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Certain Results on $\phi-$like and Starlike functions in a Parabolic Region

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Abstract

Using the concept of differential subordination, we, here, find certain results on $\phi-$like, starlike, parabolic $\phi-$like and parabolic starlike functions.

Keywords: *analytic function, parabolic ϕ-like function, ϕ-like function, starlike function, parabolic starlike function, differential subordination.*
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1 Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{E} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, Taylor's series expansion of $f \in \mathcal{A}$, is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \quad |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
Let ϕ be analytic in a domain containing $f(\mathbb{E}), \phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$like in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \quad z \in \mathbb{E}.$$

This concept was introduced by L. Brickman [2]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$like for some ϕ. Later, Ruscheweyh [5] investigated the following general class of $\phi-$like functions:
Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E}) \setminus \{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function q, $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$like in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|, \quad z \in \mathbb{E}. \tag{2}$$

Define the parabolic domain Ω as under:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Ronning [1] and Ma and Minda [9] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

maps the unit disk $\mathbb{E}$ onto the parabolic domain Ω.
Therefore, equivalently condition (2) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2005, Ravichandran et al. [8] proved the sufficient condition for a function $f \in \mathcal{A}$ to be a $\phi-$like function with respect to q and established the following result:

Theorem 1.1 *Let $\alpha \neq 0$ be any complex number and $\beta = \max\{0, -\Re\frac{1}{\alpha}\}$. Let $q(z) \neq 0$ be analytic in $\mathbb{E}$ and $Q(z) = zq'(z)q^{\frac{1}{\alpha}-1}(z)$ be starlike of order β in $\mathbb{E}$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left(1+\frac{zf''(z)}{f'(z)}-\frac{\phi'(f(z))}{\phi(f(z))}zf'(z)\right)^{\alpha} \prec q(z)\left(1+\frac{zq'(z)}{q(z)}\right)^{\alpha},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),$$

and $q(z)$ is the best dominant.

In 2010, Singh et al. [6] also proved certain results on $\phi-$like functions of above nature. Cho and Kim [3] and Shanmugam et al. [7] also contributed to the study of $\phi-$like functions.
The main aim of the present paper is to study the similar type of results as mentioned above. We, here, find certain results as sufficient conditions for $\phi-$like functions with respect to a univalent function q. Furthermore, we also discussed the results on $\phi-$like, parabolic $\phi-$like, starlike and parabolic starlike functions with different dominants. So consequently, we get certain sufficient conditions for $\phi-$like, parabolic $\phi-$like, starlike and parabolic starlike functions.

To prove our main results, we shall use the following lemma of Miller and Mocanu ([4], p.132, Theorem 3.4h).

Lemma 1.2 *Let q be a univalent in $\mathbb{E}$ and let Θ and Φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\Phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\Phi[q(z)]$, $h(z) = \Theta[q(z)] + Q(z)$ and suppose that either*
(i)h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\frac{zh'(z)}{Q(z)} = \Re\left[\frac{\Theta'[q(z)]}{\Phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] > 0.$
If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\Theta[p(z)] + zp'(z)\Phi[p(z)] \prec \Theta[q(z)] + zq'(z)\Phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

2 Main Results

In what follows, all the powers taken are principal ones.

Theorem 2.1 *Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)}\right]>0$ *and*
(ii) $\Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)}+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)q(z)\right]>0.$
If $f\in\mathcal{A}$ satisfies

$$\left[\frac{zf'(z)}{\phi(f(z))}\right]^{\beta}\left[(1-\alpha)\frac{zf'(z)}{\phi(f(z))}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}\right]^{\gamma}$$
$$\prec (q(z))^{\beta}\left((1-\alpha)q(z)+\alpha\frac{zq'(z)}{q(z)}\right)^{\gamma}, \qquad (3)$$

then

$$\frac{zf'(z)}{\phi(f(z))}\prec q(z), \quad z\in\mathbb{E},$$

where α,β,γ are complex numbers such that $\alpha,\gamma\neq 0$, and q(z) is the best dominant.

Proof Write $p(z)=\frac{zf'(z)}{\phi(f(z))}, \quad z\in\mathbb{E}.$
Then the function $p(z)$ is analytic in $\mathbb{E}$ and $p(0)=1$. Therefore, using this substitution in (2.1), we obtain:

$$(p(z))^{\beta}\left((1-\alpha)p(z)+\alpha\frac{zp'(z)}{p(z)}\right)^{\gamma}\prec (q(z))^{\beta}\left((1-\alpha)q(z)+\alpha\frac{zq'(z)}{q(z)}\right)^{\gamma}.$$

Equivalently,

$$(p(z))^{\frac{\beta}{\gamma}}\left((1-\alpha)p(z)+\alpha\frac{zp'(z)}{p(z)}\right)\prec (q(z))^{\frac{\beta}{\gamma}}\left((1-\alpha)q(z)+\alpha\frac{zq'(z)}{q(z)}\right).$$

Let us define the function Θ and Φ as follows:

$$\Theta(w)=(1-\alpha)w^{\frac{\beta}{\gamma}+1}$$

and

$$\Phi(w)=\alpha w^{\frac{\beta}{\gamma}-1}.$$

Obviously, the functions Θ and Φ are analytic in domain $\mathbb{D}=\mathbb{C}\setminus\{0\}$ and $\Phi(w)\neq 0$, $w\in\mathbb{D}$. Therefore,

$$Q(z)=\alpha q(z)^{\frac{\beta}{\gamma}-1}zq'(z)$$

and

$$h(z)=(q(z))^{\frac{\beta}{\gamma}}\left((1-\alpha)q(z)+\alpha\frac{zq'(z)}{q(z)}\right).$$

On differentiating, we obtain $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\beta}{\gamma} - 1\right)\dfrac{zq'(z)}{q(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\gamma} - 1\right)\frac{zq'(z)}{q(z)} + \left(1 + \frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)q(z).$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from Lemma 1.1.

Selecting $\alpha = \beta = 1$ in Theorem 2.1, we get the following result which represents the correct version of Theorem 1.1.

Theorem 2.2 *Let $\gamma \neq 0$ be any complex number. Let $q(z) \neq 0$ be univalent in $\mathbb{E}$ and $Q(z) = zq'(z)q^{\frac{1}{\gamma}-1}(z)$ be starlike in $\mathbb{E}$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}^{\gamma} \prec q(z)\left(\frac{zq'(z)}{q(z)}\right)^{\gamma},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E},$$

and q(z) is the best dominant.

Selecting $\phi(w) = w$ in Theorem 2.1, we get:

Theorem 2.3 *Let $q(z) \neq 0$ be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\beta}{\gamma} - 1\right)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*
(ii)$\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\beta}{\gamma} - 1\right)\dfrac{zq'(z)}{q(z)} + \left(1 + \dfrac{\beta}{\gamma}\right)\left(\dfrac{1-\alpha}{\alpha}\right)q(z)\right] > 0.$
If $f \in \mathcal{A}$ satisfies

$$\left[\frac{zf'(z)}{f(z)}\right]^{\beta}\left[(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right]^{\gamma} \prec (q(z))^{\beta}\left((1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}\right)^{\gamma},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z), \quad z \in \mathbb{E},$$

where α, β, γ are complex numbers such that $\alpha, \gamma \neq 0$, and q(z) is the best dominant.

3 Applications

Remark 3.1 *When we select the dominant* $q(z) = 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *in Theorem 2.1, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)} = \frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

$$+\left(\frac{\beta}{\gamma}-1\right)\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} \text{ and}$$

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)}+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)q(z)$$

$$=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}+\left(\frac{\beta}{\gamma}-1\right)\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right].$$

Thus for $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $\frac{-3}{4} < \frac{\beta}{\gamma} < \frac{3}{2}$, *and* $0 < \alpha \leq 1$, *we notice that q(z) satisfies the conditions (i) and (ii) of Theorem 2.1 and consequently, conditions of Theorem 2.2 and Theorem 2.3 and therefore, we, respectively, get the next three results from Theorem 2.1, Theorem 2.2 and Theorem 2.3.*

Theorem 3.2 *Let* α, β *and* γ *be real numbers such that* $\frac{-3}{4} < \frac{\beta}{\gamma} < \frac{3}{2}$, *and* $0 < \alpha \leq 1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left[\frac{zf'(z)}{\phi(f(z))}\right]^{\beta}\left[(1-\alpha)\frac{zf'(z)}{\phi(f(z))}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}\right]^{\gamma} \prec$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\beta}$$

$$\left\{(1-\alpha)\left(1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right)+\alpha\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\gamma},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E},$$

i.e. f is parabolic $\phi-$*like in* $\mathbb{E}$.

Theorem 3.3 *Let γ be real number such that $\frac{-3}{4} < \frac{1}{\gamma} < \frac{3}{2}$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}^{\gamma} \prec \left(1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right)$$
$$\left\{\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\gamma},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E},$$

i.e. f is parabolic $\phi-$like in $\mathbb{E}$.

Theorem 3.4 *Let α, β and γ be real numbers such that $\frac{-3}{4} < \frac{\beta}{\gamma} < \frac{3}{2}$, and $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\left[\frac{zf'(z)}{f(z)}\right]^{\beta}\left[(1-2\alpha)\frac{zf'(z)}{f(z)}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right]^{\gamma} \prec \left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\beta}$$
$$\left\{(1-\alpha)\left(1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right)+\alpha\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\gamma},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E},$$

i.e. f is parabolic starlike in $\mathbb{E}$.

Remark 3.5 *When we select the dominant $q(z) = \dfrac{1+z}{1-z}$ in Theorem 2.1, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}+\left(\frac{\beta}{\gamma}-1\right)\frac{2z}{1-z^2} \text{ and}$$

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)}+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)q(z) = \frac{1+z}{1-z}+\left(\frac{\beta}{\gamma}-1\right)\frac{2z}{1-z^2}$$
$$+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)\frac{1+z}{1-z}.$$

Clearly, for $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma = \beta = 1$ and $0 < \alpha \leq 1$, q(z) satisfies conditions (i) and (ii) of Theorem 2.1 and consequently, conditions of Theorem 2.2 and Theorem 2.3 and we, respectively, obtain the following three results from Theorem 2.1, Theorem 2.2 and Theorem 2.3.

Theorem 3.6 *Let $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left[(1-\alpha)\frac{zf'(z)}{\phi(f(z))}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}\right] \prec (1-\alpha)\left(\frac{1+z}{1-z}\right)^2+\frac{2\alpha z}{(1-z)^2},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E},$$

i.e. f is $\phi-$like in $\mathbb{E}$.

Theorem 3.7 *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\} \prec \frac{2z}{(1-z)^2},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E},$$

i.e. f is $\phi-$like in $\mathbb{E}$.

Theorem 3.8 *Let $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[(1-2\alpha)\frac{zf'(z)}{f(z)}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right] \prec (1-\alpha)\left(\frac{1+z}{1-z}\right)^2+\frac{2\alpha z}{(1-z)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E},$$

i.e. f is starlike in $\mathbb{E}$.

Remark 3.9 *When we select the dominant $q(z) = e^{\frac{3}{2}z}$ in Theorem 2.1, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)} = 1+\frac{3\beta}{2\gamma}z \text{ and}$$

$$1+\frac{zq''(z)}{q'(z)}+\left(\frac{\beta}{\gamma}-1\right)\frac{zq'(z)}{q(z)}+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)q(z) = 1+\frac{3\beta}{2\gamma}z+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)e^{\frac{3}{2}z}.$$

Thus for $\beta, \gamma, \alpha \in \mathbb{R}$ such that $0 \leq \frac{\beta}{\gamma} < \frac{2}{3}$, and $0 < \alpha \leq 1$, we notice that q(z) satisfies the conditions (i) and (ii) of Theorem 2.1 and consequently, conditions of Theorem 2.2 and Theorem 2.3. Therefore, we, respectively, derive the next three results from Theorem 2.1, Theorem 2.2 and Theorem 2.3.

Theorem 3.10 *Let α, β, γ be real numbers such that $0 \leq \frac{\beta}{\gamma} < \frac{2}{3}$ and $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\left[\frac{zf'(z)}{\phi(f(z))}\right]^{\beta}\left[(1-\alpha)\frac{zf'(z)}{\phi(f(z))}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}\right]^{\gamma} \prec e^{\frac{3}{2}\beta z}\left((1-\alpha)e^{\frac{3}{2}z}+\frac{3\alpha}{2}z\right)^{\gamma},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^{\frac{3}{2}z}, \quad z \in \mathbb{E},$$

i.e. f is $\phi-$like in $\mathbb{E}$.

Theorem 3.11 *Let γ be real number such that $0 \leq \frac{1}{\gamma} < \frac{2}{3}$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left\{1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}^{\gamma} \prec e^{\frac{3}{2}z}\left(\frac{3}{2}z\right)^{\gamma},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^{\frac{3}{2}z}, \quad z \in \mathbb{E},$$

i.e. f is $\phi-$like in $\mathbb{E}$.

Theorem 3.12 *Let α, β, γ be real numbers such that $0 \leq \frac{\beta}{\gamma} < \frac{2}{3}$ and $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\left[\frac{zf'(z)}{f(z)}\right]^{\beta}\left[(1-2\alpha)\frac{zf'(z)}{f(z)}+\alpha\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right]^{\gamma} \prec e^{\frac{3\beta}{2}z}\left((1-\alpha)e^{\frac{3}{2}z}+\frac{3\alpha}{2}z\right)^{\gamma},$$

then

$$\frac{zf'(z)}{f(z)} \prec e^{\frac{3}{2}z}, \quad z \in \mathbb{E},$$

i.e. f is starlike in $\mathbb{E}$.

We illustrate the results of $\phi-$like and starlike functions with the following examples obtained by selecting $\alpha = \gamma = 1$ and $\beta = 0$ in Theorem 3.10 and Theorem 3.12 respectively.

Example 3.13 *If $f \in \mathcal{A}$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))} \prec \frac{3}{2}z = h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^{\frac{3}{2}z} = q(z), \quad z \in \mathbb{E},$$

i.e. f is $\phi-$like in $\mathbb{E}$.

Example 3.14 *If $f \in \mathcal{A}$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{3}{2}z = h(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec e^{\frac{3}{2}z} = q(z), \quad z \in \mathbb{E},$$

i.e. f is starlike in $\mathbb{E}$.

Remark 3.15 *For illustration, we plot the images of unit disk under the functions $h(z) = \frac{3}{2}z$ and $q(z) = e^{\frac{3}{2}z}$, which are given by Figure 4.1 and Figure 4.2 respectively.*

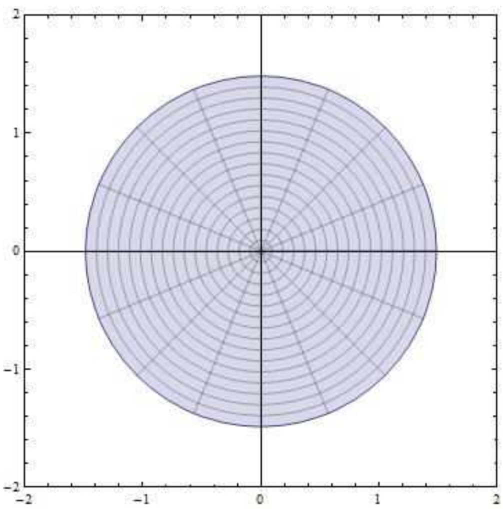

Figure 4.1

In the light of Example 3.13 when the differential operator $1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z)))'}{\phi(f(z))}$ varies in the shaded region of complex plane as given in Figure

4.1, then $\dfrac{zf'(z)}{\phi(f(z))}$ *takes values in the shaded portion as shown in Figure 4.2. Thus* $f(z)$ *is* $\phi-$*like. Similarly, according to Example 3.14, when the differential operator* $1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$ *takes values in the disk centered at origin with radius* $\dfrac{3}{2}$*, then* $\dfrac{zf'(z)}{f(z)}$ *takes values in the shaded portion of the complex plane as shown in Figure 4.2. Thus* $f(z)$ *is starlike.*

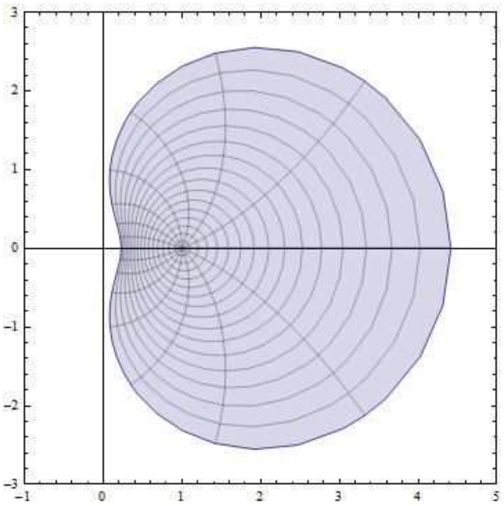

Figure 4.2

4 Open Problem

Here, the sufficient conditions for $\phi-$like, starlike, parabolic $\phi-$like and parabolic starlike functions have been obtained. One may study the operators under consideration to obtain uniformly starlikeness of normalized analytic functions.

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