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Classes of Meromorphic Multivalent Bazilevič and non-Bazilevič Functions Associated with New Operator

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Abstract

The purpose of this paper is to introduce subclasses of Bazilevič and non-Bazilevič meromorphic multivalent functions by using new operator and investigate various properties for functions of these classes.

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1 Introduction

Denote by Σ_p the class of meromorphic $p-$valent functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \ (p \in \mathbb{N} = \{1, 2, 3, ...\}), \qquad (1)$$

which are analytic in $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$.

Let $\mathcal{P}_k(\rho)$ be the class of functions $p(z)$ analytic in $\mathbb{U}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \rho}{1-\rho} \right| d\theta \leq k\pi, \tag{2}$$

where $k \geq 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan and Parvatham [10]. For $\rho = 0$, the class $\mathcal{P}_k(0) = \mathcal{P}_k$ introduced by Pinchuk [11]. Also, $\mathcal{P}_2(\rho) = \mathcal{P}(\rho)$, where $\mathcal{P}(\rho)$ is the class of functions with real part greater than ρ and $\mathcal{P}_2(0) = \mathcal{P}$, is the class of functions with positive real part. From (2), we have $p(z) \in \mathcal{P}_k(\rho)$ if and only if there exist $p_1, p_2 \in \mathcal{P}(\rho)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \ (z \in \mathbb{U}). \tag{3}$$

It is known that the class $\mathcal{P}_k(\rho)$ is a convex set (see [7]).

The Hadamard product (or convolution) of $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p}, \tag{4}$$

is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} = (g * f)(z). \tag{5}$$

For a function $f(z) \in \Sigma_p$, Frasin [3] defined the following differential operator:

$$\begin{aligned}
I_\lambda^0 f(z) &= f(z), \\
I_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) + \frac{\lambda(p+1)}{z^p}, \\
I_\lambda^2 f(z) &= (1-\lambda) I^1 f(z) + \lambda z \left(I^1 f(z)\right)' + \frac{\lambda(p+1)}{z^p}, \\
&\vdots \\
I_\lambda^m f(z) &= (1-\lambda) I^{m-1} f(z) + \lambda z \left(I^{m-1} f(z)\right)' + \frac{\lambda(p+1)}{z^p}.
\end{aligned}$$

In a general form:

$$I_\lambda^m f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left[1 + \lambda(n-p-1)\right]^m a_{n-p} z^{n-p} \ (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \lambda > 0). \tag{6}$$

For the function $I^{m*}_{\lambda,\mu}(z)$ defined by:

$$I^{m}_{\lambda}(z) * I^{m*}_{\lambda,\mu}(z) = \frac{1}{z^p(1-z)^{\mu}} \ (\mu; \lambda > 0; \ z \in \mathbb{U}^*),$$

let

$$\begin{aligned} \mathcal{L}^{m}_{\lambda,\mu} f(z) &= I^{m*}_{\lambda,\mu}(z) * f(z) \\ &= \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{1}{1+\lambda(n-p-1)} \right)^m \frac{(\mu)_n}{(1)_n} a_{n-p} z^{n-p}, \end{aligned} \tag{7}$$

where $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \begin{cases} 1 & n = 0 \\ \theta(\theta+1)...(\theta+n-1) & n \in \mathbb{N} \end{cases}.$$

It is easily verified from (7) that

$$z \left(\mathcal{L}^{m}_{\lambda,\mu} f(z) \right)' = \mu \mathcal{L}^{m}_{\lambda,\mu+1} f(z) - (\mu+p) \mathcal{L}^{m}_{\lambda,\mu} f(z). \tag{8}$$

Using the operator $\mathcal{L}^{m}_{\lambda,\mu}$, we introduce classes of meromorphic multivalent functions of Σ_p as follows:

Definition. For $k \geq 2$, $\lambda, \alpha, \gamma, \mu > 0$, $0 \leq \rho < 1$, and $m \in \mathbb{N}_0$, a function $f(z) \in \Sigma_p$ is in the class $\mathcal{M}^{p,m}_{\lambda,\mu}(\alpha, \gamma, \rho, k)$ if it satisfies the condition:

$$\left[(1-\gamma) \left(z^p \mathcal{L}^{m}_{\lambda,\mu} f(z) \right)^{\alpha} + \gamma \left(\frac{\mathcal{L}^{m}_{\lambda,\mu+1} f(z)}{\mathcal{L}^{m}_{\lambda,\mu} f(z)} \right) \left(z^p \mathcal{L}^{m}_{\lambda,\mu} f(z) \right)^{\alpha} \right] \in \mathcal{P}_k(\rho), \tag{9}$$

and is in the class $\mathcal{N}^{p,m}_{\lambda,\mu}(\alpha, \gamma, \rho, k)$ if it satisfies the condition:

$$\left[(1+\gamma) \left(\frac{1}{z^p \mathcal{L}^{m}_{\lambda,\mu} f(z)} \right)^{\alpha} - \gamma \left(\frac{\mathcal{L}^{m}_{\lambda,\mu+1} f(z)}{\mathcal{L}^{m}_{\lambda,\mu} f(z)} \right) \left(\frac{1}{z^p \mathcal{L}^{m}_{\lambda,\mu} f(z)} \right)^{\alpha} \right] \in \mathcal{P}_k(\rho). \tag{10}$$

Putting $m = 0$ and $\mu = 1$ in (9) and (10), we have, respectively

$$\mathcal{M}^{p,0}_{\lambda,1}(\alpha, \gamma, \rho, k) = \left\{ f : \left[(1-\gamma) \left(z^p f(z) \right)^{\alpha} + \gamma \left(\frac{\mathcal{L}^{0}_{\lambda,2} f(z)}{f(z)} \right) \left(z^p f(z) \right)^{\alpha} \right] \in \mathcal{P}_k(\rho) \right\}$$

and

$$\mathcal{N}^{p,0}_{\lambda,1}(\alpha, \gamma, \rho, k) = \left\{ f : \left[(1+\gamma) \left(\frac{1}{z^p f(z)} \right)^{\alpha} - \gamma \left(\frac{\mathcal{L}^{0}_{\lambda,2} f(z)}{f(z)} \right) \left(\frac{1}{z^p f(z)} \right)^{\alpha} \right] \in \mathcal{P}_k(\rho) \right\}.$$

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $k \geq 2,\ \alpha, \lambda, \gamma, \mu > 0,\ 0 \leq \rho < 1$ and $m \in \mathbb{N}_0$.

To establish our results, we need the following lemma due to Miller and Mocanu [5].

Lemma 2.1 [5]. Let $\phi(u, v)$ be a complex valued function $\phi : D \to \mathbb{C},\ D \subset \mathbb{C}^2$ and let $u = u_1 + iu_2,\ v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies
(i) $\phi(u, v)$ is continuous in D;
(ii) $(1, 0) \in D$ and $\Re\{\phi(1, 0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{n}{2}(1 + u_2^2),\ \Re\{\phi(iu_2, v_1)\} \leq 0$.
Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$ be regular in $\mathbb{U}$ such that $(p(z), zp'(z)) \in D$ for all $z \in \mathbb{U}$. If $\Re\{\phi(p(z), zp'(z))\} > 0$ for all $z \in \mathbb{U}$, then $\Re p(z) > 0$.

Employing the techniques used by Owa [9] for univalent functions, Noor and Muhammad [8] and Aouf and Seoudy [1] for multivalent functions and Mostafa et al. [6] for meromorphic multivalent functions, we prove the following theorems.

Theorem 2.1. If $f(z) \in \mathcal{M}_{\lambda,\mu}^{p,m}(\alpha, \gamma, \rho, k)$, then

$$\left(z^p \mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha \in \mathcal{P}_k(\rho_1), \tag{11}$$

where ρ_1 is given by

$$\rho_1 \leq \frac{2\alpha\mu\rho + n\gamma}{2\alpha\mu + n\gamma}\ (0 \leq \rho_1 < 1). \tag{12}$$

Proof. Let

$$\begin{aligned} \left(z^p \mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha &= (1-\rho_1)p(z) + \rho_1 \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\left[(1-\rho_1)p_1(z) + \rho_1\right] - \left(\frac{k}{4} - \frac{1}{2}\right)\left[(1-\rho_1)p_2(z) + \rho_1\right], \end{aligned} \tag{13}$$

where $p_j(z)$ is analytic in $\mathbb{U}$ with $p_j(0) = 1$ for $j = 1, 2$. Differentiating (13) with respect to z, and using identity (8) in the resulting equation, we get

$$\begin{aligned} &\left[(1-\gamma)\left(z^p \mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha + \gamma\left(\frac{\mathcal{L}_{\lambda,\mu+1}^m f(z)}{\mathcal{L}_{\lambda,\mu}^m f(z)}\right)\left(z^p \mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha\right] \\ &= \left[(1-\rho_1)p(z) + \rho_1\right] + \frac{\gamma(1-\rho_1)zp'(z)}{\mu\alpha} \in \mathcal{P}_k(\rho). \end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{\left[(1-\rho_1)p_i(z) + \rho_1\right] - \rho + \frac{\gamma(1-\rho_1)zp_i'(z)}{\mu\alpha}\right\} \in \mathcal{P}\ (z \in \mathbb{U};\ i = 1, 2).$$

Defining the function

$$\phi(u,v) = [(1-\rho_1)u + \rho_1] - \rho + \frac{\gamma(1-\rho_1)v}{\mu\alpha},$$

where $u = p_j(z) = u_1 + iu_2,\ v = zp_j'(z) = v_1 + iv_2$, we have
(i) $\phi(u,v)$ is continuous in $D = \mathbb{C}^2$;
(ii) $(1,0) \in D$ and $\Re\{\phi(1,0)\} = 1-\rho > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{n}{2}(1+u_2^2)$,

$$\begin{aligned}\Re\{\phi(iu_2,v_1)\} &= \rho_1 - \rho + \frac{\gamma(1-\rho_1)v_1}{\mu\alpha} \\ &\le \rho_1 - \rho - \frac{n\gamma(1-\rho_1)(1+u_2^2)}{2\mu\alpha} \\ &= \frac{A + Bu_2^2}{2C},\end{aligned}$$

where $A = 2(\rho_1 - \rho)\mu\alpha - n\gamma(1-\rho_1),\ B = -n\gamma(1-\rho_1),\ C = \mu\alpha > 0$. We note that $\Re\{\phi(iu_2,v_1)\} < 0$ if $A \le 0,\ B < 0$, this is true from (12). Therefore, by applying Lemma 2.1, $p_j(z) \in \mathcal{P}$ $(j = 1,2)$ and consequently $\left(z^p\mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha \in \mathcal{P}_k(\rho_1)$ for $z \in \mathbb{U}$. This completes the proof of Theorem 2.1.

Theorem 2.2. If $f(z) \in \mathcal{M}_{\lambda,\mu}^{p,m}(\alpha,\gamma,\rho,k)$, then

$$\left(z^p\mathcal{L}_{\lambda,\mu}^m f(z)\right)^{\alpha/2} \in \mathcal{P}_k(\rho_2), \tag{14}$$

where ρ_2 is given by

$$\rho_2 \le \frac{n\gamma + \sqrt{(n\gamma)^2 + 4(\mu\alpha + n\gamma)\alpha\mu\rho}}{2(\alpha\mu + n\gamma)}\ (0 \le \rho_2 < 1). \tag{15}$$

Proof. Let

$$\begin{aligned}\left(z^p\mathcal{L}_{\lambda,\mu}^m f(z)\right)^{\alpha/2} &= (1-\rho_2)p(z) + \rho_2 \\ = \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\rho_2)p_1(z)+\rho_2] &- \left(\frac{k}{4}-\frac{1}{2}\right)[(1-\rho_2)p_1(z)+\rho_2],\end{aligned} \tag{16}$$

where $p_j(z)$ is as in Theorem 2.1. Differentiating (16) with respect to z, and using identity (8) in the resulting equation, we get

$$\begin{aligned}&\left[(1-\gamma)\left(z^p\mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha + \gamma\left(\frac{\mathcal{L}_{\lambda,\mu+1}^m f(z)}{\mathcal{L}_{\lambda,\mu}^m f(z)}\right)\left(z^p\mathcal{L}_{\lambda,\mu}^m f(z)\right)^\alpha\right] \\ &= [(1-\rho_2)p(z)+\rho_2]^2 + \frac{2\gamma(1-\rho_2)[(1-\rho_2)p(z)+\rho_2]\,zp'(z)}{\alpha\mu} \in \mathcal{P}_k(\rho).\end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{\left[(1-\rho_2)p_j(z)+\rho_2\right]^2-\rho+\frac{2\gamma(1-\rho_2)\left[(1-\rho_2)p_j(z)+\rho_2\right]zp_j'(z)}{\alpha\mu}\right\}\in\mathcal{P}$$

$$(z\in\mathbb{U};\ j=1,2).$$

Defining the function

$$\psi(u,v)=\left[(1-\rho_2)u+\rho_2\right]^2-\rho+\frac{2\gamma(1-\rho_2)\left[(1-\rho_2)u+\rho_2\right]v}{\alpha\mu},$$

where $u=p_j(z)=u_1+iu_2,\ v=zp_j'(z)=v_1+iv_2$, we have
(i) $\psi(u,v)$ is continuous in $D=\mathbb{C}^2$;
(ii) $(1,0)\in D$ and $\Re\{\psi(1,0)\}=1-\rho>0$;
(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{n}{2}(1+u_2^2)$,

$$\begin{aligned}\Re\{\psi(iu_2,v_1)\} &= -(1-\rho_2)^2u_2^2+\rho_2^2-\rho+\frac{2\gamma\rho_2(1-\rho_2)v_1}{\alpha\mu}\\ &\leq -(1-\rho_2)^2u_2^2+\rho_2^2-\rho-\frac{n\gamma\rho_2(1-\rho_2)(1+u_2^2)}{\alpha\mu}\\ &= \frac{A+Bu_2^2}{C},\end{aligned}$$

where $A=\alpha\mu\rho_2^2-\alpha\mu\rho-n\gamma\rho_2(1-\rho_2),\ B=-(1-\rho_2)\left[\alpha\mu(1-\rho_2)+\gamma n\rho_2\right],\ C=\alpha\mu>0$. We note that $\Re\{\psi(iu_2,v_1)\}<0$ if $A\leq 0,\ B<0$, this is true from (14) and $0\leq\rho_2<1$. Therefore, by applying Lemma 2.1, $p_j(z)\in\mathcal{P}\ (j=1,2)$ and consequently $\left(z^p\mathcal{L}_{\lambda,\mu}^m f(z)\right)^{\alpha/2}\in\mathcal{P}_k(\rho_2)$ for $z\in\mathbb{U}$. This completes the proof of Theorem 2.2.

Theorem 2.3. If $f(z)\in\mathcal{N}_{\lambda,\mu}^{p,m}(\alpha,\gamma,\rho,k)$, then

$$\left(\frac{1}{z^p\mathcal{L}_{\lambda,\mu}^m f(z)}\right)^{\alpha}\in\mathcal{P}_k(\rho_3), \tag{17}$$

where ρ_3 is given by

$$\rho_3\leq\frac{2\alpha\mu\rho+n\gamma}{2\alpha\mu+n\gamma}\ (0\leq\rho_3<1). \tag{18}$$

Proof. Let

$$\left(\frac{1}{z^p\mathcal{L}_{\lambda,\mu}^m f(z)}\right)^{\alpha}=(1-\rho_3)p(z)+\rho_3$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left[(1-\rho_3)p_1(z)+\rho_3\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[(1-\rho_3)p_2(z)+\rho_3\right], \tag{19}$$

where $p_j(z)$ is as in Theorem 2.1. Differentiating (16) with respect to z, and using identity (8) in the resulting equation, we get

$$\begin{aligned}&\left[(1+\gamma)\left(\frac{1}{z^p\mathcal{L}^m_{\lambda,\mu}f(z)}\right)^{\alpha}-\gamma\left(\frac{\mathcal{L}^m_{\lambda,\mu+1}f(z)}{\mathcal{L}^m_{\lambda,\mu}f(z)}\right)\left(\frac{1}{z^p\mathcal{L}^m_{\lambda,\mu}f(z)}\right)^{\alpha}\right]\\ =\;&[(1-\rho_3)p(z)+\rho_3]+\frac{\gamma(1-\rho_3)zp'(z)}{\mu\alpha}\in\mathcal{P}_k(\rho).\end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{[(1-\rho_3)p_j(z)+\rho_3]-\rho+\frac{\gamma(1-\rho_3)zp_j'(z)}{\mu\alpha}\right\}\in\mathcal{P}\ (z\in\mathbb{U};\ j=1,2).$$

Defining the function

$$\phi(u,v)=[(1-\rho_3)u+\rho_3]-\rho+\frac{\gamma(1-\rho_3)v}{\mu\alpha},$$

where $u=p_i(z)=u_1+iu_2,\ v=zp_i'(z)=v_1+iv_2$.The remaning part of the proof is as in the proof of Theorem 2.1, so, we omit it.

Theorem 2.4. If $f(z)\in\mathcal{N}^{p,m}_{\lambda,\mu}(\alpha,\gamma,\rho,k)$, then

$$\left(\frac{1}{z^p\mathcal{L}^m_{\lambda,\mu}f(z)}\right)^{\alpha/2}\in\mathcal{P}_k(\rho_4), \tag{20}$$

where ρ_4 is given by

$$\rho_4\le\frac{n\gamma+\sqrt{(n\gamma)^2+4(\mu\alpha+n\gamma)\alpha\mu\rho}}{2\left(\alpha\mu+n\gamma\right)}\ (0\le\rho_4<1). \tag{21}$$

Proof. Let

$$\begin{aligned}\left(\frac{1}{z^p\mathcal{L}^m_{\lambda,\mu}f(z)}\right)^{\alpha/2}&=(1-\rho_4)p(z)+\rho_4\\ =\left(\frac{k}{4}+\frac{1}{2}\right)[(1-\rho_4)p_1(z)+\rho_4]&-\left(\frac{k}{4}-\frac{1}{2}\right)[(1-\rho_4)p_1(z)+\rho_4],\end{aligned} \tag{22}$$

where $p_j(z)$ is analytic in $\mathbb{U}$ with $p_j(0)=1$ for $j=1,2$. Differentiating (22) with respect to z, and using identity (8), we get

$$\begin{aligned}&\left[(1+\gamma)\left(\frac{1}{z^p\mathcal{L}^m_{\lambda,\mu}f(z)}\right)^{\alpha}-\gamma\left(\frac{\mathcal{L}^m_{\lambda,\mu+1}f(z)}{\mathcal{L}^m_{\lambda,\mu}f(z)}\right)\left(\frac{1}{z^p\mathcal{L}^m_{\lambda,\mu}f(z)}\right)^{\alpha}\right]\\ =\;&[(1-\rho_4)p(z)+\rho_4]^2+\frac{2\gamma(1-\rho_4)\left[(1-\rho_4)p(z)+\rho_4\right]zp'(z)}{\alpha\mu}\in\mathcal{P}_k(\rho).\end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{[(1-\rho_4)p_j(z)+\rho_4]^2-\rho+\frac{2\gamma(1-\rho_4)\left[(1-\rho_4)p_j(z)+\rho_4\right]zp_j'(z)}{\alpha\mu}\right\}\in P.$$

Defining the function

$$\psi(u,v)=[(1-\rho_4)u+\rho_4]^2-\rho+\frac{2\gamma(1-\rho_4)\left[(1-\rho_4)u+\rho_4\right]v}{\alpha\mu},$$

where $u=p_j(z)=u_1+iu_2,\ v=zp_j'(z)=v_1+iv_2$. The remaning part of the proof is as in the proof of Theorem 2.1, so, we omit it.

Remark. Putting $m=0$ and $\mu=1$ in the above results we obtain the results for the classes $\mathcal{M}_{\lambda,1}^{p,0}(\alpha,\gamma,\rho,k)$ and $\mathcal{N}_{\lambda,1}^{p,0}(\alpha,\gamma,\rho,k)$.

2 Open Problem

The authors suggest to study these classes defined by the Frasin-Darus [4] operator:

$$I^n f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}k^n a_k z^k (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, z\in\mathbb{U}^*).$$

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