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Properties of Classes of Multivalent Functions with Negative Coefficients and Starlike with Respect to Certain Points

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Abstract

In this paper, we have introduced some subclasses $S_s^*(m,p,\alpha,\beta)$***,*** $S_c^*(m,p,\alpha,\beta)$ ***and*** $S_{sc}^*(m,p,\alpha,\beta)$ ***consisting of analytic multivalent functions starlike with respect to symmetric points, starlike with respect to conjugate points and starlike with respect to symmetric conjugate points respectively and the corresponding subclasses with negative coefficients*** $S_s^*T(m,p,\alpha,\beta)$, $S_c^*T(m,p,\alpha,\beta)$ ***and*** $S_{sc}^*T(m,p,\alpha,\beta)$***. Here, we obtain coefficients inequality, growth and distrotion theorem, extreme points for the function of these subclasses. Also, we have obtained some other geometric properties and subordination result.***

Keywords: *Analytic function; multivalent function; starlike with respect to symmetric points; convex function; distortion bounds.*

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1 Introduction

Let $A(p)$ denote the class of analytic p-valent functions in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$, defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (a_{n+p} \in \mathbb{C}, p \in \mathbb{N}). \tag{1}$$

We note that $A(1) = A$, the class of analytic univalent functions. Let $T(p)$ be the subclass of $A(p)$ consisting of functions f of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (a_{n+p} \geq 0, p \in \mathbb{N}). \tag{2}$$

We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order α in U and satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in U,\ 0 \leq \alpha < 1).$$

Also, we denote by $K(\alpha)$ the subclass of A consisting of functions which are convex of order α in U and satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (z \in U,\ 0 \leq \alpha < 1).$$

The subclasses $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [14], (see also [24]). Sakagchi [16], introduced the class S_s^* of analytic univalent functions in U which are called starlike with respect to symmetric points and satisfies

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, z \in U.$$

EL-Ashwah and Thomas in [2], had introduced two other subclass namely S_c^* and S_{sc}^*. Kharinar and Rajas [9] (see also, [6, 25]) had discussed the subclass $S_s^*T(\alpha, \beta, \delta)$ of analytic multivalent functions in U and satisfying the condition

$$\left|\frac{zf'(z)}{f(z) - f(-z)} - (p + \delta)\right| < \beta\left|\frac{\alpha f'(z)}{f(z) - f(-z)} + p - \delta\right|,$$

for some $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \delta < p$ and $z \in U$.

For functions $f(z)$ belonging to the class $A(p)$, Orhan and Kiziltunc [13] defined the following differential operator which extend the Salagean operator [17]

$$\begin{aligned} D_p^0 f(z) &= f(z), \\ D_p^1 f(z) &= D_p f(z) = zf'(z) = pz^p - \sum_{n-1}^{\infty} (n+p)\, a_{n+p} z^{n+p}, \\ D_p^2 f(z) &= D_p(D_p f(z)) = p^2 z^p - \sum_{n-1}^{\infty} (n+p)^2\, a_{n+p} z^{n+p}, \\ &\ldots \\ D_p^m f(z) &= D_p\left(D_p^{m-1} f(z)\right) = p^m z^p - \sum_{n=1}^{\infty} (n+p)^m\, a_{n+p} z^{n+p}. \end{aligned}$$

Now, for $m \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $0 < \beta < 1$, $0 < \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$, we define three subclasses of $A(p)$ as follows:

Definition 1.1 *A function $f \in S_s^*(m,p,\alpha,\beta)$ is said to be p-valent starlike with respect to symmetric points if it satisfies*

$$\left| \frac{D_p^{m+1} f(z)}{D_p^m f(z) - D_p^m f(-z)} - p \right| \leq \beta \left| \frac{\alpha D_p^{m+1} f(z)}{D_p^m f(z) - D_p^m f(-z)} + p \right| \quad \text{for } \ p \in \mathbb{N} \ \text{ and } \ z \in U.$$

Definition 1.2 *A function $f \in S_c^*(m,p,\alpha,\beta)$ is said to be p-valent starlike with respect to conjugate points if it satisfies*

$$\left| \frac{D_p^{m+1} f(z)}{D_p^m f(z) + \overline{D_p^m f(\overline{z})}} - p \right| \leq \beta \left| \frac{\alpha D_p^{m+1} f(z)}{D_p^m f(z) + \overline{D_p^m f(\overline{z})}} + p \right| \quad \text{for } \ p \in \mathbb{N} \ \text{ and } \ z \in U.$$

Definition 1.3 *A function $f \in S_{sc}^*(m,p,\alpha,\beta)$ is said to be p-valent starlike with respect to symmetric conjugate points if it satisfies*

$$\left| \frac{D_p^{m+1} f(z)}{D_p^m f(z) - \overline{D_p^m f(\overline{-z})}} - p \right| \leq \beta \left| \frac{\alpha D_p^{m+1} f(z)}{D_p^m f(z) - \overline{D_p^m f(\overline{-z})}} + p \right| \quad \text{for } \ p \in \mathbb{N} \ \text{ and } \ z \in U.$$

Let $S_s^*T(m,p,\alpha,\beta) = S_s^*(m,p,\alpha,\beta) \cap T(p)$, $S_c^*T(m,p,\alpha,\beta) = S_c^*(m,p,\alpha,\beta) \cap T(p)$ and $S_{sc}^*T(m,p,\alpha,\beta) = S_{sc}^*(m,p,\alpha,\beta) \cap T(p)$.

By specializing the parameters in the above definitions, we obtain some spacial cases, as follows:

- For $p = 1$, we obtain the subclasses which introduced in ([1]);
- For $p = 1$ and $m = 1$, we obtain the subclasses which introduced in [7, 8] see also ([23] with $k = 1$ and $\sigma = 0$);
- For $p = 1$ and $m = 0$, we obtain the subclasses which introduced in [3, 5] see also ([4] with $k = 1$ and $\sigma = 0$ and [10] with $\delta = 0$);
- For $m = 0$, we obtain the subclasses which introduced in ([9] with $\delta = 0$).

In this paper, we obtain coefficients inequality, growth and distortion theorem, extreme points for the function of these subclasses. Also, we have obtained integral operator properties, integral mean and subordination result for these subclasses.

2 Coefficients estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $m \in \mathbb{N}_0$, $p \in \mathbb{N}$, $0 \leq \alpha < 1$, $0 < \beta < 1$, $0 < \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$. In this section, the authors obtained coefficients estimates.

Theorem 2.1 *Let $f(z)$ defined by (1) and satisfied the condition*

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^m \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p} a_{n+p} \leq 1, \tag{3}$$

then $f(z) \in S_s^*(m, p, \alpha, \beta)$.

Proof. Let the condition (3) is true. Then, we have

$$\left|D_p^{m+1} f(z) - pD_p^m f(z) + pD_p^m f(-z)\right| - \beta\left|\alpha D_p^{m+1} f(z) + pD_p^m f(z) - pD_p^m f(-z)\right|$$

$$= \left| p^{m+1} z^p + \sum_{n=1}^{\infty} (n+p)^{m+1} a_{n+p} z^{n+p} \right.$$
$$\left. -p\left(p^m z^p + \sum_{n=1}^{\infty} (n+p)^m a_{n+p} z^{n+p} - (-1)^p p^m z^p - \sum_{n=1}^{\infty} (-1)^{n+p} (n+p)^m a_{n+p} z^{n+p}\right)\right|$$

$$-\beta\left|\alpha p^{m+1}z^p+\alpha\sum_{n=1}^{\infty}(n+p)^m a_{n+p}z^{n+p}\right.$$

$$\left.+p\left(p^m z^p+\sum_{n=1}^{\infty}(n+p)^m a_{n+p}z^{n+p}-(-1)^p p^m z^p-\sum_{n=1}^{\infty}(-1)^{n+p}(n+p)^m a_{n+p}z^{n+p}\right)\right|$$

$$\leq \sum_{n=1}^{\infty}\left[(n+p)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)p\right](n+p)^m a_{n+p}$$

$$-\left[\beta\left(\alpha+\left(1-(-1)^P\right)\right)+(-1)^p\right]p^{m+1}\leq 0$$

Hence, by the maximum modulus theorem, we have $f\in S_s^*(m,p,\alpha,\beta)$.

Theorem 2.2 *The function $f(z)$ given by (2) is in the subclass $S_s^*T(m,p,\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m\frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}a_{n+p}\leq 1. \quad (4)$$

Proof. We only need to prove the only if part of Theorem 2.1. For function $f(z)\in T(p)$, we can write

$$\left|\frac{\frac{D_p^{m+1}f(z)}{D_p^m f(z)-D_p^m f(-z)}-p}{\frac{\alpha D_p^{m+1}f(z)}{D_p^m f(z)-D_p^m f(-z)}+p}\right|$$

$$=\left|\frac{(-1)^p p^{m+1}z^p-\sum_{n=1}^{\infty}(n+p)^m\left[(n+p)-p+(-1)^{n+p}p\right]a_{n+p}z^{n+p}}{(\alpha+1-(-1)^p)p^{m+1}z^p-\sum_{n=1}^{\infty}(n+p)^m\left[(n+p)\alpha+p-(-1)^{n+p}p\right]a_{n+p}z^{n+p}}\right|$$

$$<\beta,$$

since $\Re(z)\leq|z|$ for all z, we have

$$\Re\left\{\frac{-(-1)^p p^{m+1}z^p+\sum_{n=1}^{\infty}(n+p)^m\left[(n+p)-p+(-1)^{n+p}p\right]a_{n+p}z^{n+p}}{(\alpha+1-(-1)^p)p^{m+1}z^p-\sum_{n=1}^{\infty}(n+p)^m((n+p)\alpha+p-(-1)^{n+p}p)a_{n+p}z^{n+p}}\right\}<\beta. \quad (5)$$

Choose values of z on the real axis so that $\frac{D_p^{m+1}f(z)}{D_p^m f(z)-D_p^m f(-z)}$ is real and $D_p^m f(z)-D_p^m f(-z)\neq 0$ for $z\neq 0$. Upon clearing the denominator in (5) and letting $z\to 1^-$ through real values, we obtain.

$$-(-1)^p p^{m+1}+\sum_{n=1}^{\infty}(n+p)^m\left[(n+p)-p+(-1)^{n+p}p\right]a_{n+p}$$

$$-\beta\left[(\alpha+1-(-1)^p)p^{m+1}-\sum_{n=1}^{\infty}(n+p)^m((n+p)\alpha+p-(-1)^{n+p}p)a_{n+p}\right]$$

$$\leq 0.$$

This gives the required condition (4).

Corollary 2.3 *If $f \in S_s^* T(m,p,\alpha,\beta)$ if and only if*

$$a_{n+p} \leq \left(\frac{p}{n+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}. \tag{6}$$

The equality in (6) is attained for the function $f(z)$ given by

$$f(z)=z^p-\left(\frac{p}{n+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)} z^{n+p}. \tag{7}$$

Theorem 2.4 *Let $f(z)$ defined by (1) and satisfied the condition*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+2(\beta-1)\right]}{\beta(\alpha+2)-1} a_{n+p} \leq 1. \tag{8}$$

then $f(z) \in S_c^*(m,p,\alpha,\beta)$.

Proof. Let the condition (8) is true. Then, we have

$$\left|D_p^{m+1} f(z)-p D_p^m f(z)+p \overline{D_p^m f(\overline{z})}\right|-\beta\left|\alpha D_p^{m+1} f(z)+p D_p^m f(z)-p \overline{D_p^m f(\overline{z})}\right|$$

$$\begin{aligned}
= & \left|p^{m+1} z^p+\sum_{n=1}^{\infty}(n+p)^{m+1} a_{n+p} z^{n+p}-p\left(p^m z^p+\sum_{n=1}^{\infty}(n+p)^m a_{n+p} z^{n+p}-p^m z^p\right.\right. \\
- & \left.\left.\sum_{n=1}^{\infty}(n+p)^m a_{n+p} z^{n+p}\right)\right|-\beta\left|\alpha p^{m+1} z^p+\alpha \sum_{n=1}^{\infty}(n+p)^m a_{n+p} z^{n+p}\right. \\
& \left.+p\left(p^m z^p+\sum_{n=1}^{\infty}(n+p)^m a_{n+p} z^{n+p}-p^m z^p-\sum_{n=1}^{\infty}(n+p)^m a_{n+p} z^{n+p}\right)\right| \\
& \leq \sum_{n=1}^{\infty}\left[(n+p)(\alpha\beta+1)+2(\beta-1) p\right](n+p)^m a_{n+p}-\left[\beta(\alpha+2)+1\right] p^{m+1} \leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in S_c^*(m,p,\alpha,\beta)$.

Theorem 2.5 *The function $f(z)$ given by (2) is in the subclass $S_c^* T(m,p,\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+2(\beta-1)\right]}{\beta(\alpha+2)-1} a_{n+p} \leq 1. \tag{9}$$

Proof. We only need to prove the only if part of Theorem 2.5. For function $f(z) \in T(p)$, we can write

$$\left| \frac{\frac{D_p^{m+1} f(z)}{D_p^m f(z) + \overline{D_p^m f(\overline{z})}} - p}{\frac{\alpha D_p^{m+1} f(z)}{D_p^m f(z) + \overline{D_p^m f(\overline{z})}} + p} \right| = \left| \frac{p^{m+1} z^p - \sum_{n=1}^{\infty} (n+p)^m \left[(n+p) - 2p\right] a_{n+p} z^{n+p}}{(\alpha+2) p^{m+1} z^p - \sum_{n=1}^{\infty} (n+p)^m \left[(n+p)\alpha + 2p\right] a_{n+p} z^{n+p}} \right| < \beta,$$

since $\Re(z) \leq |z|$ for all z, we have

$$\Re \left\{ \frac{-p^{m+1} z^p + \sum_{n=1}^{\infty} (n+p)^m \left[(n+p) - 2p\right] a_{n+p} z^{n+p}}{(\alpha+2) p^{m+1} z^p - \sum_{n=1}^{\infty} (n+p)^m ((n+p)\alpha + 2p) a_{n+p} z^{n+p}} \right\} < \beta. \tag{10}$$

Choose values of z on the real axis so that $\frac{D_p^{m+1} f(z)}{D_p^m f(z) - \overline{D_p^m f(\overline{z})}}$ is real and $D_p^m f(z) - \overline{D_p^m f(\overline{z})} \neq 0$ for $z \neq 0$. Upon clearing the denominator in (10) and letting $z \to 1^-$ through real values, we obtain

$$-p^{m+1} + \sum_{n=1}^{\infty} (n+p)^m \left[(n+p) - 2p\right] a_{n+p} - \beta \left[(\alpha+2) p^{m+1} - \sum_{n=1}^{\infty} (n+p)^m ((n+p)\alpha + 2p) \right] a_{n+p} \leq 0.$$

This gives the required condition (9).

Corollary 2.6 *If $f \in S_c^* T\,(m, p, \alpha, \beta)$ then*

$$a_{n+p} \leq \left(\frac{p}{n+p} \right)^m \frac{\beta\,(\alpha+2) - 1}{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + 2\,(\beta-1)}. \tag{11}$$

The equality in (11) is attained for the function $f(z)$ given by

$$f(z) = z^p - \left(\frac{p}{n+p} \right)^m \frac{\beta\,(\alpha+2) - 1}{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + 2\,(\beta-1)} z^{n+p}. \tag{12}$$

Using the method of Theorem 2.1 and Theorem 2.2, we can prove the following theorems

Theorem 2.7 *Let $f(z)$ defined by (1) and satisfied the condition*

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p} \right)^m \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1 - (-1)^{n+p}\right) \right]}{\beta\left[\alpha + (1 - (-1)^p)\right] + (-1)^p} a_{n+p} \leq 1, \tag{13}$$

then $f(z) \in S_{sc}^*(m, p, \alpha, \beta)$.

Theorem 2.8 *A function $f \in S_{sc}^* T\,(m, p, \alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p} \right)^m \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1 - (-1)^{n+p}\right) \right]}{\beta\left[\alpha + (1 - (-1)^p)\right] + (-1)^p} a_{n+p} \leq 1.$$

Corollary 2.9 *If* $f \in S_{sc}^{*}T\left(m,p,\alpha,\beta\right)$ *then*

$$a_{n+p} \leq \left(\frac{p}{n+p}\right)^{m} \frac{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}. \tag{14}$$

The equality in (14) is attained for the function $f(z)$ given by

$$f(z)=z^{p}-\left(\frac{p}{n+p}\right)^{m} \frac{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)} z^{n+p}. \tag{15}$$

Remark 2.10

- Putting $m=0$ in the above theorems , we obtain the result obtained in ([9] with $\delta=0$);
- Putting $p=1$ in the above theorems, we obtain the result obtained in [1];
- Putting $p=1$ and $m=0$ in the above theorems, we obtain the result obtained in [3, 5] see also ([4] with $k=1$ and $\sigma=0$ and [10] with $\delta=0$);
- Putting $p=1$ and $m=1$ in the above theorems, we obtain the result obtained in [7, 8] see also ([23] with $k=1$ and $\sigma=0$).

3 Growth and Distortion Theorems

In this section, we give results concerning the growth and extreme points of the three subclasses $S_{s}^{*}T\left(m,p,\alpha,\beta\right)$, $S_{c}^{*}T\left(m,p,\alpha,\beta\right)$, $S_{sc}^{*}T\left(m,p,\alpha,\beta\right)$.

Theorem 3.1 *Let the function* f *be defined by (2) and belong to the subclass* $S_{s}^{*}T\left(m,p,\alpha,\beta\right)$. *Then for* $\{z : 0<|z|=r<1\}$,

$$r^{p}-\left(\frac{p}{1+p}\right)^{m} \frac{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{p+1}\right)} r^{1+p} \leq |f(z)|$$

$$\leq r^{p}+\left(\frac{p}{1+p}\right)^{m} \frac{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{p+1}\right)} r^{1+p}.$$

Proof. Let $f(z)$ defined by (2). From Theorem 2.2, we have

$$\left(\frac{1+p}{p}\right)^{m} \frac{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)}{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}} \sum_{n=1}^{\infty} a_{n+p}$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^{m} \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}} a_{n+p} \leq 1.$$

That is

$$\sum_{n=1}^{\infty} a_{n+p} \leq \left(\frac{p}{1+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)}. \tag{16}$$

Using (2) and (16), we have

$$\begin{aligned}
|f(z)| &\leq |z|^p + \sum_{n=1}^{\infty} a_{n+p} |z|^{n+p} \\
&\leq |z|^p + |z|^{1+p} \sum_{n=1}^{\infty} a_{n+p} \\
&\leq |z|^p + |z|^{1+p} \left(\frac{p}{1+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)} \\
&= r^p + \left(\frac{p}{1+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)} r^{1+p}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|f(z)| &\geq |z|^p - \sum_{n=1}^{\infty} a_{n+p} |z|^{n+p} \\
&\geq |z|^p - |z|^{1+p} \sum_{n=1}^{\infty} a_{n+p} \\
&\geq |z|^p - |z|^{1+p} \left(\frac{p}{1+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)} \\
&= r^p - \left(\frac{p}{1+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)} r^{1+p}.
\end{aligned}$$

This gives the proof of Theorem 3.1.

We note that result in Theorem 3.1 is sharp for the following function. Next, we state similar results for functions belongs to $S_c^* T(m,p,\alpha,\beta)$ and $S_{sc}^* T(m,p,\alpha,\beta)$.

$$f(z) = z^p - \left(\frac{p}{1+p}\right)^m \frac{\beta\left(\alpha+\left(1-(-1)^p\right)\right)+(-1)^p}{\left(\frac{1+p}{p}\right)(\beta\alpha+1)+(\beta-1)\left(1-(-1)^{1+p}\right)} z^{1+p} \text{ at } \ z = \pm r.$$

Theorem 3.2 *Let the function $f(z)$ be defined by (2) and belong to the subclass $S_c^* T(m,p,\alpha,\beta)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r^p - \left(\frac{p}{1+p}\right)^m \frac{\beta(\alpha+2)-1}{\left(\frac{1+p}{p}\right)(\alpha\beta+1)+2(\beta-1)} r^{1+p} \leq |f(z)|$$

$$\leq r^{p}+\left(\frac{p}{1+p}\right)^{m}\frac{\beta\left(\alpha+2\right)-1}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+2\left(\beta-1\right)}r^{1+p}.$$

The result is the sharp for

$$f\left(z\right)=z^{p}-\left(\frac{p}{1+p}\right)^{m}\frac{\beta\left(\alpha+2\right)-1}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+2\left(\beta-1\right)}z^{1+p},\qquad \text{at}\quad z=\pm r.$$

Theorem 3.3 *Let the function $f(z)$ be defined by (2) and belong to the subclass $S_{sc}^{*}T\left(m,p,\alpha,\beta\right)$. Then for $\left\{z:0<|z|=r<1\right\}$,*

$$r^{p}-\left(\frac{p}{1+p}\right)^{m}\frac{\beta\left(\alpha+\left(1-(-1)^{p}\right)\right)+(-1)^{p}}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+\left(\beta-1\right)\left(1-(-1)^{1+p}\right)}r^{1+p}\leq\left|f\left(z\right)\right|$$

$$\leq r^{p}+\left(\frac{p}{1+p}\right)^{m}\frac{\beta\left(\alpha+1-(-1)^{p}\right)+(-1)^{p}}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+\left(\beta-1\right)\left(1-(-1)^{1+p}\right)}r^{1+p}.$$

The result sharp for

$$f\left(z\right)=z^{p}-\left(\frac{p}{1+p}\right)^{m}\frac{\beta\left[\alpha+\left(1-(-1)^{p}\right)\right]+(-1)^{p}}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+\left(\beta-1\right)\left(1-(-1)^{1+p}\right)}z^{1+p},\qquad \text{at}\quad z=\pm r.$$

Theorem 3.4 *Let $f\in S_{s}^{*}T\left(m,p,\alpha,\beta\right)$ then for $\left\{z:0<|z|<1\right\}$*

$$pr^{p-1}-\left(\frac{p}{1+p}\right)^{m}\frac{(1+p)\left\{\beta\left[\alpha+\left(1-(-1)^{p}\right)\right]+(-1)^{p}\right\}}{\left(\frac{1+p}{p}\right)\left(\beta\alpha+1\right)+\left(\beta-1\right)\left(1-(-1)^{1+p}\right)}r^{p}\leq\left|f'\left(z\right)\right|$$

$$\leq pr^{p-1}+\left(\frac{p}{1+p}\right)^{m}\frac{(1+p)\left\{\beta\left[\alpha+\left(1-(-1)^{p}\right)\right]+(-1)^{p}\right\}}{\left(\frac{1+p}{p}\right)\left(\beta\alpha+1\right)+\left(\beta-1\right)\left(1-(-1)^{1+p}\right)}r^{p}.$$

Theorem 3.5 *Let $f\in S_{c}^{*}T\left(m,p,\alpha,\beta\right)$ then for $\left\{z:0<|z|<1\right\}$*

$$pr^{p-1}-\left(\frac{p}{1+p}\right)^{m}\frac{(1+p)\left[\beta(\alpha+2)-1\right]}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+2\left(\beta-1\right)}r^{p}\leq\left|f'\left(z\right)\right|$$

$$\leq pr^{p-1}+\left(\frac{p}{1+p}\right)^{m}\frac{(1+p)\left[\beta(\alpha+2)-1\right]}{\left(\frac{1+p}{p}\right)\left(\alpha\beta+1\right)+2\left(\beta-1\right)}r^{p}.$$

Theorem 3.6 *Let $f \in S^*_{sc}T(m,p,\alpha,\beta)$ then for $\{z : 0 < |z| < 1\}$*

$$pr^{p-1} - \left(\frac{p}{1+p}\right)^m \frac{(1+p)\{\beta[\alpha + (1-(-1)^p)] + (-1)^p\}}{\left(\frac{1+p}{p}\right)(\beta\alpha+1) + (\beta-1)\left(1-(-1)^{1+p}\right)} r^p \leq |f'(z)|$$

$$\leq pr^{p-1} + \left(\frac{p}{p+1}\right)^m \frac{(1+p)\{\beta[\alpha + (1-(-1)^p)] + (-1)^p\}}{\left(\frac{1+p}{p}\right)(\beta\alpha+1) + (\beta-1)\left(1-(-1)^{1+p}\right)} r^p.$$

Remark 3.7

- Putting $m = 0$ in the above theorems , we obtain the result obtained in ([9] with $\delta = 0$);
- Putting $p = 1$ in the above theorems, we obtain the result obtained in ([1] with $i = 1$);
- Putting $p = 1$ and $m = 0$ in the above theorems, we obtain the result obtained in [3, 5] see also ([4] with $k = 1$ and $\sigma = 0$ and [10] with $\delta = 0$);
- Putting $p = 1$ and $m = 1$ in the above theorems, we obtain the result obtained in [8].

4 Closure theorems

All three subclasses discussed here are closed under convex linear combinations. In this section, We obtained the closure theorems for the subclasses $S^*_sT(m,p,\alpha,\beta)$, $S^*_cT(m,p,\alpha,\beta)$ and $S^*_{sc}T(m,p,\alpha,\beta)$. We consider the functions $f_j(z)$ defined as

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{n+p,j} z^{n+p} \quad (a_{n,j} \geq 0,\ j = 1,2,...,l)\,. \tag{17}$$

Theorem 4.1 *Let the function $f_j(z)$ be in the subclass $S^*_sT(m,p,\alpha,\beta)$, then $g(z)$ defined as*

$$g(z) = \sum_{j=1}^{l} c_j f_j(z), \qquad \sum_{j=1}^{l} c_j = 1,$$

also belongs to the subclass $S^*_sT(m,p,\alpha,\beta)$.

Proof. Since $f_j(z)$ are in the subclass $S^*_sT(m,p,\alpha,\beta)$, it follows from Theorem 2.2 that

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^m \frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta[\alpha + (1-(-1)^p)] + (-1)^p} a_{n+p,j} \leq 1, \quad (j = 1,2,...,l).$$

Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{l}\left(\frac{n+p}{p}\right)^{m}\frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}}c_j a_{n+p,j} \\
=\;& \sum_{j=1}^{l}\left[\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^{m}\frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}}a_{n+p,j}\right]c_j \\
\leq\;& \sum_{j=1}^{l}c_j=1.
\end{aligned}$$

From Theorem 2.2, it follows that $g(z)\in S_s^{*}T\left(m,p,\alpha,\beta\right)$. This completes the proof of Theorem 4.1.

Corollary 4.2 *Let the function* $f_j(z)$ *be in the subclass* $S_c^{*}T\left(m,p,\alpha,\beta\right)$, *then* $g(z)$ *defined as*

$$g(z)=\sum_{j=1}^{l}c_j f_j(z),\qquad \sum_{j=1}^{l}c_j=1,$$

also belongs to the subclass $S_c^{*}T\left(m,p,\alpha,\beta\right)$.

Corollary 4.3 *Let the function* $f_j(z)$ *be in the subclass* $S_{sc}^{*}T\left(m,p,\alpha,\beta\right)$, *then* $g(z)$ *defined as*

$$g(z)=\sum_{j=1}^{l}c_j f_j(z),\qquad \sum_{j=1}^{l}c_j=1,$$

also belongs to the subclass $S_{sc}^{*}T\left(m,p,\alpha,\beta\right)$.

Corollary 4.4 *Let* $f_j(z)$ $(j=1,2)$ *defined by (17) be in the subclass* $S_s^{*}T\left(m,p,\alpha,\beta\right)$, *then*

$$h_1(z)=\lambda f_1(z)+(1-\lambda)f_2(z)\ (0\leq\lambda\leq 1),$$

belongs to $S_s^{*}T\left(m,p,\alpha,\beta\right)$.

5 Extrem points

In this section, we determine the extreme points of the subclasses $S_s^{*}T\left(m,p,\alpha,\beta\right)$, $S_c^{*}T\left(m,p,\alpha,\beta\right)$ and $S_{sc}^{*}T\left(m,p,\alpha,\beta\right)$.

Theorem 5.1 *Let* $f_p\left(z\right)=z^p$ *and*

$$f_{n+p}\left(z\right)=z^p-\left(\frac{p}{n+p}\right)^{m}\frac{\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}z^{n+p},\quad p\in\mathbb{N},n\geq 1,$$

then $f \in S_s^*T(m,p,\alpha,\beta)$ *if and only if it can be expressed as follows*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z),$$

where $\lambda_{n+p} \geq 0,\ \sum\limits_{n=0}^{\infty} \lambda_{n+p} = 1,\ p \in \mathbb{N}.$

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) \\ &= z^p - \sum_{n=1}^{\infty} \left(\frac{p}{n+p}\right)^m \frac{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)} \lambda_{n+p} z^{n+p}. \end{aligned}$$

Now, since $f(z) \in S_s^*T(m,p,\alpha,\beta)$

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^m \frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p} \\ &\left(\frac{p}{n+p}\right)^m \frac{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)} \lambda_{n+p} \\ = &\sum_{n=1}^{\infty} \lambda_{n+p} = 1 - \lambda_p \leq 1, \qquad p \in \mathbb{N}. \end{aligned}$$

Conversely, suppose that $f(z) \in S_s^*T(m,p,\alpha,\beta)$. Then

$$a_{n+p} \leq \left(\frac{p}{n+p}\right)^m \frac{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)}, \quad n \geq 1.$$

Set

$$\lambda_{n+p} = \left(\frac{n+p}{p}\right)^m \frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p} a_{n+p}, \quad n \geq 1, p \in \mathbb{N},$$

and $\lambda_p = 1 - \sum\limits_{n=1}^{\infty} \lambda_{n+p}$ then $f(z) = \sum\limits_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z)$. This completes the proof of Theorem 5.1.

Corollary 5.2 *The extreme points of the subclass* $S_s^*T(m,p,\alpha,\beta)$ *are the functions* $f_{n+p}(z)$ *given by Theorem 5.1.*

Method of proving next Theorems are similarly to that of Theorem 5.1 and extreme points for functions belonging to $S_c^*T(m,p,\alpha,\beta)$ and $S_{sc}^*T(m,p,\alpha,\beta)$ are obtained.

Theorem 5.3 *Let* $f_p(z) = z^p$,

$$f_{n+p}(z) = z^p - \left(\frac{p}{n+p}\right)^m \frac{\beta(\alpha+2)-1}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+2(\beta-1)} z^{n+p}, \quad p \in \mathbb{N}, n \geq 1$$

then $f \in S_c^*T(m,p,\alpha,\beta)$ *if and only if it can be expressed as follows*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z),$$

where $\lambda_{n+p} \geq 0$, $\sum\limits_{n=0}^{\infty} \lambda_{n+p} = 1$, $p \in \mathbb{N}$.

Corollary 5.4 *The extreme points of the subclass* $S_c^*T(m,p,\alpha,\beta)$ *are the functions* $f_{n+p}(z)$ *given by Theorem 5.3.*

Theorem 5.5 *Let* $f_p(z) = z^p$,

$$f_{n+p}(z) = z^p - \left(\frac{p}{n+p}\right)^m \frac{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)} z^{n+p}, \quad p \in \mathbb{N}, n \geq 1,$$

then $f \in S_{sc}^*T(m,p,\alpha,\beta)$ *if and only if it can be expressed as follows*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z)$$

where $\lambda_{n+p} \geq 0$, $\sum\limits_{n=0}^{\infty} \lambda_{n+p} = 1$, $p \in \mathbb{N}$.

Corollary 5.6 *The extreme points of the subclass* $S_{sc}^*T(m,p,\alpha,\beta)$ *are the functions* $f_{n+p}(z)$ *given by Theorem 5.5.*

6 Radii of close to convexity, starlikeness and convexity

In this section, radii of close to convexity, starlikeness and convexity for functions belonging to the subclass $S_s^*T(m,p,\alpha,\beta)$ are obtained.

Theorem 6.1 *Let the function $f(z)$ defined by (2) be in the subclass $S_s^*T(m,p,\alpha,\beta)$ then $f(z)$ is close to convex of order δ in $|z|<r_1$, where*

$$r_1=\inf_{n\geq 1}\left\{\left(\frac{p-\delta}{n+p}\right)\left(\frac{n+p}{p}\right)^m\frac{\binom{n+p}{p}(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right\}^{\frac{1}{n}}, \tag{18}$$

$0\leq\delta<1, n\geq 1.$

The result is sharp with the extremal function given by (7).

Proof. For close to convexity, it is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\delta$ for $|z|<r_1$, we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq\sum_{n=1}^{\infty}(n+p)\,a_{n+p}\,|z|^n.$$

Thus, $\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\delta$ if

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p-\delta}\right)a_{n+p}\,|z|^n\leq 1. \tag{19}$$

According to Theorem 2.2, we have

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m\frac{\binom{n+p}{p}(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}a_{n+p}\leq 1.$$

Hence (19) will be true if

$$\left(\frac{n+p}{p-\delta}\right)|z|^n\leq\left(\frac{n+p}{p}\right)^m\frac{\binom{n+p}{p}(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}$$

or, if

$$|z|\leq\left\{\left(\frac{p-\delta}{n+p}\right)\left(\frac{n+p}{p}\right)^m\frac{\binom{n+p}{p}(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right\}^{\frac{1}{n}}. \tag{20}$$

The result follows easily from (20). This completes the proof of Theorem 6.1.

Theorem 6.2 *Let the function $f(z)$ define by (2) be in the subclass $S_s^*T(m,p,\alpha,\beta)$. Then $f(z)$ is starlike in $|z|<r_2$, where*

$$r_2=\inf_{n\geq 1}\left\{\left(\frac{p-\delta}{n+p-\delta}\right)\left(\frac{n+p}{p}\right)^m\frac{\binom{n+p}{p}(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right\}^{\frac{1}{n}}, 0\leq\delta<1, n\geq 1. \tag{21}$$

The result is sharp with the extremal function given by (7).

Proof. To find the required result, it is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\delta,\qquad |z|\leq r_2,$$

where r_2 is given by (21). Now

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq\frac{\sum\limits_{n=1}^{\infty}na_{n+p}|z|^n}{1-\sum\limits_{n=1}^{\infty}a_{n+p}|z|^n}.$$

The above expression is less then $p-\delta$, if

$$\sum_{n=1}^{\infty}\left(\frac{n+p-\delta}{p-\delta}\right)a_{n+p}|z|^n\leq 1. \tag{22}$$

Using the fact $f(z)\in S_s^*T(m,p,\alpha,\beta)$ if and only if

$$\left(\frac{n+p}{p}\right)^m\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}a_{n+p}\leq 1.$$

Hence (22) will be true if

$$\left(\frac{n+p-\delta}{p-\delta}\right)|z|^n\leq\left(\frac{n+p}{p}\right)^m\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}.$$

Or, equivalently,

$$|z|\leq\left\{\left(\frac{p-\delta}{n+p-\delta}\right)\left(\frac{n+p}{p}\right)^m\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right\}^{\frac{1}{n}}. \tag{23}$$

The result follows easily from (23). This completes the proof of Theorem 6.2.

Theorem 6.3 *Let the function $f(z)$ defined by (2) be in the subclass $S_s^*T(m,p,\alpha,\beta)$, then $f(z)$ is convex in $|z|<r_3$, where*

$$r_3=\inf_{n\geq 1}\left\{\left(\frac{n+p}{p}\right)^m\left(\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right)\left(\frac{p(p-\delta)}{(n+p)(n+p-\delta)}\right)\right\}^{\frac{1}{n}} \tag{24}$$

The result is sharp with the extremal function given by (7).

Proof. It is sufficient to prove that

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|\le p-\delta,\ \ |z|\le r_3,$$

where r_3 given by (24). Indeed, we find that

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|\le \frac{\sum\limits_{n=1}^{\infty} n(n+p)\,a_{n+p}\,|z|^n}{p-\sum\limits_{n=1}^{\infty}(n+p)\,a_{n+p}\,|z|^n}.$$

Thus $\left|\frac{zf''(z)}{f'(z)}+1-p\right|\le p-\delta$, if

$$\sum_{n=1}^{\infty}\frac{(n+p)(n+p-\delta)\,a_{n+p}\,|z^n|}{p(p-\delta)}\le 1. \tag{25}$$

Using Theorem 2.2 then (25) will be true if

$$\frac{(n+p)(n+p-\delta)}{p(p-\delta)}|z|^n\le\left(\frac{n+p}{p}\right)^m\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}.$$

Or, equivalent

$$|z|\le\left\{\left(\frac{n+p}{p}\right)^m\left(\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right)\left(\frac{p(p-\delta)}{(n+p)(n+p-\delta)}\right)\right\}^{\frac{1}{n}},$$

$$(n\ge 1). \tag{26}$$

The theorem follows easily from (26).

7 Integral Mean inequalities

In this section, integral means for functions belonging to the subclass $S_s^*T(m,p,\alpha,\beta)$ are obtained. In [18], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [19] and settled in [20]. In this section, we prove Silverman's conjecture for functions in the subclass $S_s^*T(m,p,\alpha,\beta)$.

Lemma 7.1 *[11] let $f,g\in A$ if $f\prec g$, then for $z=re^{i\theta}\,(0<r<1)$ and $\delta>0$, we have*

$$\int\limits_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^\delta d\theta\le\int\limits_0^{2\pi}\left|g\left(re^{i\theta}\right)\right|^\delta d\theta,$$

Theorem 7.2 *Suppose* $f \in S_s^*T(m,p,\alpha,\beta)$, $\delta > 0$ *and* $0 < r < 1$ *then*

$$\int_0^{2\pi} |f(z)|^\delta d\theta \leq \int_0^{2\pi} |f_{p+1}(z)|^\delta d\theta \qquad (0 < r < 1, \delta > 0), \tag{27}$$

where

$$f_{p+1}(z) = z^p - \left(\frac{p}{1+p}\right)^m \frac{\beta[\alpha + (1-(-1)^p)] + (-1)^p}{\left(\frac{1+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{1+p}\right)} z^{p+1}. \tag{28}$$

Proof. For function $f(z)$ given by (2) the inequality (27) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=1}^\infty a_{n+p} z^n\right|^\delta d\theta \leq \int_0^{2\pi} \left|1 - \left(\frac{p}{1+p}\right)^m \frac{\beta[\alpha + (1-(-1)^p)] + (-1)^p}{\left(\frac{1+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{1+p}\right)} z\right|^\delta d\theta$$

by Lemma 7.1, it suffices to show that

$$1 - \sum_{n=1}^\infty a_{n+p} z^n \prec 1 - \left(\frac{p}{1+p}\right)^m \frac{\beta[\alpha + (1-(-1)^p)] + (-1)^p}{\left(\frac{1+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{1+p}\right)} z.$$

Thus, by setting

$$\sum_{n=1}^\infty a_{n+p} z^n = \left(\frac{p}{1+p}\right)^m \frac{\beta[\alpha + (1-(-1)^p)] + (-1)^p}{\left(\frac{1+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{1+p}\right)} \omega(z),$$

and use Theorem 2.2

$$\begin{aligned} |\omega(z)| &= \left|\sum_{n=1}^\infty \left(\frac{1+p}{p}\right)^m \frac{\left(\frac{1+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{1+p}\right)}{\beta[\alpha + (1-(-1)^p)] + (-1)^p} a_{n+p} z^n\right| \\ &\leq |z| \sum_{n=1}^\infty \left(\frac{n+p}{p}\right)^m \frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)}{\beta[\alpha + (1-(-1)^p)] + (-1)^p} a_{n+p} \\ &\leq |z|. \end{aligned}$$

This completes the proof of Theorem 7.2.

8 A family of integral operator

Saitoh et al.[15] defined the integral operator $J_{c,p}$ by

$$\begin{aligned} J_{c,p} &= \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, \qquad (f(z) \in S, c > -p, p \in N) \\ &= z^p + \sum_{n=1}^\infty \frac{c+p}{c+p+n} a_{n+p} z^{n+p}. \end{aligned}$$

In this section, a family of integral operators for functions belonging to the subclass $S_s^*T\left(m,p,\alpha,\beta\right)$ are discussed.

Theorem 8.1 *Let $f(z)$ defined by (2) be in the subclass $S_s^*T\left(m,p,\alpha,\beta\right)$ and c be a real number such that $c>-p$. Then the function $F\left(z\right)$ defined by*

$$F\left(z\right)=\frac{c+p}{z^c}\int\limits_0^z t^{c-1}f\left(t\right)dt, \tag{29}$$

also belongs to the subclass $S_s^*T\left(m,p,\alpha,\beta\right)$.

Proof. From the representation of $F\left(z\right)$, it follows that

$$F\left(z\right)=z^p-\sum_{n=1}^{\infty}b_{n+p}z^{n+p},\qquad b_{n+p}=\frac{c+p}{c+n+p}a_{n+p}.$$

Therefore

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]b_{n+p}\\
=&\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]\frac{c+p}{c+n+p}a_{n+p}\\
\leq&\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^m\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]a_{n+p}\\
\leq&\ \beta\left[\alpha+\left(1-(-1)^p\right)\right]+(-1)^p,
\end{aligned}$$

since $f\left(z\right)\in S_s^*T\left(m,p,\alpha,\beta\right)$. By Theorem 2.2, $F\left(z\right)\in S_s^*T\left(m,p,\alpha,\beta\right)$.

Theorem 8.2 *Let c be a real number such that $c>-p$. If $F\left(z\right)\in S_s^*T\left(m,p,\alpha,\beta\right)$, then the function $F\left(z\right)$ define by (29) is p-valent in $|z|<r^*$, where*

$$r^*=\inf_{n\geq1}\left\{\frac{(c+p)}{(n+p)(c+n+p)}\left(\frac{n+p}{p}\right)^m\frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+\left(1-(-1)^p\right)\right]+(-1)^p}\right\}^{\frac{1}{n}}.$$

The result is sharp.

Proof. Let $F(z)$ given by (2). It follows from (29) that

$$f\left(z\right)=\frac{z^{1-c}}{c+p}\left[z^cF\left(z\right)\right]'=z^p-\sum_{n=1}^{\infty}\frac{c+n+p}{c+p}a_{n+p}z^{n+p},c>-p.$$

In order to obtain the required result, it suffices to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < 1$ in $|z| < r^*$. Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \sum_{n=1}^{\infty} \frac{(n+p)(c+n+p)}{(c+p)} a_{n+p} z^n.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| < 1$, if

$$\sum_{n=1}^{\infty} \frac{(n+p)(c+n+p)}{(c+p)} a_{n+p} z^n < 1. \tag{30}$$

Hence, by using (3) and (30) will be satisfied if

$$\frac{(n+p)(c+n+p)}{(c+p)} |z|^n < \left(\frac{n+p}{p}\right)^m \frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}$$

i.e., if

$$|z| < \left\{\left(\frac{c+p}{(n+p)(c+n+p)}\right)\left(\frac{n+p}{p}\right)^m \frac{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)}{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}\right\}^{\frac{1}{n}}.$$

Therefore $F(z)$ in p-valent in $|z| < r^*$. Sharpness follows, if we take

$$f(z) = z^p - \left(\frac{(n+p)(c+n+p)}{c+p}\right)\left(\frac{p}{n+p}\right)^m \frac{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}{\left(\frac{n+p}{p}\right)(\alpha\beta+1) + (\beta-1)\left(1-(-1)^{n+p}\right)} z^{n+p}.$$

9 Partial sums

Silvia [22] studies the partial sums of convex functions of order α $(0 \le \alpha < 1)$. Later on, Silverman [21] and several researchers studied and generalized the results on partial sums for various subclasses of analytic functions. In this section, inequalities involving partial sums of $f(z) \in A(p)$ have discussed Let non zero partial sums of $f(z) \in A(p)$ of the form (1) define as follows.

$$f_p(z) = z^p, \quad f_k(z) = z^p + \sum_{n=1}^{k} a_{n+p} z^{n+p}, \qquad k \ge 1.$$

Theorem 9.1 *Let $f(z) \in S_s^*(m,p,\alpha,\beta)$ be given by* (1) *and satisfies the condition (3) and*

$$c_{n+p} \ge \begin{cases} 1, & n = 1,2,...,k; \\ c_{k+p+1} & n = k+1, k+2, ..., \end{cases}$$

where

$$c_{n+p}=\left(\frac{n+p}{p}\right)^{m}\frac{\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}}. \tag{31}$$

Then

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\}>1-\frac{1}{c_{k+p+1}}\,(z\in U;n\in\mathbb{N}) \tag{32}$$

and

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\}>\frac{c_{k+p+1}}{1+c_{k+p+1}} \tag{33}$$

Proof. For the coefficients c_{n+p} given by (31), it is not difficult to verify that

$$c_{n+p+1}>c_{n+p}>1.$$

Therefore, we have

$$\sum_{n=1}^{k}|a_{n+p}|+c_{k+p+1}\sum_{n=k+1}^{\infty}|a_{n+p}|\leq\sum_{n=1}^{\infty}c_{n+p}|a_{n+p}|\leq1. \tag{34}$$

Set

$$\begin{aligned} g_1(z) &= c_{k+p+1}\left\{\frac{f(z)}{f_k(z)}-\left(1-\frac{1}{c_{k+p+1}}\right)\right\} \\ &= 1+\frac{c_{k+p+1}\sum\limits_{n=k+1}^{\infty}a_{n+p}z^{n}}{1+\sum\limits_{n=1}^{k}a_{n+p}z^{n}}, \end{aligned}$$

and applying (34), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right|\leq\frac{c_{n+p+1}\sum\limits_{n=k+1}^{\infty}|a_{n+p}|}{2-2\sum\limits_{n=1}^{k}|a_{n+p}|-c_{k+p+1}\sum\limits_{n=k+1}^{\infty}|a_{n+p}|}\leq1,$$

if

$$\sum_{n=1}^{\infty}|a_{n+p}|+c_{k+p+1}\sum_{n=k+1}^{\infty}|a_{n+p}|\leq1.$$

From the condition (4), it is sufficient to show that

$$\sum_{n=1}^{\infty}|a_{n+p}|+c_{k+p+1}\sum_{n=k+1}^{\infty}|a_{n+p}|\leq\sum_{n=1}^{\infty}c_{n+p}|a_{n+p}|$$

which is equivalent to

$$\sum_{n=1}^{k}\left(c_{n+p}-1\right)\left|a_{n+p}\right|+\sum_{n=k+1}^{\infty}\left(c_{n+p}-c_{k+p+1}\right)\left|a_{n+p}\right|\geq 0,$$

which readily yields the assertion (32) of Theorem 9.1. In order to see that

$$f(z)=z^{p}+\frac{z^{k+p+1}}{c_{k+p+1}} \tag{35}$$

gives sharp result, we observe that for $z=re^{\frac{i\pi}{k+1}}$ that $\frac{f(z)}{f_k(z)}=1+\frac{z^{k+1}}{c_{k+p+1}}\to 1-\frac{1}{c_{k+p+1}}$ as $z\to 1^{-}$. Similarly, if we take

$$\begin{aligned} g_2\left(z\right) &= \left(1+c_{k+p+1}\right)\left\{\frac{f_k\left(z\right)}{f\left(z\right)}-\frac{c_{k+p+1}}{1+c_{k+p+1}}\right\} \\ &= 1-\frac{\left(1+c_{k+p+1}\right)\left(\sum\limits_{n=k+1}^{\infty}a_{n+p}z^{n}\right)}{1+\sum\limits_{n=1}^{\infty}a_{n+p}z^{n}} \end{aligned}$$

and making use of (34), we find that

$$\left|\frac{g_2\left(z\right)-1}{g_2\left(z\right)+1}\right|\leq\frac{\left(1+c_{k+p+1}\right)\sum\limits_{n=k+1}^{\infty}\left|a_{n+p}\right|}{2-2\sum\limits_{n=1}^{k}\left|a_{n+p}\right|-\left(1-c_{k+p+1}\right)\sum\limits_{n=k+1}^{\infty}\left|a_{n+p}\right|}$$

which leads us immediately to the assertion (33) of Theorem 9.1.

Theorem 9.2 *Let $f\left(z\right)\in S_s^{*}\left(m,p,\alpha,\beta\right)$ be given by* (1) *and satisfies the condition (3). Then*

$$\Re\left\{\frac{f'\left(z\right)}{f_k'\left(z\right)}\right\}\geq 1-\frac{k+p+1}{c_{k+p+1}} \tag{36}$$

and

$$\Re\left\{\frac{f_k'\left(z\right)}{f'\left(z\right)}\right\}\geq\frac{c_{k+p+1}}{k+p+1+c_{k+p+1}}, \tag{37}$$

where c_{n+p}defined by (31) and satisfies the condition

$$c_{n+p}\geq\begin{cases}\frac{n+p}{p}, & n=1,2,...,k,p\in\mathbb{N};\\ \left(\frac{c_{k+p+1}}{k+p+1}\right)\left(\frac{n+p}{p}\right) & n=k+1,k+2,...,\end{cases}$$

The results are sharp with the function $f(z)$ given by (35).

Proof. Set

$$g(z) = \frac{c_{k+p+1}}{k+p+1}\left\{\frac{f'(z)}{f_k'(z)} - \left(1 - \frac{k+p+1}{c_{k+p+1}}\right)\right\}$$
$$= \frac{1 + \frac{c_{k+p+1}}{k+p+1}\sum\limits_{n=k+1}^{\infty}(\frac{n+p}{p})a_{n+p}z^n + \sum\limits_{n=1}^{k}(\frac{n+p}{p})a_{n+p}z^n}{1 + \sum\limits_{n=1}^{k}(\frac{n+p}{p})a_{n+p}z^n},$$

then, we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\frac{c_{k+p+1}}{k+p+1}\sum\limits_{n=k+1}^{\infty}(\frac{n+p}{p})\left|a_{n+p}\right|}{2 - 2\sum\limits_{n=1}^{k}(\frac{n+p}{p})\left|a_{n+p}\right| - \frac{c_{k+p+1}}{k+p+1}\sum\limits_{n=k+1}^{\infty}(\frac{n+p}{p})\left|a_{n+p}\right|} \le 1,$$

if

$$\sum_{n=1}^{k}(\frac{n+p}{p})\left|a_{n+p}\right| + \frac{c_{k+p+1}}{k+p+1}\sum_{n=k+1}^{\infty}(\frac{n+p}{p})\left|a_{n+p}\right| \le 1. \tag{38}$$

Since the left side of (38) is bounded above by $\sum\limits_{n=1}^{\infty} c_{n+p}\left|a_{n+p}\right|$ if

$$\sum_{n=1}^{k}\left(c_{n+p} - \left(\frac{n+p}{p}\right)\right)\left|a_{n+p}\right| + \sum_{n=k+1}^{\infty}\left(c_{n+p} - \frac{c_{k+p+1}}{k+p+1}\left(\frac{n+p}{p}\right)\right)\left|a_{n+p}\right| \ge 0$$

and the proof of (36) is completed.

To prove the result (37), we define the function $h(z)$ by

$$h(z) = \frac{k+p+1+c_{k+p+1}}{k+p+1}\left\{\frac{f_k'(z)}{f'(z)} - \frac{c_{k+p+1}}{k+p+1+c_{k+p+1}}\right\}$$
$$= 1 - \frac{\left(1 + \frac{c_{k+p+1}}{k+p+1}\right)\sum\limits_{n=k+1}^{\infty}(\frac{n+p}{p})a_{n+p}z^n}{1 + \sum\limits_{n=1}^{\infty}(\frac{n+p}{p})a_{n+p}z^n},$$

then, we find that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \le \frac{\left(1 + \frac{c_{k+p+1}}{k+p+1}\right)\sum\limits_{n=k+1}^{\infty}(\frac{n+p}{p})a_{n+p}}{2 - 2\sum\limits_{n=1}^{k}(\frac{n+p}{p})\left|a_{n+p}\right| - \left(1 + \frac{c_{k+p+1}}{k+p+1}\right)\sum\limits_{n=k+1}^{\infty}(\frac{n+p}{p})\left|a_{n+p}\right|} \le 1,$$

which leads us to the assertion (37) of Theorem 9.2.

10 Subordination Result

In this section we obtain subordination theorem for the subclass $S_s^*(m,p,\alpha,\beta)$. To prove our result we need the following definitions and lemma.

Definition 10.1 *[12] For $f,g\in A$, we say that the function f is subordinate to g, if there exists a Schwarz function w, with $w(0)=0$ and $|w(z)|<1, z\in U$, such that $f(z)=g(w(z))$ for all $z\in U$. This subordination is usually denoted by $f(z)\prec g(z)$. It is well-known that, if the function g is univalent in U, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.*

Definition 10.2 *[26] A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence, if for each function f of the form $f(z)=\sum_{n=1}^{\infty}a_nz^n$ is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{n=1}^{\infty}b_na_nz^n\prec f(z)\qquad(z\in U, a_1=1).$$

Lemma 10.3 *[26] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{n=1}^{\infty}b_nz^n\right)>0\qquad(z\in U).\tag{39}$$

Theorem 10.4 *Let $f\in S_s^*(m,p,\alpha,\beta)$ and $g\in K$ then*

$$\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)p\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)p\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}$$
$$\frac{f(z)}{z^{p-1}}*g(z)\prec g(z),\qquad(z\in U),\tag{40}$$

and

$$\Re\left(\frac{f(z)}{z^{p-1}}\right)\tag{41}$$
$$>-\frac{\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]}.$$

The constant factor

$$\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}$$

cannot be replaced by a larger number.

Proof. Let a function f of the form (1) belong to the subclass $S_s^*(m,p,\alpha,\beta)$ and suppose that a function g of the form

$$g(z)=\sum_{n=1}^{\infty}c_nz^n \qquad (c_1=1; z\in U),$$

belongs to the subclass K. Then

$$\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}$$

$$\left(\frac{f(z)}{z^{p-1}}*g(z)\right)=\sum_{n=1}^{\infty}b_nc_nz^n.$$

where

$$b_n=\begin{cases}\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta[\alpha+(1-(-1)^p)]+(-1)^p\right\}} & \text{if } n=1,\\ \frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta[\alpha+(1-(-1)^p)]+(-1)^p\right\}}a_{n+p+1} & \text{if } n>1.\end{cases}$$

Thus, by Definition the subordination result (39) holds true if $\{b_n\}_{n=1}^{\infty}$ is the subordinating factor sequence. Since

$$\begin{aligned}&\left(\frac{n+p}{p}\right)^m\left[\left(\frac{n+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]a_{n+p}\\ \geq\ &\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]a_{n+p}, \quad (n\geq 1, n\in\mathbb{N}),\end{aligned}$$

we have

$$\begin{aligned}&\Re\left(1+2\sum_{n=1}^{\infty}b_nz^n\right)\\ \geq\ & 1-\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right](\beta-1)\left(1-(-1)^{1+p}\right)}{\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}r\\ &-\sum_{n=2}^{\infty}\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+\right]}{\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}\\ &\left|a_{n+p}\right|r^{n+p}.\end{aligned}$$

Thus, by using Theorem 2.1 we obtain

$$\begin{aligned}
&\Re\left(1+2\sum_{n=1}^{\infty}b_n z^n\right)\\
\geq\ & 1-\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]}{\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}r\\
&-\frac{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}{\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}r\\
=\ & 1-r>0 \quad (|z|=r<1).
\end{aligned}$$

This evidently proves the inequality (39) and hence the subordination result inequality (40). Inequality (41) follows from (40) by taking

$$g(z)=\frac{z}{1-z}=\sum_{n=1}^{\infty}z^n \qquad (z\in U)$$

Next, we observe that the function

$$f(z)=z-\left(\frac{p}{n+p}\right)^m\frac{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}{\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{1+p}\right)}z^{1+p} \qquad (z\in U)$$

clearly $f(z)$ belongs to the subclass $S_s^*(m,p,\alpha,\beta)$ for this function (40) becomes

$$\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}\frac{f(z)}{z^{p-1}}\prec\frac{z}{1-z}$$

it is easily verified that

$$\min\left\{\Re\left(\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}\frac{f(z)}{z^{p-1}}\right)\right\}=-\frac{1}{2}$$

and the constant

$$\frac{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]}{2\left\{\left(\frac{1+p}{p}\right)^m\left[\left(\frac{1+p}{p}\right)(\alpha\beta+1)+(\beta-1)\left(1-(-1)^{n+p}\right)\right]+\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right\}}$$

can not be replaced by any larger one.

11 Open Problem

Discuss all the classes properties using integral operator.

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