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Emden-Fowler Equation and Inverse Analysis of Simple Flows through Variable Permeability Porous Layers

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Abstract

In this work we introduce an inverse method to analyze simple flows through variable permeability porous layers. Assuming that the velocity distribution is given, or specified as a function of the permeability, the governing equation is solved for the permeability distribution, then the velocity function is then recovered. Poiseuille-type flow involving Brinkman's equation is considered together with other flow problems involving coupled parallel flow through composite layers. In case of flow through a Brinkman-Forchheimer layer over a Darcy layer, the governing equation was transformed into an Emden-Fowler equation whose solution provides a method for determining Beavers and Joseph slip parameter.

Keywords: *Brinkman-Forchheimer, Variable Permeability, Emden-Fowler Equation, Slip Parameter.*

1 Introduction

Many excellent reviews of flow through and over porous layers are available in the literature, (cf. [2,5,8,11]) and discuss all aspects of the flow phenomena, including the various flow models and their validity, applications, solutions and analysis of the model equation.

Modelling fluid flow through porous layers with variable permeability can be argued to be more realistic in representing natural phenomena as compared to flow through constant permeability layers [1], and finds applications in industry and nature, and has implications in the analysis of the transition layer [9]. However, analysis of this type of flow presents challenges at more than one front, including modelling permeability variations and solutions to the resulting governing equations. The momentum equation in this type of unidirectional flow involves two functions to be solved for: velocity and permeability. If permeability distribution is specified then velocity can be solved for.

A number of variable permeability models have been introduced and successfully analyzed in the literature [1], and are based mainly on specifying the permeability distribution and solving the resulting governing equation for the velocity function (cf. [1,6] and the references therein). In a recent article however, [6], the authors used a non-dimensionalizing procedure that resulted in a variable permeability distribution, for Brinkman's equation, that is tied to the velocity distribution. We capitalize on this idea in the current work where we consider simple flows in which the velocity is pre-defined in terms of the variable permeability in order to obtain a permeability equation that can be solved for the permeability distribution and subsequently the velocity is recovered. We illustrate this inverse approach by considering five flow configurations that involve Brinkman's equation and the Brinkman-Forchheimer equation and obtain a permeability equation that is easily solved. In case of the Brinkman-Forchheimer equation, we reduce the permeability equation to the Emden-Fowler equation, [4,10], whose solution is readily available. Inverse analysis in this case can provide information on the slip parameter of the Beavers and Jospeh condition, [3].

2 Problem Formulation

The steady flow of a viscous, incompressible fluid, in the absence of body forces, through porous media is governed by the continuity equation, namely

$$\nabla \bullet \vec{v} = 0 \qquad \ldots(1)$$

and the following general momentum equation that incorporates Darcian and non-Darcian effects, known as the Brinkman- Forchheimer equation:

$$\rho(\vec{v} \bullet \nabla \vec{v}) = -\nabla p + \mu_{eff} \nabla^2 \vec{v} - \left\{ \frac{\mu}{k} \vec{v} + \frac{\rho C_f}{\sqrt{k}} \vec{v} |\vec{v}| \right\} \qquad \ldots(2)$$

wherein p is the pressure, k is the permeability, $\vec{v}$ is the velocity vector, ρ is the fluid density, C_f is the Forchheimer drag coefficient, μ is the base fluid viscosity, and μ_{eff} is the effective viscosity of the fluid saturating the porous medium.

For parallel, unidirectional flow through a porous layer with variable permeability, governing equations (1) and (2) reduce to:

$$\frac{d^2u}{dy^2} - \frac{\mu}{\mu_{eff} k(y)} u - \frac{\rho C_f}{\mu_{eff}} u^2 = \frac{1}{\mu_{eff}} \frac{dp}{dx} \qquad \ldots(3)$$

where $u = u(y)$ is the tangential velocity component.

If the permeability distribution $k(y)$ is given, equation (3) can be solved for the velocity distribution $u(y)$. On the other hand, if the velocity distribution $u(y)$ is given as a function of the permeability, equation (3) gives an equation that the permeability has to satisfy. Once the permeability distribution is solved for, the associated velocity distribution can be obtained. Thus, if

$$u(y) = f(k(y)) \qquad \ldots(4)$$

then (3) takes the form

$$k''(y)f'(k(y)) + k'(y)f''(k(y)) - \frac{\mu}{\mu_{eff}k(y)} f(k(y)) - \frac{\rho C_f}{\mu_{eff}}[f(k(y))]^2 = \frac{1}{\mu_{eff}}\frac{dp}{dx}. \qquad \ldots(5)$$

Clearly, with the knowledge of either $k(y)$ or $f(k(y))$, equation (5) can be solved for the other function. In what follows we consider simple flows in which $f(k(y))$ is given.

3 Simple Flows

3.1 Flow through a variable-permeability Brinkman porous layer between parallel plates

Consider the unidirectional flow through a Brinkman porous layer between solid, parallel plates located at $y = 0$ and $y = h$. On the solid walls, the no-slip velocity condition $u(0) = u(h) = 0$ and the non-penetrability condition $k(0) = k(h) = 0$ are imposed. We assume that the channel is "squeezed" to the point that permeability is continuously varying. The flow is governed by Brinkman's equation which takes the following form for the configuration in **Fig. 1,** obtained by setting C_f to zero in equation (3):

$$\frac{d^2u}{dy^2} - \frac{\mu}{\mu_{eff}k(y)}u = \frac{1}{\mu_{eff}}\frac{dp}{dx}. \qquad \ldots(6)$$

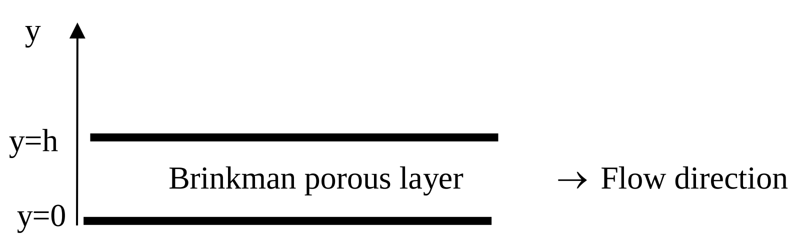

Fig. 1. Representative sketch of flow between parallel plates

Assuming that equation (4) is valid for this flow, equation (5) reduces to:

$$k''(y)\,f'(k(y)) + k'(y)\,f''(k(y)) - \frac{\mu}{\mu_{eff}k(y)}\,f(k(y)) = \frac{1}{\mu_{eff}}\frac{dp}{dx}. \qquad \ldots(7)$$

Equation (7) can be solved for $k(y)$ if given the form of $f(k(y))$. For the sake of illustration, assume that

$$u(y) = f(k(y)) = Ak \qquad \ldots(8)$$

where A is a constant. Using (8) in (7), we obtain

$$k''(y) = \frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}. \qquad \ldots(9)$$

Solution to (9) satisfying $k(0) = k(h) = 0$ is given by

$$k(y) = [\frac{\mu}{2\mu_{eff}} + \frac{1}{2A\mu_{eff}}\frac{dp}{dx}](y^2 - hy) \qquad \ldots(10)$$

and velocity distribution (8) takes the form

$$u(y) = [\frac{A\mu}{2\mu_{eff}} + \frac{1}{2\mu_{eff}}\frac{dp}{dx}](y^2 - hy). \qquad \ldots(11)$$

Equations (10) and (11) give maximum permeability, $k_{\max}$, and maximum velocity, $u_{\max}$, at $y = h/2$ respectively as

$$k_{\max} = k(\frac{h}{2}) = -\frac{h^2}{8}[\frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}] \qquad \ldots(12)$$

$$u_{\max} = u(\frac{h}{2}) = -\frac{h^2}{8}A[\frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}] = Ak_{\max}. \qquad \ldots(13)$$

We note that equation (10) is the dimensional form of the permeability function obtained in [6] through non-dimensionalizing.

3.2 Flow through a variable-permeability finite Brinkman porous layer over a semi-infinite Darcy layer

Consider the unidirectional flow through a Brinkman porous layer between $y = 0$ and $y = h$, over a semi-infinite Darcy layer of constant permeability, depicted in **Fig. 2**. The flow is governed by Brinkman's equation (6) in the Brinkman layer and in the Darcy layer by Darcy's law, of the form

$$u_{1D} = -\frac{k_1}{\mu}\frac{dp}{dx} \qquad \ldots(14)$$

where u_{1D} is the Darcy velocity, k_1 is the constant permeability in the Darcy layer and $\frac{dp}{dx} < 0$ is the common driving pressure gradient.

On the solid wall at $y = h$, the no-slip velocity condition $u(h) = 0$ and the non-penetrability condition $k(h) = 0$ are imposed. At the interface, $y = 0$, the following Beavers and Joseph condition [3] is valid

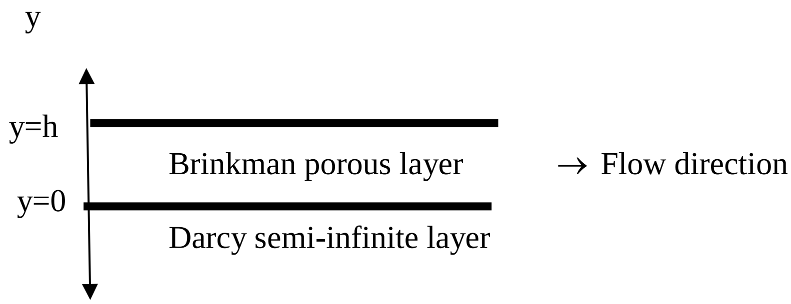

Fig. 2. Representative sketch of flow through a finite Brinkman layer over a semi-infinite Darcy layer

$$u' = \frac{\alpha}{\sqrt{k_1}}(u_B - u_{1D}) \quad \text{at } y = 0 \qquad \ldots(15)$$

where

$$u_B = u(0^+) . \qquad \ldots(16)$$

Assuming that (8) is valid, then using (8) in (6) results in

$$k''(y) = \frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx} . \qquad \ldots(17)$$

Solution to (17) takes the form

$$k(y) = [\frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}]\frac{y^2}{2} + c_1 y + c_2 . \qquad \ldots(18)$$

where c_1, c_2 are arbitrary constants. Velocity in the Brinkman layer is thus given by

$$u(y) = Ak = A[\frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}]\frac{y^2}{2} + c_1 Ay + c_2 A \qquad \ldots(19)$$

and its derivative is given by

$$u'(y) = A[\frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}]y + c_1 A. \qquad \ldots(20)$$

Using $u(h) = 0$, in (19), we obtain

$$[\frac{\mu}{\mu_{eff}} + \frac{1}{A\mu_{eff}}\frac{dp}{dx}]\frac{h^2}{2} + c_1 h + c_2 = 0. \qquad \ldots(21)$$

From (16) and (19) we obtain

$$u(0) = u_B = c_2 A \qquad \ldots(22)$$

and from (20) we obtain

$$u'(0) = c_1 A. \qquad \ldots(23)$$

Using (14), (22) and (23) in (15) we obtain

$$c_1 = \frac{\alpha}{\sqrt{k_1}}c_2 + \frac{\alpha\sqrt{k_1}}{A\mu}\frac{dp}{dx}. \qquad \ldots(24)$$

Equations (21) and (24) are solved for the arbitrary constants c_1 and c_2 and yield

$$c_2 = -\frac{h\sqrt{k_1}}{A[\alpha h + \sqrt{k_1}]}\left[(\frac{2\mu_{eff}\alpha\sqrt{k_1} + \mu h}{2\mu_{eff}\mu})\frac{dp}{dx} + \frac{Ah\mu}{2\mu_{eff}}\right] \qquad \ldots(25)$$

$$c_1 = \frac{\alpha}{A[\alpha h + \sqrt{k_1}}\left[(\frac{2\mu_{eff}k_1 - \mu h^2}{2\mu_{eff}\mu})\frac{dp}{dx} - \frac{Ah^2\mu}{2\mu_{eff}}\right]. \qquad \ldots(26)$$

With c_1 and c_2 determined, the flow quantities are described as follows.

$$k(y)=[\frac{\mu}{\mu_{eff}}+\frac{1}{A\mu_{eff}}\frac{dp}{dx}]\frac{y^2}{2}+\frac{\alpha}{A[\alpha h+\sqrt{k_1}]}\left[(\frac{2\mu_{eff}k_1-\mu h^2}{2\mu_{eff}\mu})\frac{dp}{dx}-\frac{Ah^2\mu}{2\mu_{eff}}\right]y$$
$$-\frac{h\sqrt{k_1}}{A[\alpha h+\sqrt{k_1}]}\left[(\frac{2\mu_{eff}\alpha\sqrt{k_1}+\mu h}{2\mu_{eff}\mu})\frac{dp}{dx}+\frac{Ah\mu}{2\mu_{eff}}\right]. \quad \ldots(27)$$

$$u(y)=[\frac{A\mu}{\mu_{eff}}+\frac{1}{\mu_{eff}}\frac{dp}{dx}]\frac{y^2}{2}+\frac{\alpha}{[\alpha h+\sqrt{k_1}}\left[(\frac{2\mu_{eff}k_1-\mu h^2}{2\mu_{eff}\mu})\frac{dp}{dx}-\frac{Ah^2\mu}{2\mu_{eff}}\right]y$$
$$-\frac{h\sqrt{k_1}}{[\alpha h+\sqrt{k_1}]}\left[(\frac{2\mu_{eff}\alpha\sqrt{k_1}+\mu h}{2\mu_{eff}\mu})\frac{dp}{dx}+\frac{Ah\mu}{2\mu_{eff}}\right]. \quad \ldots(28)$$

$$u'(0)=\frac{\alpha}{[\alpha h+\sqrt{k_1}}\left[(\frac{2\mu_{eff}k_1-\mu h^2}{2\mu_{eff}\mu})\frac{dp}{dx}-\frac{Ah^2\mu}{2\mu_{eff}}\right] \quad \ldots(29)$$

$$u_B=-\frac{h\sqrt{k_1}}{[\alpha h+\sqrt{k_1}]}\left[(\frac{2\mu_{eff}\alpha\sqrt{k_1}+\mu h}{2\mu_{eff}\mu})\frac{dp}{dx}+\frac{Ah\mu}{2\mu_{eff}}\right]. \quad \ldots(30)$$

3.3 Shear-driven flow in a semi-infinite Brinkman-Forchheimer porous layer

Consider the flow through a semi-infinite porous layer, shown in **Fig. 3**. The flow is generated by an applied pressure gradient and a moving plate, with velocity U, located at $y=h$. The flow is governed by equation (3), valid on the interval $h<y<+\infty$. Velocity conditions to be satisfied are:

$$u(h)=U\,;\;\; u(+\infty)=0\,;\;\; u'(+\infty)=0\,. \quad \ldots(31)$$

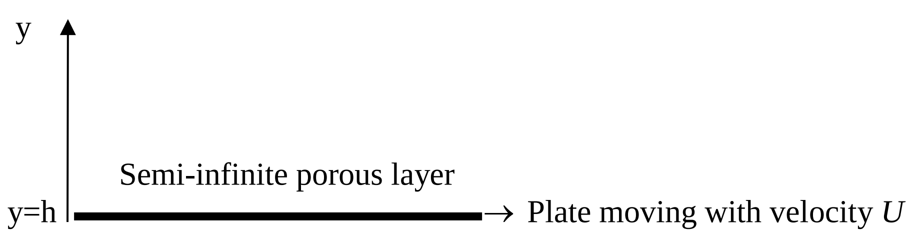

Fig. 3. Representative sketch of moving plate

In order to satisfy (4), (5) and (31) we assume that

$$u = f(k) = -\frac{1}{\mu}\frac{dp}{dx}k(y)\,. \qquad \ldots(32)$$

Equation (5) thus reduces to

$$k''(y) = \beta k^{3/2} \qquad \ldots(33)$$

where

$$\beta = -\frac{\rho C_f}{\mu\mu_{eff}}\frac{dp}{dx}\,. \qquad \ldots(34)$$

Comparing (33) with the Emden-Fowler equation, [4,10], namely

$$k''(y) = \beta y^n k^m \qquad \ldots(35)$$

whose solution is given by

$$k = \lambda y^{(n+2)/(1-m)} \qquad \ldots(36)$$

where

$$\lambda = [\frac{(n+2)(n+m+1)}{\beta(m-1)^2}]^{1/(m-1)} \qquad \ldots(37)$$

we see that $n = 0$ and $m = \frac{3}{2}$, and solution to (33) takes the form

$$k = (\frac{20}{\beta})^2 y^{-4}\,. \qquad \ldots(38)$$

Velocity distribution thus takes the form

$$u = -\frac{1}{\mu}\frac{dp}{dx}k(y) = -\frac{1}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 y^{-4} \qquad \ldots(39)$$

with

$$u' = \frac{4}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 y^{-5}\,. \qquad \ldots(40)$$

Using (39), we obtain velocity of the moving plate as

$$u(h) = U = -\frac{1}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 h^{-4}\,. \qquad \ldots(41)$$

Conditions (31) are thus satisfied, as can be seen from (39) and (40), wherein $u(+\infty) = u'(+\infty) = 0$. Furthermore, equation (41) shows the parameters that the velocity of the moving plate depends on in order to have the permeability distribution given by (38).

3.4 Flow through a Brinkman-Forchheimer layer over a Darcy layer

Consider the coupled, parallel flow through a variable permeability Brinkman-Forchheimer porous layer of semi-infinite extent ($h < y < +\infty$) over a constant permeability Darcy porous layer of semi-infinite extent ($-\infty < y < h$), shown in **Fig. 4**.

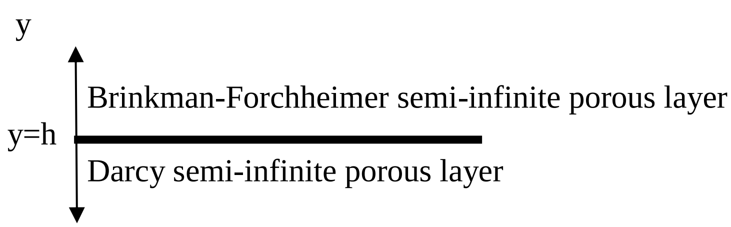

Fig. 4. Representative sketch of coupled parallel flow

The assumingly sharp interface between the layers is located at $y = h$. In the Brinkman-Forchheimer layer, the flow is governed by equation (3), and in the Darcy layer the flow is governed by Darcy's law, written as:

$$u_{1D} = -\frac{k_1}{\mu}\frac{dp}{dx} \qquad \ldots(42)$$

where u_{1D} is the Darcy velocity and k_1 is the constant permeability in the Darcy layer. The flow in both layers is driven by a common constant pressure gradient, $\frac{dp}{dx} < 0$. At the interface, $y = h$, Beavers and Joseph condition [3,7] is assumed to be valid, namely

$$u' = \frac{\alpha}{\sqrt{k_1}}(u_{1B} - u_{1D}) \quad \text{at } y = h \qquad \ldots(43)$$

where α is a slip parameter, $u_{1B} = u(y = h^+)$ is the velocity at the interface obtained from solution to (3). Permeability and velocity distributions in the Brinkman-Forchheimer layer are obtained from solution to equation (3) and are given by (21) and (22), from which we obtain

$$u_{1B} = u(h) = -\frac{1}{\mu}\frac{dp}{dx}\left(\frac{20}{\beta}\right)^2 h^{-4} \qquad \ldots(44)$$

$$u'(h) = \frac{4}{\mu}\frac{dp}{dx}\left(\frac{20}{\beta}\right)^2 h^{-5} . \qquad \ldots(45)$$

Upon using (42), (44) and (45) in (43) we obtain the following expression for the slip parameter α :

$$\alpha = \frac{4(20)^2\sqrt{k_1}}{h[k_1\beta^2h^4 - (20)^2]} . \qquad \ldots(46)$$

Expression (46) indicates that the Beavers and Joseph slip parameter α depends on the Darcy constant permeability, viscosity of the base fluid, effective viscosity of the fluid saturating the porous layer, density of the fluid, the pressure gradient, location of the interface, thickness of the porous layer and the Forchheimer drag coefficient. In order to further illustrate dependence of the slip parameter on thickness of the porous layer we consider the situation in the next section.

3.5 Flow through a Brinkman-Forchheimer layer sandwiched between two Darcy layers

An interesting variation of the above problem of flow over a Darcy layer is the flow through a Brinkman-Forchheimer layer of variable permeability that is bounded from above and below by two semi-infinite Darcy layers of constant permeability, as shown in **Fig. 5**.

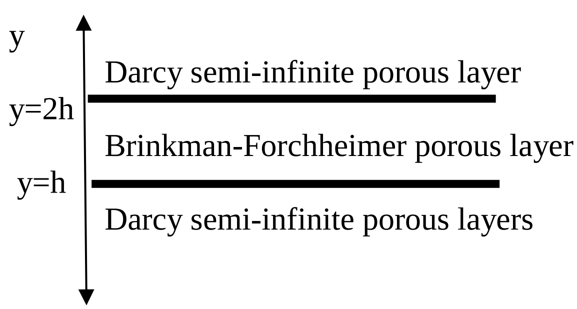

Fig. 5. Representative sketch of a Brinkman-Forchheimer porous core

The Brinkman-Forchheimer layer spans $h < y < 2h$ while the Darcy layers span $-\infty < y < h$ and $2h < y < +\infty$. Equations governing the flow through the given configuration are equation (3) in the Brinkman-Forchheimer layer, equation (42) in the lower Darcy layer, and

$$u_{2D} = -\frac{k_2}{\mu}\frac{dp}{dx} \qquad \ldots(47)$$

in the upper Darcy layer. Conditions at the interfaces between layers are given by

$$u' = \frac{\alpha_1}{\sqrt{k_1}}(u_{1B} - u_{1D}) \quad \text{at } y = h, \qquad \ldots(48)$$

and

$$u' = \frac{\alpha_2}{\sqrt{k_2}}(u_{2B} - u_{2D}) \quad \text{at } y = 2h. \qquad \ldots(49)$$

In equations (47)-(49), α_1 and α_2 are the slip parameters associated with the lower and upper interfaces, respectively, u_{1B} and u_{2B} are the lower and upper interfacial velocities, respectively, u_{1D} and u_{2D} are the Darcy velocities in the lower and upper Darcy layers, respectively, and k_1 and k_2 are the constant permeabilities in the lower and upper Darcy layers, respectively.

Permeability and velocity distributions in the Brinkman-Forchheimer layer are obtained from solution to equation (3) and are given by (38) and (39), which yield the following expressions for the lower and upper interfacial velocities, respectively:

$$u_{1B} = u(h) = -\frac{1}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 h^{-4} \qquad \ldots(50)$$

$$u_{2B} = u(2h) = -\frac{1}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 (2h)^{-4} \qquad \ldots(51)$$

and the following shear stress expressions at the lower and upper interfaces, respectively:

$$u'(h) = \frac{4}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 h^{-5} \qquad \ldots(52)$$

$$u'(2h) = \frac{4}{\mu}\frac{dp}{dx}(\frac{20}{\beta})^2 (2h)^{-5}. \qquad \ldots(53)$$

Upon using (42), (47) and (50) to (53) in (48) and (49), we can solve for the following expression for the slip parameters α_1 and α_2:

$$\alpha_1 = \frac{4(20)^2\sqrt{k_1}}{h[k_1\beta^2 h^4 - (20)^2]}. \qquad \ldots(54)$$

$$\alpha_2 = \frac{4(20)^2\sqrt{k_2}}{(2h)[k_2\beta^2 (2h)^4 - (20)^2]}. \qquad \ldots(55)$$

It is clear from (54) and (55) that the slip parameters depend on fluid and medium properties, and on the location of the interfaces and the thickness of the Brinkman-Forchheimer porous layer.

4 Conclusion

In this work we introduced an inverse method to analyze simple, unidirectional flows in variable permeability porous layers. Rather than specifying a permeability distribution and then solving the momentum equation for the velocity, we imposed a velocity distribution that is a function of the permeability and solved the momentum equation for the necessary permeability distribution. We analyzed five situations that involve simple flows through porous layers where the flows were governed by Brinkman's equation and the Brinkman-Forchheimer equation. In the latter case, the governing equation was transformed into an Emden-Fowler equation whose solution is well-documented in the literature. The analysis in this work provided some insights into the determination of the Beavers and Joseph slip parameter.

5 Open Problem

In most of the inverse analysis above, we relied on the particular solution of the Emden-Fowler equation to provide a variable permeability distribution that satisfies a prescribed velocity distribution. When the flow domain is composed of a Brinkman-Forchheimer variable permeability porous layer that is bounded by a lower and an upper solid wall on which no-slip and no-penetration conditions are imposed, the Brinkman-Forchheimer equation is reduced to the Emden-Fowler equation (33). Particular solution to this equation does not satisfy the no-slip, no-penetration conditions on the solid boundary. There is a need to construct a solution that satisfies this Poiseuille-type flow.

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