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On The Weinstein Equations in Spaces of Type $\mathcal{D}^p_{\alpha,d}$

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Abstract. In this paper, we consider the Weinstein operator $\Delta_W^{\alpha,d}$, we introduce new function spaces that are denoted by $\mathcal{D}^p_{\alpha,d}$, $1 \leq p \leq \infty,\ \alpha \geq \frac{-1}{2}$. Some properties of these spaces are studied. We study the convolutors and the surjective Weinstein convolution operator acting on $(\mathcal{D}^p_{\alpha,d})'$, $1 \leq p \leq \infty$. In the case $p = 2$, we obtain complete characterization.

1. INTRODUCTION

In [11], L. Schwartz has introduced the space $D_{L^p},\ 1 \leq p \leq \infty$, of all C^∞-functions ψ on $\mathbb{R}$ such that for all $n \in \mathbb{N},\ D^n\psi$ is in $L^p(\mathbb{R})$ and the map $\psi \mapsto D^n\psi$ from D_{L^p} into $L^p(\mathbb{R})$ is continuous. These spaces are studied by many authors (see [1], [2], [6], [10]).

In this paper we introduce for every $1 \leq p \leq \infty,\ \alpha > \frac{-1}{2}$, function spaces, denoted by $\mathcal{D}^p_{\alpha,d}$, similar to D_{L^p} but replacing the usual derivative D by the Weinstein operator $\Delta_W^{\alpha,d}$ defined on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times]0,\ +\infty[$, by:

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha,\ \alpha > -\frac{1}{2}, \tag{1.1}$$

where Δ_d is the Laplacian for the d first variables and L_α is the Bessel operator for the last variable defined on $]0,\ +\infty[$ by :

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}}\frac{\partial u}{\partial x_{d+1}} = \frac{1}{x_{d+1}^{2\alpha+1}}\frac{\partial}{\partial x_{d+1}}\left[x_{d+1}^{2\alpha+1}\frac{\partial u}{\partial x_{d+1}}\right].$$

The main result of this paper consists to give a new characterization of the dual space $(\mathcal{D}^p_{\alpha,d})'$ of $\mathcal{D}^p_{\alpha,d}$ and a description of its bounded subsets.

The Weinstein kernel $\Lambda_{\alpha,d}$ is the function given by :

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d}(x,y) = e^{-i\langle x',y'\rangle} j_\alpha(x_{d+1}y_{d+1}),$$

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where $x = (x', x_{d+1})$, $x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α.
The function $\Lambda_{\alpha,d}$ can be written in the form :

$$\Lambda_{\alpha,d}(x, y) = a_\alpha e^{-i\langle x', y' \rangle} \int_0^1 \left(1 - t^2\right)^{\alpha - \frac{1}{2}} \cos(t x_{d+1} y_{d+1}) dt, \tag{1.2}$$

where a_α is the constant given by the relation :

$$a_\alpha = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma\left(\alpha + \frac{1}{2}\right)}. \tag{1.3}$$

Using the Weinstein kernel $\Lambda_{\alpha,d}$, we define the Weinstein transform $\mathcal{F}_W^{\alpha,d}$ by :

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathcal{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x) \Lambda_{\alpha,d}(x, \lambda) d\mu_{\alpha,d}(x),$$

where $f \in L^1(\mathbb{R}_+^{d+1}, \mu_{\alpha,d}(x))$ and $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by:

$$d\mu_{\alpha,d}(x) = C_{\alpha,d} x_{d+1}^{2\alpha+1} dx, \tag{1.4}$$

dx is the Lebesgue measure on $\mathbb{R}^{d+1}$ and $C_{\alpha,d}$ is the constant given by

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}} 2^\alpha \Gamma(\alpha + 1)}. \tag{1.5}$$

If $T \in (\mathcal{D}_{\alpha,d}^p)'$, we define the Weinstein transform $\mathcal{F}_W^{\alpha,d}(T)$ as following:

$$\forall \phi \in \mathcal{D}_{\alpha,d}^p,\ \langle \mathcal{F}_W^{\alpha,d}(T), \phi \rangle = \langle T, \mathcal{F}_W^{\alpha,d}(\phi) \rangle.$$

We analyze the behaviour of the Weinstein transform $\mathcal{F}_W^{\alpha,d}$ on the spaces $\mathcal{D}_{\alpha,d}^p$ and $(\mathcal{D}_{\alpha,d}^p)'$. We study the Weinstein convolutors on $\mathcal{D}_{\alpha,d}^p$, that is, the functional $T \in (\mathcal{D}_{\alpha,d}^p)'$ such that $T *_W \phi \in \mathcal{D}_{\alpha,d}^p$ for every $\phi \in \mathcal{D}_{\alpha,d}^p$. We show that the convolutors of $\mathcal{D}_{\alpha,d}^1$ or of $\mathcal{D}_{\alpha,d}^\infty$ are the elements of $(\mathcal{D}_{\alpha,d}^\infty)'$ and we characterize the convolutors of $\mathcal{D}_{\alpha,d}^2$. We prove S is a convolutor in $\mathcal{D}_{\alpha,d}^2$ if and only if there exists $l \in \mathbb{N}$ such that

$$(1 + ||\xi||^2)^{-l} \mathcal{F}_W^{\alpha,d}(S) \in L_\alpha^\infty\left(\mathbb{R}_+^{d+1}\right).$$

On the other hand, we prove that every convolutor of $\mathcal{D}_{\alpha,d}^p$ is also a convolutor of $\mathcal{D}_{\alpha,d}^q$ for every q satisfying :

$$\min\{p, p'\} \leq q \leq \max\{p, p'\}.$$

The surjectivity of the Weinstein convolution operator on $\mathcal{D}_{\alpha,d}^2$ is characterized. Moreover, we show that such a surjective operator admits a continuous linear right inverse. A partial result concerning the surjectivity of the Dunkl convolution operator on $\mathcal{D}_{\alpha,d}^p$ is also obtained.
The contents of the paper is as follows :

In the second section, we recapitulate some results related to the harmonic analysis associated with the Weinstein operator $\Delta_W^{\alpha,d}$ given by the relation (1.1).

The section 3 is devoted to studied the space $\mathcal{D}^p_{\alpha,d}$ and its dual $(\mathcal{D}^p_{\alpha,d})'$. We give somes property of theme. In particular, we prove that T is in $(\mathcal{D}^p_{\alpha,d})'$ if and only if there exist $r \in \mathbb{N}$ and $\psi_k \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$, $k = 0, 1, ..., r$, for which

$$T = \sum_{k=0}^{r} \left(\triangle_W^{\alpha,d}\right)^k \psi_k, \text{ on } \mathcal{D}^p_{\alpha,d}.$$

In the last section, we investigate the convolutors in $\mathcal{D}^p_{\alpha,d}$, where their surjectivity in $\mathcal{D}^p_{\alpha,d}$ is descussed at is the functionals $T \in \left(\mathcal{D}^p_{\alpha,d}\right)'$ such that $T *_W \varphi \in \mathcal{D}^p_{\alpha,d}$ for every $\varphi \in \mathcal{D}^p_{\alpha,d}$.

2. **Preliminaires**

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Weinstein operator $\Delta_W^{\alpha,d}$ defined on $\mathbb{R}^{d+1}_+$ by the relation (1.1).

Let us begin by the following result, which gives the eigenfunction $\Psi_\lambda^{\alpha,d}$ of the Weinstein operator $\Delta_W^{\alpha,d}$.

Proposition 1. *(see* [3, 4]*)*
For all $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the system

$$\text{(2.1)} \quad \begin{cases} \frac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), \ \text{if} \ 1 \le j \le d \\ L_\alpha u(x) = -\lambda_{d+1}^2 u(x), \\ u(0) = 1, \ \frac{\partial u}{\partial x_{d+1}}(0) = 0 \ \text{and} \ \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \ \text{if} \ 1 \le j \le d. \end{cases}$$

has a unique solution $\Psi_\lambda^{\alpha,d}$ given by :

$$\text{(2.2)} \qquad \forall z \in \mathbb{C}^{d+1}, \ \Psi_\lambda^{\alpha,d}(z) = e^{-i\langle z', \lambda' \rangle} j_\alpha(\lambda_{d+1} z_{d+1}),$$

where $z = (z', x_{d+1})$, $z' = (z_1, z_2, ..., z_d)$ and j_α is the normalized Bessel function of index α, defined by :

$$\forall \xi \in \mathbb{C}, \ j_\alpha(\xi) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\alpha+1)} (\frac{\xi}{2})^{2n}.$$

The Weinstein kernel $\Lambda_{\alpha,d} : (\lambda, z) \mapsto \Psi_\lambda^{\alpha,d}(z)$ has a unique extention to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ and satisfies the following properties.

Proposition 2. *(see* [3, 4, 5]*)*
i) For all λ, $z \in \mathbb{C}^{d+1}$ and $t \in \mathbb{R}$, we have

$$\Lambda_{\alpha,d}(\lambda, 0) = 1, \ \Lambda_{\alpha,d}(\lambda, z) = \Lambda_{\alpha,d}(z, \lambda) \ \text{and} \ \Lambda_{\alpha,d}(\lambda, tz) = \Lambda_{\alpha,d}(t\lambda, z).$$

ii) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}_+^{d+1}$ *and* $z \in \mathbb{C}^{d+1}$, *we have*

$$|D_z^\nu \Lambda_{\alpha,d}(x,z)| \leq \|x\|^{|\nu|} \exp(\|x\| \, \|\operatorname{Im} z\|), \tag{2.3}$$

where $D_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1} ... \partial z_{d+1}^{\nu_{d+1}}}$ *and* $|\nu| = \nu_1 + ... + \nu_{d+1}$. *In particular*

$$\forall x, y \in \mathbb{R}_+^{d+1}, \ |\Lambda_{\alpha,d}(x,y)| \leq 1. \tag{2.4}$$

Notations. In what follows, we need the following notations:
- $C_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $C_{*,c}(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$ with compact support, even with respect to the last variable.
- $C_{*,0}^0(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable and vanishing to 0 when $\|x\| \to +\infty$.
- $C_*^p(\mathbb{R}^{d+1})$, the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{D}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\begin{aligned} \|f\|_{\alpha,p} &= \left[\int_{\mathbb{R}_+^{d+1}} |f(x)|^p d\mu_{\alpha,d}(x)\right]^{\frac{1}{p}} < +\infty, \text{ if } 1 \leq p < +\infty, \\ \|f\|_{\alpha,\infty} &= \operatorname*{ess\,sup}_{x \in \mathbb{R}_+^{d+1}} |f(x)| < +\infty, \end{aligned}$$

where $\mu_{\alpha,d}$ is the measure given by the relation (1.4).
- $\mathcal{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.

Definition 1. *The Weinstein transform is given for* $f \in L_\alpha^1(\mathbb{R}_+^{d+1})$ *by :*

$$\forall \lambda \in \mathbb{R}_+^{d+1}, \ \mathcal{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x) \Lambda_{\alpha,d}(x,\lambda) d\mu_{\alpha,d}(x). \tag{2.5}$$

where $\mu_{\alpha,d}$ *is the measure on* $\mathbb{R}_+^{d+1}$ *given by the relation (1.2).*

Some basic properties of the transform $\mathcal{F}_W^{\alpha,d}$ are summarized in the following results.

Proposition 3. *(see* [3, 4, 5]*)*
i) For all $f \in L^1_\alpha(\mathbb{R}^{d+1}_+)$*, we have*

$$\|\mathcal{F}^{\alpha,d}_W(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}. \tag{2.6}$$

ii) Let $m \in \mathbb{N}$ *and* $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$*, for all* $y \in \mathbb{R}^{d+1}_+$*, we have*

$$\mathcal{F}^{\alpha,d}_W\left[\left(\triangle^{\alpha,d}_W\right)^m f\right](y) = (-1)^m \|y\|^{2m}\mathcal{F}^{\alpha,d}_W(f)(y). \tag{2.7}$$

iii) For all $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$ *and* $m \in \mathbb{N}$*, we have*

$$\forall\lambda \in \mathbb{R}^{d+1}_+,\ \left(\triangle^{\alpha,d}_W\right)^m\left[\mathcal{F}^{\alpha,d}_W(f)\right](\lambda) = \mathcal{F}^{\alpha,d}_W(P_m f)(\lambda), \tag{2.8}$$

where $P_m(\lambda) = (-1)^m \|\lambda\|^{2m}$.

Theorem 1. *(see* [3, 4, 5]*)*
i) The Weinstein transform $\mathcal{F}^{\alpha,d}_W$ *is a topological isomorphism from* $\mathcal{S}_*(\mathbb{R}^{d+1})$ *onto itself and from* $\mathcal{D}_*(\mathbb{R}^{d+1})$ *onto* $\mathcal{H}_*(\mathbb{C}^{d+1})$*.*
ii) Let $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$*. The inverse transform* $\left(\mathcal{F}^{\alpha,d}_W\right)^{-1}$ *is given by :*

$$\forall x \in \mathbb{R}^{d+1}_+,\ \left(\mathcal{F}^{\alpha,d}_W\right)^{-1}(f)(x) = \mathcal{F}^{\alpha,d}_W(f)(-x). \tag{2.9}$$

iii) Let $f \in L^1_\alpha(\mathbb{R}^{d+1}_+)$*. If* $\mathcal{F}^{\alpha,d}_W(f) \in L^1_\alpha(\mathbb{R}^{d+1}_+)$*, then we have*

$$f(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)(y)\,\Lambda_{\alpha,d}(-x,y)d\mu_{\alpha,d}(y),\ a.e\ x \in \mathbb{R}^{d+1}_+. \tag{2.10}$$

Theorem 2. *(see* [3, 4, 5]*)*
i) For all $f, g \in \mathcal{S}_*(\mathbb{R}^{d+1})$*, we have the following Parseval formula :*

$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)(\lambda)\overline{\mathcal{F}^{\alpha,d}_W(g)(\lambda)}d\mu_{\alpha,d}(\lambda). \tag{2.11}$$

ii) (Plancherel formula).
For all $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$*, we have :*

$$\int_{\mathbb{R}^{d+1}_+} |f(x)|^2\,d\mu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}_+} \left|\mathcal{F}^{\alpha,d}_W(f)(\lambda)\right|^2 d\mu_{\alpha,d}(\lambda). \tag{2.12}$$

iii) (Plancherel Theorem) :
The transform $\mathcal{F}^{\alpha,d}_W$ *extends uniquely to an isometric isomorphism on* $L^2_\alpha(\mathbb{R}^{d+1}_+)$*.*

Proposition 4. *Let* f *be in* $L^p_\alpha\left(\mathbb{R}^{d+1}_+\right)$, $p \in [1,2]$*. Then* $\mathcal{F}^{\alpha,d}_W f$ *belongs to* $L^{p'}_\alpha\left(\mathbb{R}^{d+1}_+\right)$*, with* p' *the conjugate exponent of* p*, that is* $\frac{1}{p} + \frac{1}{p'} = 1$ *and we have*

$$\left\|\mathcal{F}^{\alpha,d}_W(f)\right\|_{L^{p'}_\alpha\left(\mathbb{R}^{d+1}_+\right)} \leqslant \|f\|_{L^p_\alpha\left(\mathbb{R}^{d+1}_+\right)}. \tag{2.13}$$

Proof. From the relation (2.6) and the Theorem 2 iii), we deduce that the relation (2.13) is true in the cases $p = 1$ and $p = 2$.
Hence from the Riez-Thorin interpolation (see [12] and [13]) , deduce that $\mathcal{F}_W^{\alpha,d}$ can be extended as a continuous mapping from $L^p_\alpha\left(\mathbb{R}^{d+1}_+\right)$ into $L^{p'}_\alpha\left(\mathbb{R}^{d+1}_+\right)$ and we have the relation (2.13). □

Definition 2. *The translation operator T_x, $x \in \mathbb{R}^{d+1}_+$, associated with the Weinstein operator $\Delta_W^{\alpha,d}$ is defined on $C_*(\mathbb{R}^{d+1})$, for all $y \in \mathbb{R}^{d+1}_+$, by :*
(2.14)
$$T_x f\left(y\right) = \frac{a_\alpha}{2}\int_0^\pi f\left(x' + y',\ \sqrt{x^2_{d+1} + y^2_{d+1} + 2x_{d+1}y_{d+1}\cos\theta}\right)\left(\sin\theta\right)^{2\alpha} d\theta,$$
where $x' + y' = (x_1 + y_1, ..., x_d + y_d)$ and a_α is the constant given by the relation (1.3).

Proposition 5. *(see [3, 4, 5])*
i) For $f \in C_*(\mathbb{R}^{d+1})$, we have
$$\forall x,\ y \in \mathbb{R}^{d+1}_+,\ T_x f\left(y\right) = T_y f\left(x\right) \ \text{ and } T_0 f = f.$$
ii) For all $f \in \mathcal{E}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}^{d+1}_+$, the function $x \mapsto T_x f\left(y\right)$ belongs to $\mathcal{E}_*(\mathbb{R}^{d+1})$.
iii) We have
$$\forall x \in \mathbb{R}^{d+1}_+,\ \Delta_W^{\alpha,d} \circ T_x = T_x \circ \Delta_W^{\alpha,d}. \tag{2.15}$$
iv) Let $f \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1 \leq p \leq +\infty$ and $x \in \mathbb{R}^{d+1}_+$. Then $T_x f$ belongs to $L^p_\alpha(\mathbb{R}^{d+1}_+)$ and we have
$$\|T_x f\|_{\alpha,p} \leq \|f\|_{\alpha,p}. \tag{2.16}$$
v) The function $\Lambda_{\alpha,d}\left(.,\lambda\right),\ \lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}^{d+1}_+$ the following product formula:
$$\forall y \in \mathbb{R}^{d+1}_+,\ \Lambda_{\alpha,d}\left(x,\lambda\right)\Lambda_{\alpha,d}\left(y,\lambda\right) = T_x\left[\Lambda_{\alpha,d}\left(.,\lambda\right)\right]\left(y\right). \tag{2.17}$$

vi) Let $f \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $p = 1$ or 2 and $x \in \mathbb{R}^{d+1}_+$, we have
$$\forall y \in \mathbb{R}^{d+1}_+,\ \mathcal{F}_W^{\alpha,d}\left(T_x f\right)\left(y\right) = \Lambda_{\alpha,d}\left(x,y\right)\mathcal{F}_W^{\alpha,d}\left(f\right)\left(y\right). \tag{2.18}$$
vii) The space $\mathcal{S}_*(\mathbb{R}^{d+1})$ is invariant under the operators T_x, $x \in \mathbb{R}^{d+1}_+$.

Definition 3. *The Weinstein convolution product of $f, g \in L^1_\alpha(\mathbb{R}^{d+1}_+)$ is given by :*
$$\forall x \in \mathbb{R}^{d+1}_+,\ f *_W g\left(x\right) = \int_{\mathbb{R}^{d+1}_+} T_x f\left(y\right) g\left(y\right) d\mu_{\alpha,d}(y). \tag{2.19}$$

Proposition 6. *(see [3, 4, 5])*
i) Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$.
Then for all $f \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ and $g \in L^q_\alpha(\mathbb{R}^{d+1}_+)$, the function $f *_W g$ belongs to $L^r_\alpha(\mathbb{R}^{d+1}_+)$ and we have

$$\|f *_W g\|_{\alpha,r} \leq \|f\|_{\alpha,p}\|g\|_{\alpha,q}. \tag{2.20}$$

ii) For all $f, g \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, $\big(resp.\ \mathcal{S}_*(\mathbb{R}^{d+1})\big)$, $f *_W g \in L^1_\alpha(\mathbb{R}^{d+1}_+)$ $\big(resp.\ \mathcal{S}_*(\mathbb{R}^{d+1})\big)$ and we have

$$\mathcal{F}^{\alpha,d}_W(f *_W g) = \mathcal{F}^{\alpha,d}_W(f)\mathcal{F}^{\alpha,d}_W(g). \tag{2.21}$$

3. The spaces $\mathcal{D}^p_{\alpha,d}$ and $(\mathcal{D}^p_{\alpha,d})'$

In this section, we introduce new function spaces that are denoted by $\mathcal{D}^p_{\alpha,d}$, $1 \leq p \leq \infty$, $\alpha > \frac{-1}{2}$. Some properties of these spaces are studied. We study the convolutors and the surjective Weinstein convolution operator acting on the dual space of $\mathcal{D}^p_{\alpha,d}$ denoted by $(\mathcal{D}^p_{\alpha,d})'$. In the case $p = 2$, we obtain complete characterization.
Now, we define the new spaces $\mathcal{D}^p_{\alpha,d}$, $1 \leq p \leq \infty$.

Definition 4. *i) The space $\mathcal{D}^p_{\alpha,d}$, $1 \leq p < \infty$ is the set of all C^∞-functions φ in $\mathbb{R}^{d+1}$ such that, for all $n \in \mathbb{N}$, $\left(\triangle^{\alpha,d}_W\right)^n \varphi$ is in $L^p_\alpha(\mathbb{R}^{d+1}_+)$ which is equipped with the topology generated by the countable norms*

$$\forall m \in \mathbb{N},\ \mu^{m,p}_{\alpha,d}(\varphi) = \left(\sum_{n=0}^{m} || \left(\triangle^{\alpha,d}_W\right)^n \varphi||^p_{L^p_\alpha(\mathbb{R}^{d+1}_+)}\right)^{\frac{1}{p}}. \tag{3.1}$$

ii) A function $u \in \mathcal{E}(\mathbb{R}^{d+1})$ is in $\mathcal{B}^\infty_{\alpha,d}$ when for each $m \in \mathbb{N}$, $\mu^{m,\infty}_{\alpha,d}(u) < \infty$, where

$$\mu^{m,\infty}_{\alpha,d}(\varphi) = \sum_{n=0}^{m} || \left(\triangle^{\alpha,d}_W\right)^n \varphi||_{L^\infty_\alpha(\mathbb{R}^{d+1})}. \tag{3.2}$$

iii) We denote by $\mathcal{D}^\infty_{\alpha,d}$ the subspace of $\mathcal{B}^\infty_{\alpha,d}$ that consists of all those functions $u \in \mathcal{B}^\infty_{\alpha,d}$ for which $\lim\limits_{\|x\|\to+\infty} \left(\triangle^{\alpha,d}_W\right)^m u(x) = 0$ for each $m \in \mathbb{N}$.

The space $\mathcal{B}^\infty_{\alpha,d}$ is endowed with the topology generated by the system $\{\mu^{m,\infty}_{\alpha,d}\}_{m\in\mathbb{N}}$.
In the following results, we give some topological properties of the spaces $\mathcal{D}^p_{\alpha,d}$.

Proposition 7. *i) For every $1 \leq p \leq \infty$, $\mathcal{D}^p_{\alpha,d}$ is a Fréchet space.*
ii) Let $1 \leq p \leq 2 \leq q \leq \infty$. Then the space $\mathcal{D}^p_{\alpha,d}$ is continuously contained in $\mathcal{D}^q_{\alpha,d}$.

iii) If $1 < p < \infty$ then $\mathcal{D}^p_{\alpha,d}$ is a reflexive space.
iv) $\mathcal{B}^\infty_{\alpha,d}$ is the strong bidual of $\mathcal{D}^\infty_{\alpha,d}$.
v) For every $1 \le p \le \infty$, the space $\mathcal{D}(\mathbb{R}^{d+1}_+)$ is dense in $\mathcal{D}^p_{\alpha,d}$.

Proof. i) Let $1 \le p \le \infty$ and $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}^p_{\alpha,d}$. Since $L^p_\alpha(\mathbb{R}^{d+1}_+)$ is a Banach space, then there exists $\psi_m \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ such that for each $m \in \mathbb{N}$, $\left(\triangle^{\alpha,d}_W\right)^m \varphi_n \to \psi_m$, as $n \to \infty$, in $L^p_\alpha(\mathbb{R}^{d+1}_+)$. On the other hand, it is easy to see that

$$\forall m \in \mathbb{N},\ \left(\triangle^{\alpha,d}_W\right)^m \psi_0 = \psi_m.$$

Hence this implies that $(\varphi_n)_{n\in\mathbb{N}}$ converge to ψ_0 in $\mathcal{D}^p_{\alpha,d}$.
ii) Let $1 \le p \le 2 \le q \le \infty$, p' the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in \mathcal{D}^p_{\alpha,d}$. For all $n \in \mathbb{N}$, the function $\lambda \to \lambda^{2n}\mathcal{F}^{\alpha,d}_W(f)(\lambda)$ belongs to the space $L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$. By applying Holder's inequality it follows that this last function belongs to the space $L^{q'}_\alpha(\mathbb{R}^{d+1})$ where q' the conjugate exponent of q. On the other hand, for all $x \in \mathbb{R}^{d+1}_+$, we have

$$\begin{aligned}\left(\triangle^{\alpha,d}_W\right)^n f(x) &= \int_{\mathbb{R}^{d+1}_+} \lambda^{2n}\mathcal{F}^{\alpha,d}_W(f)(\lambda)\Lambda_{\alpha,d}(x,\lambda)d\mu_{\alpha,d}(x)\\ &= \left(\mathcal{F}^{\alpha,d}_W\right)^{-1}(\lambda^{2n}\mathcal{F}^{\alpha,d}_W(f))(x)\end{aligned}$$

and this implies that for all $n \in \mathbb{N}$, the function $\triangle^{\alpha,d}_W f$ belongs to the space $L^q_\alpha\left(\mathbb{R}^{d+1}_+\right)$.
iii) To see iii) it is sufficient to argue like in [11].
iv) See [11].
v) If $1 \le p < \infty$, it sufficient to observe that $\mathcal{D}(\mathbb{R}^{d+1}_+)$ is dense in $L^p_\alpha\left(\mathbb{R}^{d+1}_+\right)$.
On the other hand, if $p = \infty$, the result follows immediately from the definition of $\mathcal{D}^\infty_{\alpha,d}$ and the fact that the space $\mathcal{D}(\mathbb{R}^{d+1}_+)$ is dense in $C^0_{*,0}(\mathbb{R}^{d+1})$. □

As usual, by $(\mathcal{D}^p_{\alpha,d})'$ we represent the dual space of $\mathcal{D}^p_{\alpha,d}$.

In the following proposition, we give a representation for the elements of $\mathcal{D}^p_{\alpha,d}$.

Proposition 8. *Let T be a functional on $\mathcal{D}^p_{\alpha,d}$, $1 \le p < \infty$ and p' the conjugate of p. Then T is in $(\mathcal{D}^p_{\alpha,d})'$ if and only if, there exist $r \in \mathbb{N}$ and $\psi_k \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$, $k = 0, 1, ..., r$, for which*

$$T = \sum_{k=0}^{r} \left(\triangle^{\alpha,d}_W\right)^k \psi_k,\ \text{on } \mathcal{D}^p_{\alpha,d}. \tag{3.3}$$

Proof. Suppose that $T \in (\mathcal{D}^p_{\alpha,d})'$. Then there exist an integer r and a positive constant C such that

$$\forall \phi \in \mathcal{D}^p_{\alpha,d},\ |\langle T, \phi\rangle| \leq C \max_{k\leq r} \mu^{\alpha}_{k,p}(\phi). \tag{3.4}$$

We put $E^{r+1}_p = L^p_\alpha(\mathbb{R}^{d+1}_+) \times ...^{r+1}. \times L^p_\alpha(\mathbb{R}^{d+1}_+)$, we define the mappings :

$$\begin{array}{rcl} J : \mathcal{D}^p_{\alpha,d} & \longrightarrow & E^{r+1}_p \\ \phi & \longmapsto & (\left(\triangle^{\alpha,d}_W\right)^k \phi)^r_{k=0} \end{array}$$

and

$$\begin{array}{rcl} L : J\mathcal{D}^p_{\alpha,d} & \longrightarrow & \mathbb{C} \\ (\left(\triangle^{\alpha,d}_W\right)^k \phi)^r_{k=0} & \longmapsto & \langle T, \phi\rangle. \end{array}$$

Note that, since J is one to one, the mapping L is well defined. On the other hand, according to (3.4), L is a continuous linear mapping when in $J\mathcal{D}^p_{\alpha,d}$ we consider the topology induced by E^{r+1}_p. Then by invoking Hahn-Banach theorem, we can extended L continuously to E^{r+1}_p as an element of $(E^{r+1}_p)'$. Then, there exists $u_k \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$, $k = 0, ..., r$ such that (3.3) holds.
Conversely, if T takes the form (3.3), for some $u_k \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$ where $k = 0, ..., r$ and $r \in \mathbb{N}$, using the Hölder's inequality, we deduce that $T \in (\mathcal{D}^p_{\alpha,d})'$. □

Definition 5. *If T is in $(\mathcal{D}^p_{\alpha,d})'$, we define the Weinstein transform $\mathcal{F}^{\alpha,d}_W(T)$ as following :*

$$\forall \phi \in \mathcal{D}^p_{\alpha,d},\ \langle \mathcal{F}^{\alpha,d}_W(T), \phi\rangle = \langle T, \mathcal{F}^{\alpha,d}_W(\phi)\rangle. \tag{3.5}$$

Now we analyze the behaviour of the Weinstein transform $\mathcal{F}^{\alpha,d}_W$ on the spaces $\mathcal{D}^p_{\alpha,d}$ and $(\mathcal{D}^p_{\alpha,d})'$.

Proposition 9. *i) If $u \in \mathcal{D}^p_{\alpha,d}$ with $1 \leq p \leq 2$, then for every polynomial P, we have $P(\|\xi\|^2)\mathcal{F}^{\alpha,d}_W(u) \in L^p_\alpha(\mathbb{R}^{d+1}_+)$.*
ii) If $T \in (\mathcal{D}^p_{\alpha,d})'$, with $2 \leq p < \infty$, then $\mathcal{F}^{\alpha,d}_W(T) = P(\|\xi\|^2)F$, where P is a polynomial and $F \in L^p_\alpha(\mathbb{R}^{d+1}_+)$.

Proof. i) Let $u \in \mathcal{D}^p_{\alpha,d}$, with $1 \leq p \leq 2$. Then from the relation (2.13), it follows that

$$\forall k \in \mathbb{N},\ \mathcal{F}^{\alpha,d}_W(\left(\triangle^{\alpha,d}_W\right)^k u) \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$$

where p' the conjugate exponent of p.
On the other hand, from relation (2.7), we have

$$\forall \xi \in \mathbb{R}_+^{d+1},\ \mathcal{F}_W^{\alpha,d}\left[\left(\triangle_W^{\alpha,d}\right)^k u\right](\xi) = (-1)^k \|\xi\|^{2k}\mathcal{F}_W^{\alpha,d}(u)(\xi).$$

This gives the result.
ii) Let $T \in (\mathcal{D}_{\alpha,d}^p)'$, with $2 \leq p < \infty$. From Proposition **??** there exist $r \in \mathbb{N}$ and u_k belongs to $L_\alpha^{p'}(\mathbb{R}_+^{d+1})$, $k = 0, ..., r$, such that

$$T = \sum_{k=0}^{r} \left(\triangle_W^{\alpha,d}\right)^k u_k.$$

Hence, from the relation (2.7), we obtain :

$$\mathcal{F}_W^{\alpha,d}(T) = \sum_{k=0}^{r} \mathcal{F}_W^{\alpha,d}(\left(\triangle_W^{\alpha,d}\right)^k u_k) = \left(1 + \|\xi\|^2\right)^r F(y)$$

where the function F defined by :

$$F(\xi) = \sum_{n=0}^{r} \frac{(-1)^k \|\xi\|^{2k}\mathcal{F}_W^{\alpha,d}(u_k)(\xi)}{\left(1 + \|\xi\|^2\right)^r}.$$

It is clear that $F \in L_\alpha^p(\mathbb{R}_+^{d+1})$. □

In the following we give a necessary and sufficient condition for that a distribution T belongs to $(\mathcal{D}_{\alpha,d}^2)'$.

Proposition 10. *Let $T \in \mathcal{S}'(\mathbb{R}_+^{d+1})$. Then $T \in (\mathcal{D}_{\alpha,d}^2)'$, if and only if, there exist a polynomial P and a function F in $L_\alpha^2(\mathbb{R}_+^{d+1})$ such that $\mathcal{F}_W^{\alpha,d}(T) = P(\|\xi\|^2)F$.*

Proof. Assume that $\mathcal{F}_W^{\alpha,d}(T) = P(\|\xi\|^2)F$, where P is a polynomial and $F \in L_\alpha^2(\mathbb{R}_+^{d+1})$. Then according to the relation (2.9),

$$T = \left(\mathcal{F}_W^{\alpha,d}\right)^{-1}(P(\|\xi\|^2)F) = P(-\triangle_W^{\alpha,d})\left(\mathcal{F}_W^{\alpha,d}\right)^{-1}(F).$$

Since $\left(\mathcal{F}_W^{\alpha,d}\right)^{-1}(F) \in L_\alpha^2(\mathbb{R}_+^{d+1})$, from Proposition 8, we deduce that $T \in (\mathcal{D}_{\alpha,d}^2)'$.
The conversely is immediately from Proposition 9 ii). □

Proposition 11. *Let $1 \leq p \leq \infty$, $S \in (\mathcal{D}_{\alpha,d}^p)'$ and $\varphi \in \mathcal{D}\left(\mathbb{R}_+^{d+1}\right)$ be given. Then $S *_W \varphi \in L_\alpha^{p'}(\mathbb{R}_+^{d+1})$, where p' is the conjugate exponent of p.*

Proof. Firstly we take $1 \leq p < \infty$. Let $S \in (\mathcal{D}^p_{\alpha,d})'$. According to Proposition 8, there exist $r \in \mathbb{N}$ and $\psi_k \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$, $k = 0, 1, ..., r$, for which

$$S = \sum_{k=0}^{r} \left(\triangle^{\alpha,d}_W\right)^k \psi_k.$$

For every $\varphi \in \mathcal{D}\left(\mathbb{R}^{d+1}_+\right)$, we put $\widetilde{\varphi}(x) = \varphi(-x)$. For all $x \in \mathbb{R}^{d+1}_+$, we have

$$\begin{aligned} S *_W \varphi(x) = \langle S, T_x\widetilde{\varphi}\rangle &= \sum_{k=0}^{r}\langle\left(\triangle^{\alpha,d}_W\right)^k \psi_k, T_x\widetilde{\varphi}\rangle = \sum_{k=0}^{r}\langle\psi_k, \left(\triangle^{\alpha,d}_W\right)^k T_x\widetilde{\varphi}\rangle \\ &= \sum_{k=0}^{r}\langle\psi_k, \left(\triangle^{\alpha,d}_W\right)^k T_x\widetilde{\varphi}\rangle = \sum_{k=0}^{r}(-1)^k\langle\psi_k, T_x\left(\triangle^{\alpha,d}_W\right)^k \widetilde{\varphi}\rangle \\ &= \sum_{k=0}^{r}(-1)^k\psi_k *_W \left(\left(\triangle^{\alpha,d}_W\right)^k \widetilde{\varphi}\right). \end{aligned}$$

Since for each $\varphi \in \mathcal{D}\left(\mathbb{R}^{d+1}_+\right)$, $\left(\triangle^{\alpha,d}_W\right)^k \widetilde{\varphi} \in \mathcal{D}\left(\mathbb{R}^{d+1}_+\right) \subset L^1_\alpha(\mathbb{R}^{d+1}_+)$, then from the relation (2.20), we deduce that $S *_W \varphi$ belongs to $L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$. Suppose now that $p = \infty$ and for every $x \in \mathbb{R}^{d+1}_+$, we have $S *_W \varphi(x) \in \mathbb{R}$. We define two open sets U_+ and U_- as follows:

$$U_+ = \left\{x \in \mathbb{R}^{d+1}_+, v *_W \varphi(x) > 0\right\}$$

and

$$U_- = \left\{x \in \mathbb{R}^{d+1}_+, v *_W \varphi(x) < 0\right\}.$$

If $K \neq \emptyset$ is a compact subset of U_+, we choose $\phi \in D(\mathbb{R}^{d+1}_+)$ such that $\phi \equiv 1$ on K, $0 \leq \phi \leq 1$ and $supp\,\phi \subset U_+$. Then

$$\begin{aligned} \int_K |S *_W \varphi(x)| d\mu_{\alpha,d}(x) &\leq \int_{\mathbb{R}^{d+1}_+} (S *_W \varphi(x))\phi(x) d\mu_{\alpha,d}(x) \\ &\leq C\mu^\alpha_{m,1}(\varphi)||\phi||_{L^\infty_\alpha(\mathbb{R}^{d+1}_+)}. \end{aligned}$$

Hence $S *_W \varphi \in L^1_\alpha(U_+)$.
In a similar way, we show that $S *_W \varphi \in L^1_\alpha(U_-)$. Then $S *_W \varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$.
In the general case it is sufficient to consider the real and imaginary part of $S *_W \varphi$ to conclude. □

Proposition 12. *For every $p \in \mathbb{N}$ and $\varepsilon > 0$, there exist $m_0 \in \mathbb{N}$ such that for any $m \in \mathbb{N}$, $m \geq m_0$, we can find two functions $\gamma_m \in \mathcal{D}\left(\mathbb{R}^{d+1}_+\right)$*

and $\Gamma_m \in \mathcal{D}^{2p}(B\,(o,\varepsilon))$ *such that*

$$\delta = (I - \triangle_W^{\alpha,d})^m \Gamma_m + \gamma_m \tag{3.6}$$

where δ *is the Dirac distribution and*

$$B\,(o,\varepsilon) = \left\{ x \in \mathbb{R}^{d+1},\ \|x\| \leq \varepsilon \right\}.$$

Proof. The proposition can be proved in the same way of Proposition 3.6 in [9](see also [8]) □

Proposition 13. *Let* $1 \leq p \leq q \leq \infty$ *,the space* $\mathcal{D}^p_{\alpha,d}$ *is continuously contained in* $\mathcal{D}^q_{\alpha,d}$.

Proof. Using the relations (3.6), (2.15) and (2.20), we deduce the result. □

Proposition 14. *Let* $1 \leq p \leq \infty$ *and* $S \in D'(\mathbb{R}^{d+1}_+)$ *be given. Suppose that* $S *_W \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ *for every* $\varphi \in D(\mathbb{R}^{d+1}_+)$*. Then, there exist* $m \in \mathbb{N}$ *and* $f, g \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ *for which* $S = (I - \triangle_W^{\alpha,d})^m f + g$.

Proof. Let $1 \leq p \leq \infty$, $S \in D'(\mathbb{R}^{d+1}_+)$ and $V_{p'}$ be the unit ball of $L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$.
Assume that $\varphi \in V_{p'} \bigcap D(\mathbb{R}^{d+1}_+)$. We have

$$\forall \phi \in D(\mathbb{R}^{d+1}_+),\ |\langle S *_W \widetilde{\varphi}, \widetilde{\phi}\rangle| = |\langle S *_W \phi, \varphi\rangle|.$$

On the other hand, using the Hölder inequality, we obtain

$$|\langle S *_W \phi, \varphi\rangle| \leq ||S *_W \phi||_{L^p_\alpha(\mathbb{R}^{d+1}_+)} ||\varphi||_{L^{p'}_\alpha(\mathbb{R}^{d+1}_+)} \leq ||S *_W \phi||_{L^p_\alpha(\mathbb{R}^{d+1}_+)}. \tag{3.7}$$

Thus the set

$$\left\{ S *_W \widetilde{\varphi} : \quad \varphi \in V_{p'} \bigcap D(\mathbb{R}^{d+1}_+) \right\}$$

is bounded in $D'(\mathbb{R}^{d+1}_+)$ when we consider in $D'(\mathbb{R}^{d+1}_+)$ the weak topology, and hence, equicontinuous on $D'(\mathbb{R}^{d+1}_+)$. Therefore, if $K_1 = \overline{B}\,(o,2) = \left\{ x \in \mathbb{R}^{d+1},\ \|x\| \leq 2 \right\}$, we can find $m \in \mathbb{N}$ such that

$$\forall \phi \in D(K_1) \text{ and } \varphi \in V_{p'} \bigcap D(\mathbb{R}^{d+1}_+),\ |\langle S *_W \widetilde{\varphi}, \widetilde{\phi}\rangle| \leq C ||\phi||_{K_1,m}$$

where

$$||\phi||_{K_1,m} = \sup_{\substack{x \in K_1 \\ k \leq m}} |\left(\triangle_W^{\alpha,d}\right)^k \phi(x)|.$$

Hence, for every $\phi \in D(K_1)$ and $\varphi \in D(\mathbb{R}^{d+1}_+)$, we have

$$|\langle S *_W \widetilde{\varphi}, \widetilde{\phi}\rangle| \leq C ||\phi||_{K_1,m} ||\varphi||_{L^{p'}_\alpha(\mathbb{R}^{d+1}_+)}. \tag{3.8}$$

Now we want to prove that, if we take $K_2 = \overline{B}\,(o,1) = \left\{x \in \mathbb{R}^{d+1},\ \|x\| \leq 1\right\}$, then

$$\forall \phi \in D^{2m}(K_2),\ S *_W \phi \in L^p_\alpha(\mathbb{R}^{d+1}_+).$$

Let $\psi \in D(K_2)$ such that $0 \leq \psi \leq 1$ and $\displaystyle\int_{\overline{B}(o,1)} \psi(x) d\mu_{\alpha,d}(x) = 1$.

Let $0 < \varepsilon < 1$, we put :

$$\forall x \in \mathbb{R}^{d+1},\ \psi_\varepsilon(x) = \varepsilon^{-2\alpha-d-2}\psi\left(\frac{x}{\varepsilon}\right).$$

For $\phi \in D^{2m}(K_2)$, $\phi *_W \psi_\varepsilon \in D(K_1)$ and $\phi *_W \psi_\varepsilon \to \phi$, as $\varepsilon \to 0^+$, in $D^m(K_1)$. Consequently, $S *_W (\phi *_W \psi_\varepsilon) \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ and we deduce from (3.8) that

$$||S *_W (\phi *_W \psi_\varepsilon)||_{L^p_\alpha(\mathbb{R}^{d+1}_+)} \leq C||\phi *_W \psi_\varepsilon||_{K_1,m} \leq C||\phi||_{K_1,m}. \tag{3.9}$$

Observe that we also get from (3.9) that $S *_W (\phi *_W \psi_\varepsilon)$ is a Cauchy net in $L^p_\alpha(\mathbb{R}^{d+1}_+)$, thus convergent. Since $S \in D'(\mathbb{R})$, there exist $l \in \mathbb{N}$ and $C > 0$ such that

$$\forall \varphi \in D(K_2),\ |\langle S, \varphi\rangle| \leq C||\varphi||_{K_2,l}.$$

Then S can be continuously extended to $D^l(K_2)$. Hence, if m large enough, we conclude that for all $x \in \mathbb{R}^{d+1}_+$, we have

$$|(S *_W (\phi *_W \psi_\varepsilon)(x) - S *_W \phi)(x)| \leq C||\phi *_W \psi_\varepsilon - \phi||_{K_2,l} \leq C||\phi *_W \psi_\varepsilon - \phi||_{K_1,m}.$$

Thus $S *_W (\phi *_W \psi_\varepsilon) \to S *_W \phi$, as $\varepsilon \to 0^+$, in $C_b(\mathbb{R}^{d+1})$.
From (3.9), we deduce that $S *_W \phi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$.
According to Proposition 12, we can write

$$\delta = (I - \triangle^{\alpha,d}_W)^m \Gamma_m + \gamma_m,$$

where $\gamma_m \in D(K_2)$ and $\Gamma_m \in D^{2m}(K_2)$.
Then, we obtain :

$$S = S *_W \delta = S *_W \Big((I - \triangle^{\alpha,d}_W)^m \Gamma_m + \gamma_m\Big) = (I - \triangle^{\alpha,d}_W)^m f + g$$

where $f = S *_W \Gamma_m$ and $g = S *_W \gamma_m$. □

Now we can establish the following property, where we present as a necessary and sufficient condition in order that a distribution belongs to $(\mathcal{D}^p_{\alpha,d})'$.

Theorem 3. *Let $1 \leq p \leq \infty$ and $S \in D'(\mathbb{R}^{d+1}_+)$. The following assertions are equivalent:*

i) $S \in (\mathcal{D}^p_{\alpha,d})'$.

ii) $S *_W \phi \in L^{p'}_\alpha((\mathbb{R}^{d+1}_+)$ for every $\phi \in D(\mathbb{R}^{d+1}_+)$.

iii) There exist $m \in \mathbb{N}$ *and* $f_m \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$ *such that* $S = (I - \triangle^{\alpha,d}_W)^m f_m$.

Proof. The results follow directly from Proposition 14. □

We give now an allternative description of the space $\mathcal{D}^p_{\alpha,d}$. that will be useful in the sequel.

Proposition 15. *Let* $1 \leq p \leq \infty$. *The family of seminorms*

$$\Gamma = \left\{ q^{m,p}_{\alpha,d} : \quad m \in \mathbb{N} \right\}$$

where for all $m \in \mathbb{N}$ *and* $\phi \in \mathcal{D}^p_{\alpha,d}$

$$q^{m,p}_{\alpha,d}(\phi) = ||(I - \triangle^{\alpha,d}_W)^m \phi||_{L^p_\alpha(\mathbb{R}^{d+1}_+)} \tag{3.10}$$

generates the topology of $\mathcal{D}^p_{\alpha,d}$. *Moreover, every continuous seminorm* $\mu^{m,p}_{\alpha,d}$ *is dominated by some* $q^{m,p}_{\alpha,d} \in \Gamma$.

Proof. It is clear that the family Γ defines on $\mathcal{D}^p_{\alpha,d}$ a topology weaker than the one associated with $\{\mu^{n,p}_{\alpha,d}\}_{n\in\mathbb{N}}$. Let $n \in \mathbb{N}$. There exist a positive constant C and a bounded subset B of $(\mathcal{D}^p_{\alpha,d})'$ for which

$$\forall \varphi \in \mathcal{D}^p_{\alpha,d},\ \mu^{n,p}_{\alpha,d}(\varphi) \leq C \sup_{S\in B} |\langle S, \varphi\rangle|.$$

From Theorem 3 there exists $m \in \mathbb{N}$ and a positive constant C such that, for every $S \in B$, we can find $f_S \in L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$, satisfying

$$S = (I - \triangle^{\alpha,d}_W)^m f_S, \quad ||f_S||_{L^{p'}_\alpha(\mathbb{R})} \leq C.$$

On the other hand, for all $\varphi \in \mathcal{D}^p_{\alpha,d}$, we have

$$\begin{aligned} |\langle S, \varphi\rangle| &\leq \Big| \int_{\mathbb{R}} f_S(x)(I - \triangle^{\alpha,d}_W)^m \varphi(x) d\mu_{\alpha,d}(x) \Big| \\ &\leq C||(I - \triangle^{\alpha,d}_W)^m \varphi||_{L^p_\alpha(\mathbb{R}^{d+1}_+)}. \end{aligned}$$

Thus there exists $m \in \mathbb{N}$ such that

$$\forall \varphi \in \mathcal{D}^p_{\alpha,d},\ \mu^{n,p}_{\alpha,d}(\varphi) \leq C||(I - \triangle^{\alpha,d}_W)^m \varphi||_{L^p_\alpha(\mathbb{R}^{d+1}_+)}.$$

Then we conclude that Γ generates the topology of $\mathcal{D}^p_{\alpha,d}$. □

From the previous proposition we deduce an interesting characterization of the functions in $\mathcal{D}^p_{\alpha,d}$ as follows :

Proposition 16. *i) Let* $1 \leq p < \infty$. *A function* $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ *is in* $\mathcal{D}^p_{\alpha,d}$ *if and only if*

$$\forall m \in \mathbb{N},\ (I - \triangle^{\alpha,d}_W)^m \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+).$$

ii) A function $\phi \in L^\infty_\alpha(\mathbb{R}^{d+1}_+)$ is in $\mathcal{B}^\infty_{\alpha,d}$ if and only if

$$\forall m \in \mathbb{N},\ (I - \triangle^{\alpha,d}_W)^m \phi \in L^\infty_\alpha(\mathbb{R}^{d+1}_+).$$

4. Convolutors in $\mathcal{D}^p_{\alpha,d}$

In this section we study the convolutors in $\mathcal{D}^p_{\alpha,d}$, $1 \le p \le \infty$, where their surjectivity in $\mathcal{D}^p_{\alpha,d}$ is descussed at is the functionals $T \in \left(\mathcal{D}^p_{\alpha,d}\right)'$ such that $T *_W \varphi \in \mathcal{D}^p_{\alpha,d}$ for every $\varphi \in \mathcal{D}^p_{\alpha,d}$.

Definition 6. *The generalized convolution of $S \in (\mathcal{D}^p_{\alpha,d})'$, $1 \le p \le \infty$ and $\varphi \in \mathcal{D}^p_{\alpha,d}$ is given by :*

$$\forall x \in \mathbb{R}^{d+1}_+,\ S *_W \varphi(x) = \langle S,\ T_x\widetilde{\varphi}\rangle \tag{4.1}$$

where $\widetilde{\varphi}(x) = \varphi(-x)$.
The functionnel $S \in (\mathcal{D}^p_{\alpha,d})'$ is called convolutor in $\mathcal{D}^p_{\alpha,d}$ if for every $\varphi \in \mathcal{D}^p_{\alpha,d}$, we have $S *_W \varphi \in \mathcal{D}^p_{\alpha,d}$.

Remark 1. *Using the fact that for all $x \in \mathbb{R}^{d+1}_+$ and $\varphi \in \mathcal{D}^p_{\alpha,d}$, we have $T_x\widetilde{\varphi} \in \mathcal{D}^p_{\alpha,d}$, we deduce that the Definition 6 is meaningful.*

Proposition 17. *Let $S \in (\mathcal{D}^p_{\alpha,d})'$, $1 \le p \le \infty$, be a convolutor in $\mathcal{D}^p_{\alpha,d}$. Then the mapping F_S defined by :*

$$\forall \varphi \in \mathcal{D}^p_{\alpha,d},\ F_S(\varphi) = S *_W \varphi$$

is continuous from $\mathcal{D}^p_{\alpha,d}$ into itself.

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}^p_{\alpha,d}$ such that $\varphi_n \to \varphi$, as $n \to \infty$, and $F_S(\varphi_n) \to \phi$, as $n \to \infty$, in $\mathcal{D}^p_{\alpha,d}$, for certain $\varphi, \phi \in \mathcal{D}^p_{\alpha,d}$. Since, for every $x \in \mathbb{R}^{d+1}_+$, the mapping $\varphi \to T_x\widetilde{\varphi}$ is continuous from $\mathcal{D}^p_{\alpha,d}$ into $\mathcal{D}^p_{\alpha,d}$, then for every $x \in \mathbb{R}^{d+1}_+$, we have $S *_W \varphi_n(x) \to S *_W \varphi(x)$ as $n \to \infty$, . Then $F_S(\varphi) = \phi$ and the closed graph theorem implies that F_S is continuous. □

Now, using Definition 6 and the proposition 17, the following definition have a sense.

Definition 7. *Let $S \in (\mathcal{D}^p_{\alpha,d})'$,$1 \le p \le \infty$ and $T \in (\mathcal{D}^{p'}_{\alpha,d})'$ where p' the conjugate of p. Then the Weinstein convolution $S *_W T$ is the functional given by :*

$$\forall \varphi \in \mathcal{D}^p_{\alpha,d},\ \langle S *_W T, \varphi\rangle = \langle S, \widetilde{T} *_W \varphi\rangle, \tag{4.2}$$

where $\langle \widetilde{T}, \varphi\rangle = \langle T, \widetilde{\varphi}\rangle)$.

Proposition 18. *i) Let $S \in (\mathcal{D}^p_{\alpha,d})'$, $1 \le p \le \infty$ and $T \in (\mathcal{D}^{p'}_{\alpha,d})'$, where p' the conjugate of p, that $S *_W T = T *_W S$.*
ii) Let $S, T \in (\mathcal{D}^\infty_{\alpha,d})'$, then $S *_W T \in (\mathcal{D}^\infty_{\alpha,d})'$.
iii) Let $S \in (\mathcal{D}^\infty_{\alpha,d})'$ and $T \in (\mathcal{D}^1_{\alpha,d})'$, then $S *_W T \in (\mathcal{D}^1_{\alpha,d})'$.
iv) For every $T \in \mathcal{B}^\infty_{\alpha,d}$ and $S \in (\mathcal{D}^\infty_{\alpha,d})'$, we have $S *_W T \in \mathcal{B}^\infty_{\alpha,d}$.
v) Let $T \in (\mathcal{D}^1_{\alpha,d})'$ and $\varphi \in \mathcal{D}^1_{\alpha,d}$, then $T *_W \varphi \in \mathcal{B}^\infty_{\alpha,d}$.

Proof. i) By a standard argument, it is easy to see i)
ii) We deduce this result by Theorem 3 and i)
iii) The proof is similar to that of part ii).
iv) Since $\mathcal{B}^\infty_{\alpha,d}$ is contained in $(\mathcal{D}^1_{\alpha,d})'$, then from Theorem 3 and Proposition 17 we get the conclusion.
v) The proof is similar to that of part iv). □

As a consequence of Theorem 3, we characterize $(\mathcal{D}^\infty_{\alpha,d})'$ as the space of convolutors in $\mathcal{D}^1_{\alpha,d}$ and in $\mathcal{D}^\infty_{\alpha,d}$.

Proposition 19. *Let $S \in (\mathcal{D}^1_{\alpha,d})'$. Then $S \in (\mathcal{D}^\infty_{\alpha,d})'$ if and only if $S *_W \varphi \in \mathcal{D}^1_{\alpha,d}$ for every $\varphi \in \mathcal{D}^1_{\alpha,d}$. Moreover, for each $1 \le p \le \infty$, we have $S *_W \varphi \in \mathcal{D}^p_{\alpha,d}$ whenever $S \in (\mathcal{D}^\infty_{\alpha,d})'$ and $\varphi \in \mathcal{D}^p_{\alpha,d}$.*

In the following result, we characterize the Weinstein convolution in $\mathcal{D}^2_{\alpha,d}$ via the Weinstein transform.

Proposition 20. *i) Let S be a convolutor in $\mathcal{D}^2_{\alpha,d}$ and $\varphi \in \mathcal{D}^2_{\alpha,d}$. Then we have*

$$\mathcal{F}^{\alpha,d}_W(S *_W \varphi) = \mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(\varphi). \tag{4.3}$$

ii) Let $S, T \in (\mathcal{D}^2_{\alpha,d})'$. Then $\mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(T) \in \mathcal{S}'\left(\mathbb{R}^{d+1}_+\right)$. If moreover, S is a convolutor in $\mathcal{D}^2_{\alpha,d}$ then

$$\mathcal{F}^{\alpha,d}_W(S *_W T) = \mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(T). \tag{4.4}$$

Proof. i) The results follow immediatly from the relations (4.1) and (2.18).
ii) Using the relations (4.2) and (4.3), we get the relation (4.4) □

Theorem 4. *Let $S \in (\mathcal{D}^2_{\alpha,d})'$. We have S is a convolutor in $\mathcal{D}^2_{\alpha,d}$ if and only if there exists $l \in \mathbb{N}$ such that $(1 + ||\xi||^2)^{-l}\mathcal{F}^{\alpha,d}_W(S) \in L^\infty_\alpha\left(\mathbb{R}^{d+1}_+\right)$.*

Proof. From Theorem 3, there exist $f \in L^2_\alpha\left(\mathbb{R}^{d+1}_+\right)$ and $l \in \mathbb{N}$ such that $S = (I - \triangle^{\alpha,d}_W)^l f$. Then $\mathcal{F}^{\alpha,d}_W(S) = (1 + ||\xi||^2)^l\mathcal{F}^{\alpha,d}_W(f)$. Assume that S is a convolutor in $\mathcal{D}^2_{\alpha,d}$, that is for each $\varphi \in \mathcal{D}^2_{\alpha,d}$, we have $S *_W \varphi \in \mathcal{D}^2_{\alpha,d}$.

Then, according to the relations (4.3), (2.13) and Proposition 15 for all $\varphi \in \mathcal{D}^2_{\alpha,d}$, we can write

(4.5)
$$||\mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(\varphi)||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} = ||S*_W\varphi||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} \leq C||(I-\triangle^{\alpha,d}_W)^l\varphi||_{L^2_\alpha(\mathbb{R}^{d+1}_+)}$$

where $C > 0$.

Let now $g \in L^2_\alpha(\mathbb{R}^{d+1}_+)$, it is not hard to see that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $D(\mathbb{R}^{d+1}_+)$ such that $(1 + \|\xi\|^2)^l\mathcal{F}^{\alpha,d}_W(\varphi_n) \to g$, as $n \to \infty$, in $L^2_\alpha(\mathbb{R}^{d+1}_+)$.

From (4.5), we deduce that

$$||\frac{\mathcal{F}^{\alpha,d}_W(S)g}{(1 + \|\xi\|^2)^l}||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} \leq C||g||_{L^2_\alpha(\mathbb{R}^{d+1}_+)}.$$

Hence, for certain $C > 0$ and $l \in \mathbb{N}$, we have

$$\forall\xi \in \mathbb{R}^{d+1}_+,\ |\mathcal{F}^{\alpha,d}_W(S)(\xi)| \leq C(1 + \|\xi\|^2)^l.$$

Conversely assume now that $(1 + \|\xi\|^2)^{-l}\mathcal{F}^{\alpha,d}_W(S) \in L^\infty_\alpha(\mathbb{R}^{d+1}_+)$ for some $l \in \mathbb{N}$. Let $m \in \mathbb{N}$, for all $\varphi \in \mathcal{D}^2_{\alpha,d}$, we have

$$\begin{aligned}||(I - \triangle^{\alpha,d}_W)^m(S *_W \varphi)||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} &= ||(1 + \|\xi\|^2)^m\mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(\varphi)||_{L^2_\alpha(\mathbb{R}^{d+1}_+)}\\ &\leq C||(1 + \|\xi\|^2)^{l+m}\mathcal{F}^{\alpha,d}_W(\varphi)(\xi)||_{L^2_\alpha(\mathbb{R}^{d+1}_+)}.\end{aligned}$$

Hence, we obtain

$$\forall\varphi \in \mathcal{D}^2_{\alpha,d},\ ||(I-\triangle^{\alpha,d}_W)^m(S*_W\varphi)||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} \leq C||(I-\triangle^{\alpha,d}_W)^{m+l}\varphi||_{L^2_\alpha(\mathbb{R}^{d+1}_+)}.$$

Then we conclude that S is a convolutor in $\mathcal{D}^2_{\alpha,d}$. □

Proposition 21. *Let* $1 \leq p \leq \infty$. *Assume that* $S \in (\mathcal{D}^p_{\alpha,d})'$ *is a convolutor in* $\mathcal{D}^p_{\alpha,d}$. *Then for every* $\min\{p, p'\} \leq q \leq \max\{p, p'\}$, *S is a convolutor of* $\mathcal{D}^q_{\alpha,d}$.

Proof. The cases $p = 1$ and $p = \infty$ are proved in Proposition 19.

We first prove that S is a convolutor in $(\mathcal{D}^{p'}_{\alpha,d})'$.

Since for every $\varphi \in D(\mathbb{R}^{d+1}_+)$, $S *_W \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, then from Theorem 3, we deduce that $S \in (\mathcal{D}^{p'}_{\alpha,d})'$.

We now take $T \in (\mathcal{D}^{p'}_{\alpha,d})'$ and we show that $S *_W T \in (\mathcal{D}^{p'}_{\alpha,d})'$.

Since T is a convolutor in $\mathcal{D}^p_{\alpha,d}$ and $S *_W \varphi \in \mathcal{D}^p_{\alpha,d}$, we deduce that for all $\varphi \in D(\mathbb{R}^{d+1}_+)$, we have

$$(S *_W T) *_W \varphi = T *_W (S *_W \varphi) \in L^p_\alpha(\mathbb{R}^{d+1}_+). \tag{4.6}$$

Then the Theorem 3 implies that $S *_W T \in (\mathcal{D}^{p'}_{\alpha,d})'$. Thus we have seen that the mapping F_S defined by :

$$\forall T \in (\mathcal{D}^{p'}_{\alpha,d})',\ F_S(T) = T *_W S$$

maps $(\mathcal{D}^{p'}_{\alpha,d})'$ into itself. Moreover, F_S has a sequentially closed graph. We apply the closed graph theorem (see [7]) to conclude that F_S is continuous.
By Proposition 7, the mapping F_S^*, transposed of F_S is continuous from $(\mathcal{D}^{p'}_{\alpha,d})'$ into itself. On the other hand, it follows from Definition 7 that

$$\forall \varphi \in D(\mathbb{R}^{d+1}_+),\ F_S^*(\varphi) = \widetilde{S} *_W \varphi.$$

Hence, for every $m \in \mathbb{N}$, there exist $C > 0$ and $n \in \mathbb{N}$ such that

$$\forall \varphi \in D(\mathbb{R}^{d+1}_+),\ \mu^\alpha_{m,p'}(\widetilde{S} *_W \varphi) \leq C\mu^\alpha_{n,p'}(\varphi). \tag{4.7}$$

Since $D(\mathbb{R}^{d+1}_+)$ is a dense subspace of $\mathcal{D}^{p'}_{\alpha,d}$, (4.7) implies that $\widetilde{S}$ and hence S is a convolutor in $\mathcal{D}^{p'}_{\alpha,d}$.
To finish the proof of this proposition, we will assume that $p > 2$ and we will prove that

$$\forall S \in \mathcal{D}^q_{\alpha,d},\ T *_W S \in (\mathcal{D}^q_{\alpha,d})'.$$

Let $f, g \in L^{q'}_\alpha(\mathbb{R}^{d+1}_+)$ and $m \in \mathbb{N}$ such that $T = (I - \triangle^{\alpha,d}_W)^m f$.
We now observe that for every $\varphi \in D(\mathbb{R}^{d+1}_+)$, we have $S *_W (I - \triangle^{\alpha,d}_W)^m \varphi$ is a convolutor in $L^{q'}_\alpha(\mathbb{R}^{d+1}_+)$. In fact, for every $g \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, we have $(I - \triangle^{\alpha,d}_W)^m \varphi *_W g \in \mathcal{D}^p_{\alpha,d}$ and

$$g *_W (S *_W (I - \triangle^{\alpha,d}_W)^m \varphi) = S *_W ((I - \triangle^{\alpha,d}_W)^m \varphi *_W g) \in L^p_\alpha(\mathbb{R}^{d+1}_+).$$

Then $S *_W (I - \triangle^{\alpha,d}_W)^m \varphi$ is a convolutor in $L^{p'}_\alpha(\mathbb{R}^{d+1}_+)$.
By applying the Riesz-Thorin interpolation theorem, we deduce that $S *_W (I - \triangle^{\alpha,d}_W)^m \varphi$ is a convolutor in $L^{q'}_\alpha(\mathbb{R}^{d+1}_+)$.
Finally, for every $\varphi \in D(\mathbb{R}^{d+1}_+)$, we obtain

$$(T *_W S) *_W \varphi = f *_W (S *_W (I - \triangle^{\alpha,d}_W)^m \varphi) \in L^{q'}_\alpha(\mathbb{R}^{d+1}_+)$$

and $T *_W S \in (\mathcal{D}^q_{\alpha,d})'$. □

Remark 2. *A consequence immediate of Proposition 21 is the following:*
if $S \in (\mathcal{D}^p_{\alpha,d})'$ is a convolutor in $\mathcal{D}^p_{\alpha,d}$ for some $1 \leq p \leq \infty$, then S is a convolutor in $\mathcal{D}^2_{\alpha,d}$.

Corollary 1. *Let $1 \leq p \leq \infty$. If $T \in (\mathcal{D}^p_{\alpha,d})'$ is a convolutor of $\mathcal{D}^p_{\alpha,d}$ then there exists $m \in \mathbb{N}$ such that*

$$(1 + ||\xi||^2)^{-m} \mathcal{F}^{\alpha,d}_W(T) \in L^\infty_\alpha(\mathbb{R}^{d+1}_+).$$

Proof. The result follows directly from Proposition 21 and Theorem 4. □

Now, we study the convolutors and the surjective Weinstein convolution operator acting on $(\mathcal{D}^p_{\alpha,d})'$, $1 \leq p \leq \infty$. In the case $p = 2$, we obtain complete characterization.

Theorem 5. *Let $S \in (\mathcal{D}^2_{\alpha,d})'$ be a convolutor in $\mathcal{D}^2_{\alpha,d}$. The following assertions are equivalent:*
i) $S *_W \mathcal{D}^2_{\alpha,d} = \mathcal{D}^2_{\alpha,d}$.
ii) $S *_W (\mathcal{D}^2_{\alpha,d})' = (\mathcal{D}^2_{\alpha,d})'$.
iii)There exists a convolutor R in $\mathcal{D}^2_{\alpha,d}$ such that $S *_W R = \delta$.
iv) There exist $M > 0$ and $l \in \mathbb{N}$ such that

$$|\mathcal{F}^{\alpha,d}_W(S)(\xi)| \geq M(1 + ||\xi||^2)^{-l}, \ \textit{a.e.}\ \xi \in \mathbb{R}^{d+1}_+.$$

Proof. i) $\Longrightarrow$ ii) ? Firstly it is not hard to see that $\mathcal{F}^{\alpha,d}_W(S)(\xi) \neq 0$, a.e. $\xi \in \mathbb{R}^{d+1}_+$.
Assume now $S *_W \varphi = 0$, where $\varphi \in \mathcal{D}^2_{\alpha,d}$. Then $\mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(\varphi) = 0$ and $\varphi = 0$. Thus the Weinstein convolution operator defined by S is one-to-one on $\mathcal{D}^2_{\alpha,d}$. It easily follows that the Weinstein convolution operator defined by $\widetilde{S}$ is an automorphism of $\mathcal{D}^2_{\alpha,d}$. Then we obtain $S *_W (\mathcal{D}^2_{\alpha,d})' = (\mathcal{D}^2_{\alpha,d})'$.
ii) $\Longrightarrow$ iii) ? From the hypothesis ii), we deduce that there exists $R \in (\mathcal{D}^2_{\alpha,d})'$ such that $S *_W R = \delta$. Let now $\varphi \in \mathcal{D}^2_{\alpha,d}$. We choose $\phi \in \mathcal{D}^2_{\alpha,d}$ such that $\varphi = S *_W \phi$. Then

$$R *_W \varphi = R *_W (S *_W \phi) = (R *_W S) *_W \phi = \delta *_W \phi = \phi.$$

Thus R is a convolutor in $\mathcal{D}^2_{\alpha,d}$ and iii) is established.
iii) $\Longrightarrow$ *iν*) ? Let $R \in (\mathcal{D}^2_{\alpha,d})'$ be a convolutor in $\mathcal{D}^2_{\alpha,d}$ such that $S *_W R = \delta$. Then $\mathcal{F}^{\alpha,d}_W(S)\mathcal{F}^{\alpha,d}_W(R) = 1$. By using now Theorem 4, we conclude that there exist $M > 0$ and $l \in \mathbb{N}$ for which

$$|\mathcal{F}^{\alpha,d}_W(S)(\xi)| \geq M(1 + ||\xi||^2)^{-l} \text{ a.e. } \xi \in \mathbb{R}^{d+1}_+.$$

iv) $\Longrightarrow$ i)? Let $\varphi \in \mathcal{D}^2_{\alpha,d}$. We define $\psi = \dfrac{\mathcal{F}_W^{\alpha,d}(\varphi)}{\mathcal{F}_W^{\alpha,d}(S)}$. If iv) holds then ψ is a measurable function and for all $m \in \mathbb{N}$, we have

$$\begin{aligned}||(1+||\xi||^2)^m\psi||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} &\le C||\mathcal{F}_W^{\alpha,d}((I-\triangle_W^{\alpha,d})^{m+l}\psi)||_{L^2_\alpha(\mathbb{R}^{d+1}_+)}\\ &\le C||(I-\triangle_W^{\alpha,d})^{m+l}\psi||_{L^2_\alpha(\mathbb{R}^{d+1}_+)} < \infty.\end{aligned}$$

Then using the Theorem 2 and Corollary 1, we deduce that the function $\phi = \left(\mathcal{F}_W^{\alpha,d}\right)^{-1}(\psi)$ is in $\mathcal{D}^2_{\alpha,d}$ and $S *_W \phi = \varphi$. Thus the proof of i) is completed. □

Proposition 22. *Let $1 \le p \le 2$. Assume that $S \in (\mathcal{D}^p_{\alpha,d})'$ is a convolutor in $\mathcal{D}^p_{\alpha,d}$. We consider the following assertions :*

i) $S *_W \mathcal{D}^p_{\alpha,d} = \mathcal{D}^p_{\alpha,d}$.

ii)The Weinstein convolution operator defined by S is an automorphism of $\mathcal{D}^p_{\alpha,d}$.

iii) There exists a convolutor R in $\mathcal{D}^p_{\alpha,d}$ such that $S *_W R = \delta$.

iv) There exist $M > 0$ and $l \in \mathbb{N}$ such that

$$|\mathcal{F}_W^{\alpha,d}(S)(\xi)| \ge M(1+||\xi||^2)^{-l}, \ a.e.\ \xi \in \mathbb{R}^{d+1}_+.$$

Then, we have i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Rightarrow$ iv).

Proof. The proof of this results is in the same spirit with Theorem5. □

Proposition 23. *Let $2 \le p \le \infty$. Assume that $S \in (\mathcal{D}^p_{\alpha,d})'$ is a convolutor in $\mathcal{D}^p_{\alpha,d}$. We consider the following assertions :*

i) $S *_W (\mathcal{D}^p_{\alpha,d})' = (\mathcal{D}^p_{\alpha,d})'$.

ii) The Weinstein convolution operator defined by S is an automorphism of $\mathcal{D}^p_{\alpha,d}$,

iii) There exists a convolutor R in $\mathcal{D}^p_{\alpha,d}$ such that $S *_W R = \delta$.

iv) There exist $M > 0$ and $l \in \mathbb{N}$ such that

$$|\mathcal{F}_W^{\alpha,d}(S)(\xi)| \ge M(1+||\xi||^2)^{-l}, \ a.e.\ \xi \in \mathbb{R}^{d+1}_+.$$

Then, we have i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Rightarrow$ iv).

Proof. The proof of this results is in the same spirit with Theorem5. □

References

[1] **S. Abdullah,** *On convolution operator and multipliers of distributions of L^p-growth, J. Math. Anal. Appl. 183 (1994),p. 196-207.*

[2] **S. Abdullah,** *Solvability of convolution equations in D'_{L^p}, Hokkaido Math. J. 29 (2000), p. 247-254.*

[3] **H. Ben Mohamed,N. Bettaibi and S.H. Jah** *Sobolev type spaces associated with the Weinstein operator, Int. Journal of Math. Analysis, Vol.5,Nr.28,(2011), p.1353-1373.*

[4] **H. Ben Mohamed,B. Ghribi** *Weinstein-Sobolev spaces of exponential type and applications* . Acta Mathematica Sinica, English Series,Vol.29, Nr.3,(2013), p.591-608.

[5] **H. Ben Mohamed, A. Gasmi and N. Bettaibi. "** Inversion of the *Weinstein intertwining operator and its dual using Weinstein wavelets. ". An. St. Univ. Ovidius Constanta. Vol. 1,Nr1,(2016), p. 1-19.*

[6] **J.J. Betancor and B.J. Gonzalez,** Spaces of D_{L^p} type and Hankel convolution, Proc. Amer. Math. Soc. V. 129 N.1 (2001) 219-228.

[7] **H. Jarchow,** *Locally convex spaces, Teubner, Stuttgart, (1981).*

[8] **L. Hörmander,** On the division of distributions by polynomials, Arkiv för Mat. **3** (1958) 555- 568.

[9] **H. Mejjaoli and A. Ould Ahmed Salem.** *On the Dunkl convolution equations in spaces D^p_α type. Integ. Transf. and Special Func. Vol. 26, N 8,(2015), p. 619-634.*

[10] **D.H. Pahak,** *On the convolution equations in the spaces of distributions of L_p-growth, Proc. Amer. Math. Soc, 94 (1985), p.81-86.*

[11] **L.Schwartz,** *Théories des distributions, Hermann (Paris) 1966.*

[12] **E.M. Stein.** *Interpolation of linear operators. Amer. Math. Soc. Vol. 83,(1956), p. 482-492.*

[13] **E.M. Stein and G. Weiss.** *Introduction to Fourier Analysis on Euclidian spaces, Princeton Univ. Press, Princeton,NJ. (1971).*

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