lambda\left(1-\alpha\right)+1-\beta\right)+\left(\lambda+\mu\right)\left(1-\alpha\right)+1+\beta\Big)\frac{\left(-c/4\right)^{2}}{m\left(m+1\right)}\\
&\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k-1}}{\left(m\right)_{k-1}\left(k-1\right)!}+\Big(\lambda\left(1+\beta\right)\left(\mu+1\right)+\mu\left(\lambda\left(1-\alpha\right)+1-\beta\right)+\left(\lambda+\mu+1\right)\left(1-\alpha\right)\\
&+\lambda\left(1+\mu\right)\left(2+\beta-\alpha\right)+2\left(1+\beta\right)\Big)\frac{\left(-c/4\right)}{m}\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k}}{\left(m+1\right)_{k}k!}+\left(1-\alpha\right)\sum_{k=0}^{\infty}\frac{\left(-c/4\right)^{k+1}}{\left(m\right)_{k+1}\left(k+1\right)!}
\end{aligned}$$

$$\begin{aligned}
&= \lambda\mu\,(1{+}\beta)\,\frac{(-c/4)^4}{m\,(m+1)\,(m+2)\,(m+3)}u_{p+4}(1)\\
&\quad+\Big(\lambda\,(1{+}\beta)\,(7\mu{+}1)+\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)\Big)\,\frac{(-c/4)^3}{m\,(m+1)\,(m+2)}u_{p+3}(1)\\
&\quad+\Big(2\lambda\,(1{+}\beta)\,(5\mu{+}2)+4\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)+(\lambda{+}\mu)\,(1{-}\alpha)+1{+}\beta\Big)\,\frac{(-c/4)^2}{m\,(m{+}1)}u_{p+2}(1)\\
&\quad+\Big(\lambda\,(1{+}\beta)\,(\mu{+}1)+\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)+(\lambda{+}\mu{+}1)\,(1{-}\alpha)+\lambda\,(1{+}\mu)\,(2{+}\beta{-}\alpha)\\
&\quad+2\,(1{+}\beta)\Big)\,\frac{(-c/4)}{m}u_{p+1}(1)+(1{-}\alpha)\,\Big(u_p(1)-1\Big)\\
&= \lambda\mu\,(1{+}\beta)\,u_p^{(4)}(1)+\Big(\lambda\,(1{+}\beta)\,(7\mu{+}1)+\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)\Big)\,u_p'''(1)\\
&\quad+\Big(2\lambda\,(1{+}\beta)\,(5\mu{+}2)+4\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)+(\lambda{+}\mu)\,(1{-}\alpha)+1{+}\beta\Big)\,u_p''(1)\\
&\quad+\Big(\lambda\,(1{+}\beta)\,(\mu{+}1)+\mu\,(\lambda\,(1{-}\alpha)+1{-}\beta)+(\lambda{+}\mu{+}1)\,(1{-}\alpha)+\lambda\,(1{+}\mu)\,(2{+}\beta{-}\alpha)\\
&\qquad+2\,(1{+}\beta)\Big)\,u_p'(1)+(1{-}\alpha)\,\Big(u_p(1)-1\Big),
\end{aligned}$$

We conclude that the expression $\Psi(\alpha,\beta,\lambda,\mu,c,m)$ is bounded above by $(1-\alpha)$ if (3.4) is satisfied. This completes the proof of Theorem 9.

Theorem 10. *Let* $c<0$ *and* $m>0$. *If the inequality* (2.9) *holds true, then* $(\mathcal{N}f)(z)$ *maps the class* K *into the class* $S(\lambda,\alpha,\beta)$.
Proof. By virtue of Lemma 1, it is suffices to show that

$$\sum_{k=2}^{\infty}[k\,(1{+}\beta)-(\alpha{+}\beta)]\,[1{+}\lambda\,(k{-}1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(1)_{k-1}}\,[1{+}\mu(k{-}1)]\,|a_k|\le(1{-}\alpha);\;\; f\in K. \tag{3.5}$$

By using Bieberbach conjecture ($f\in K \Longrightarrow |a_k|\le 1$) and some computations, we have

$$\begin{aligned}
&\Psi(\alpha,\beta,\lambda,\mu,c,m)\\
&= \sum_{k=2}^{\infty}[k\,(1+\beta)-(\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!}\,[1{+}\mu(k{-}1)]\,|a_k|\\
&\le \sum_{k=2}^{\infty}[k\,(1+\beta)-(\alpha+\beta)]\,[1+\lambda\,(k-1)]\,\frac{(-c/4)^{k-1}}{(m)_{k-1}\,(k-1)!}\,[1{+}\mu(k{-}1)]\,,
\end{aligned}$$

which is exactly the left hand side of (2.10). Therefore, we see that the expression $\Psi(\alpha,\beta,\lambda,\mu,c,m)$ is bounded above by $(1-\alpha)$ if (2.9) is satisfied, this completes the proof of Theorem 10.

4 Integral Operators

In order to introduce the results of this section we recall the subclass $L(\lambda,\alpha,\beta)$ consisting of functions of the form (1.1) and satisfy the following inequality for all $z \in U$ (see):

$$\begin{aligned}&\operatorname{Re}\left\{\frac{f'(z)+(1+2\lambda)\,zf''(z)+\lambda z^2 f'''(z)}{f'(z)+\lambda zf''(z)}-\alpha\right\}\\ > \;& \beta\left|\frac{f'(z)+(1+2\lambda)\,zf''(z)+\lambda z^2 f'''(z)}{f'(z)+\lambda zf''(z)}-1\right|, \qquad (4.1)\end{aligned}$$

also let the subclass $TL(\lambda,\alpha,\beta)$ defined by

$$TL(\lambda,\alpha,\beta)=T\cap L(\lambda,\alpha,\beta).$$

We note that

$$f(z)\in TL(\lambda,\alpha,\beta)\Longleftrightarrow zf'(z)\in TS(\lambda,\alpha,\beta). \qquad (4.2)$$

Now, using (1.7)-(1.9), let us define the following integral operators $\mathcal{I}_i(i=1,2,3,4)$ as following:

$$\mathcal{I}_1(z)=\int_0^z \frac{v_p(t)}{t}dt. \qquad (4.3)$$

$$\mathcal{I}_2(z)=\int_0^z \left(2-\frac{v_p(t)}{t}\right)dt, \qquad (4.4)$$

$$\mathcal{I}_3(z)=\int_0^z \frac{w_p(t)}{t}dt. \qquad (4.5)$$

$$\mathcal{I}_4(z)=\int_0^z \left(2-\frac{w_p(t)}{t}\right)dt. \qquad (4.6)$$

Theorem 11. *If $c<0$ and $m>0$, then $\mathcal{I}_2(z)\in TL(\lambda,\alpha,\beta)$ if and only if the condition (2.4) is satisfied.*

Proof. Using (4.4), we have

$$\mathcal{I}_2(z)=z-\sum_{k=2}^{\infty}\frac{(-c/4)^{k-1}}{(m)_{k-1}}\frac{z^k}{k!},$$

moreover,

$$\mathcal{I}_2(z)\in TL(\lambda,\alpha,\beta)\Longleftrightarrow z\mathcal{I}_2'(z)\in TS(\lambda,\alpha,\beta),$$

by virtue of Theorem 2, we have $z\mathcal{I}_2'(z)=z\left(2-\frac{v_p(z)}{z}\right)\in TS(\lambda,\alpha,\beta)$ if and only if (2.4) is satisfied. This completes the proof of Theorem 11.

Similarly, by using Theorem 4, we can obtain the following:

Theorem 12. *If $c < 0$ and $m > 0$, then $\mathcal{I}_4(z) \in TL(\lambda, \alpha, \beta)$ if and only if the condition (2.9) is satisfied.*
Theorem 13. *If $c < 0$ and $m > 0$, then $\mathcal{I}_1(z) \in K(\alpha)$ if the condition (2.4.c) is satisfied.*
Proof. From (4.5), it is easily to obtain that

$$1 + \frac{z\mathcal{I}_1''(z)}{\mathcal{I}_1'(z)} = 1 + \frac{z\left(\frac{v_p(z)}{z}\right)'}{\frac{v_p(z)}{z}} = \frac{zv_p'(z)}{v_p(z)}. \tag{4.7}$$

Using (4.3), It is clear that the convexity results of $\mathcal{I}_1(z)$ is direct consequence from the starlikeness of $v_p(z)$. The proof is completed
Using the same arguments, we have the following:
Theorem 14. *If $c < 0$ and $m > 0$, then $\mathcal{I}_3(z) \in K(\alpha)$ if the condition (2.9.c) is satisfied.*

Remark 3. Specializing the parameters α, β, λ and μ in various theorems in Sections 3, 4, as done in Section 2, we can obtain new results of the well-known subclasses of starlike, convex, uniformly starlike and uniformly convex functions.

5 Open Problem

The authors suggest to obtain the corresponding results of some other special function, such as, Lommel function, Struve function and Wright function.

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