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On Starlike and Convex Functions in a Parabolic Region

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Abstract

Using the technique of differential subordination, we, here, find the best dominant having parabolic image for an operator which is a combined form of starlike and convex operators. As special cases of our main results, we obtain the sufficient conditions for parabolic starlike and uniformly convex functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{E} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, Taylor's series expansion of $f \in \mathcal{A}$, is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \quad |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{E}. \tag{2}$$

The class of parabolic starlike functions is denoted by $\mathcal{S}_{\mathcal{P}}$. A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{E}$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathbb{E}. \tag{3}$$

Let UCV denote the class of all such functions. Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}.$$

Note that the conditions (2) and (3) are equivalent to the condition that $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f'(z)}$ take values in the parabolic domain Ω respectively.
Ronning [1] and Ma and Minda [6] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{4}$$

maps the unit disk $\mathbb{E}$ onto the parabolic domain Ω. Therefore, the condition (2) is equivalent to

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \prec q(z), z \in \mathbb{E}, \tag{5}$$

and condition (3) is same as

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z), z \in \mathbb{E}, \tag{6}$$

where q(z) is given by (4).
Irmak et al. [2] studied the class $T_\lambda(\alpha)$ consisting of functions $f \in \mathcal{A}$ satisfying the following condition

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec 1+(1-\alpha)z, 0 \leq \alpha < 1, z \in \mathbb{E},$$

and obtained certain conditions for $f \in \mathcal{A}$ to be a member of class $T_\lambda(\alpha)$ and consequently, they get some sufficient conditions for starlike and convex functions. The work of Irmak et al. ([2], [3], [4]) is the main source of motivation for the present paper.

2 Preliminaries

To prove our main results, we shall use the following lemma of Miller and Mocanu ([5], p.132, Theorem 3.4h).

Lemma 2.1 *Let q be a univalent in $\mathbb{E}$ and let Θ and Φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\Phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\Phi[q(z)]$, $h(z) = \Theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\frac{zh'(z)}{Q(z)} = \Re\left[\frac{\Theta'[q(z)]}{\Phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] > 0.$
If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\Theta[p(z)] + zp'(z)\Phi[p(z)] \prec \Theta[q(z)] + zq'(z)\Phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

3 Main results

Theorem 3.1 *If $f \in \mathcal{A}$, satisfies the differential subordination*

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\left[\beta\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)+\gamma+\right.$$

$$\left.\alpha\left\{\frac{(1+2\lambda)zf''(z)+f'(z)+\lambda z^2 f'''(z)}{f'(z)+\lambda zf''(z)} - \frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\}\right]$$

$$\prec \beta\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2 + \gamma\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), \tag{7}$$

where $\alpha, \beta, \gamma, \lambda \in \mathbb{R}$ *such that* $\alpha \neq 0, \frac{\beta}{\alpha} > 0, \frac{\gamma}{\alpha} > 0, 0 \leq \lambda \leq 1$ *then*

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E}.$$

Proof. Let us define the function θ and ϕ as follows:

$$\theta(w) = \beta w^2 + \gamma w$$

and

$$\phi(w) = \alpha.$$

Define the functions Q and h as follows:

$$Q(z) = zq'(z)\phi(q(z)) = \alpha z q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = \beta q^2(z) + \gamma q(z) + \alpha z q'(z).$$

Further, select the functions $p(z) = \dfrac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}, f \in \mathcal{A}$ and $q(z) = 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, we obtain (7) reduces to

$$\beta p^2(z) + \gamma p(z) + \alpha z p'(z) \prec \beta q^2(z) + \gamma q(z) + \alpha z q'(z) = h(z). \tag{8}$$

Now,

$$Q(z) = \frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}.$$

It can easily be verified that $\Re\dfrac{zQ'(z)}{Q(z)} > 0$ in $\mathbb{E}$ and hence Q is starlike in $\mathbb{E}$. Also we have

$$h(z) = \beta\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2 + \gamma\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} + \frac{2\beta}{\alpha}\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} + \frac{\gamma}{\alpha}.$$

Using given conditions, we have $\Re\dfrac{zh'(z)}{Q(z)} > 0$.
The proof, now, follows from (8) by the use of Lemma 2.1.

Theorem 3.2 *If $f \in \mathcal{A}$, satisfies the differential subordination*

$$\beta\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}\right) + \alpha\left\{\frac{(1+2\lambda)zf''(z)+f'(z)+\lambda z^2 f'''(z)}{f'(z)+\lambda z f''(z)} - \frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}\right\}$$

$$\prec \beta\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} + \frac{\frac{4\alpha}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z \in \mathbb{E}, \quad (9)$$

where $\alpha, \beta, \lambda \in \mathbb{R}$ such that $\alpha \neq 0, \frac{\beta}{\alpha} > 0, 0 \leq \lambda \leq 1$ then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E}.$$

Proof. Let us define the function θ and ϕ as follows:

$$\theta(w) = \beta w$$

and

$$\phi(w) = \frac{\alpha}{w}.$$

Obviously, the function θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Define the functions Q and h as follows:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha z q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \beta q(z) + \frac{\alpha z q'(z)}{q(z)}.$$

Further, select the functions $p(z) = \dfrac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}$, $f \in \mathcal{A}$ and $q(z) = 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, we obtain (9) reduces to

$$\beta p(z) + \frac{\alpha z p'(z)}{p(z)} \prec \beta q(z) + \frac{\alpha z q'(z)}{q(z)} = h(z). \quad (10)$$

Now,

$$Q(z)=\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

and

$$\frac{zQ'(z)}{Q(z)}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

It can easily be verified that $\Re\frac{zQ'(z)}{Q(z)}>0$ in $\mathbb{E}$ and hence Q is starlike in $\mathbb{E}$. Also we have

$$h(z)=\beta\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+\frac{\frac{4\alpha}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

and

$$\frac{zh'(z)}{Q(z)}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$
$$+\frac{\beta}{\alpha}\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}.$$

Using given conditions, we have $\Re\frac{zh'(z)}{Q(z)}>0$.
The proof, now, follows from (10) by the use of Lemma 2.1.

4 Conditions for Parabolic Starlikeness

Setting $\lambda=0$ in Theorem 3.1 and Theorem 3.2, we respectively get the following results:

Corollary 4.1 *If $f\in\mathcal{A}$, satisfies the differential subordination*

$$(\beta-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^2+(\alpha+\gamma)\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)}\prec\beta\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$
$$+\gamma\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),z\in\mathbb{E},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $\alpha \neq 0, \frac{\beta}{\alpha} > 0, \frac{\gamma}{\alpha} > 0$ *then*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E}.$$

Corollary 4.2 *If* $f \in \mathcal{A}$, *satisfies the differential subordination*

$$(\beta - \alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \beta\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+\frac{\frac{4\alpha}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z \in \mathbb{E},$$

where $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha \neq 0, \frac{\beta}{\alpha} > 0$ *then*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E}.$$

5 Conditions for Uniform Convexity

Setting $\lambda = 1$ in Theorem 3.1 and Theorem 3.2, we respectively get the following results for uniform convexity:

Corollary 5.1 *If* $f \in \mathcal{A}$, *satisfies the differential subordination*

$$(\beta - \alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(1 + \frac{z^2 f'''(z)}{f'(z)} + \frac{3zf''(z)}{f'(z)}\right)$$

$$\prec \beta\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2 + \gamma\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), z \in \mathbb{E},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $\alpha \neq 0, \frac{\beta}{\alpha} > 0, \frac{\gamma}{\alpha} > 0$ *then*

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E}.$$

Corollary 5.2 *If* $f \in \mathcal{A}$, *satisfies the differential subordination*

$$(\beta - \alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(\frac{3zf''(z) + f'(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)$$

$$\prec \beta\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+\frac{\frac{4\alpha}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z\in\mathbb{E},$$

where $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha \neq 0, \frac{\beta}{\alpha} > 0$ *then*

$$1+\frac{zf''(z)}{f'(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z\in\mathbb{E}.$$

6 Open Problem

The sufficient conditions for parabolic starlikeness and uniform convexity of normalized analytic functions have been obtained in terms of the combination of starlike and convex operators. One may also investigate other combinations of these differential operators for uniformly starlikeness and convexity of normalized analytic functions.

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