*Int. J. Open Problems Compt. Math., Vol.10, No.3, November 2017*
*ISSN 1998-6262; Copyright ©ICSRS Publication, 2017*
*www.i-csrs.org*

# On a Furdui problem

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**Abstract**

***In this note, we give some series representations for integrals involving sine and cosine functions.***

**Keywords:** *integral; series representation; gamma function.*
**2010 Mathematics Subject Classification:** 33B10.

## 1 Introduction

In 2017, V. Furdui posed the following problem in Jozsef Widt International Mathematical Competition:

**W11**. Calculate $\int_0^{\pi/2} \frac{\sin x}{\left(1+\sqrt{\sin 2x}\right)^2} dx$.

Motivated by this problem, we consider four similar integrals involving sine and cosine functions. For the sake of convenience, let us denote

$$I = \int_0^{\pi/2} \frac{\sin x}{\left(1+\sqrt{\sin 2x}\right)^2} dx,$$
$$J = \int_0^{\pi/2} \frac{\cos x}{\left(1+\sqrt{\sin 2x}\right)^2} dx,$$
$$M = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\left(1+\sqrt{\cos 2x}\right)^2} dx,$$
$$N = \int_{-\pi/4}^{\pi/4} \frac{\cos x}{\left(1+\sqrt{\cos 2x}\right)^2} dx.$$

Very obvious, the substitution $x = \frac{\pi}{2} - t$ yields $I = J$. Since the function $\frac{\sin x}{\left(1+\sqrt{\cos 2x}\right)^2}$ is odd on $[\frac{\pi}{4}, -\frac{\pi}{4}]$, so, we easily get $M = 0$. In addition, simple computation results in

$$\begin{aligned} I &= \tfrac{I+J}{2} = \tfrac{1}{2}\int_0^{\pi/2} \tfrac{\sin x+\cos x}{\left(1+\sqrt{\sin 2x}\right)^2}dx \\ &= \tfrac{\sqrt{2}}{2}\int_0^{\pi/2} \tfrac{\cos(x-\pi/4)}{\left(1+\sqrt{\sin 2x}\right)^2}dx \int_0^{\pi/2} \tfrac{\cos(x-\pi/4)}{\left(1+\sqrt{\sin 2x}\right)^2}dx \\ &= \sqrt{2}\int_0^{\pi/4} \tfrac{\cos u}{\left(1+\sqrt{\cos 2u}\right)^2}du \\ &= \tfrac{\sqrt{2}}{2}N \end{aligned}$$

by using substitution $x - \frac{\pi}{4} = u$. Comprehensive the above, we only think about the first integral.

# 2 main results

**Theorem 2.1** *The following identity holds true:*

$$I = \sum_{n=0}^{\infty} \frac{T_n}{(2n+1)4^{n+1}} \tag{2.1}$$

*where* $T_n = \frac{2\cdot 6\cdot 10\cdots(4n-10)}{3\cdot 4\cdot 5\cdots(n-1)}$.

**Proof.** Simple computation yields

$$\begin{aligned} I &= \tfrac{I+J}{2} = \sqrt{2}\int_0^{\pi/4} \tfrac{\cos u}{\left(1+\sqrt{\cos 2u}\right)^2}du \\ &= \sqrt{2}\int_0^{\pi/4} \tfrac{\cos u}{1+2\sqrt{\cos 2u}+\cos 2u}du \\ &= \sqrt{2}\int_0^{\pi/4} \tfrac{\cos u}{2-2\sin^2 u+2\sqrt{1-2\sin^2 u}}du. \end{aligned}$$

By using substitution $\sin u = t$, we have

$$I = \frac{\sqrt{2}}{2}\int_0^{\sqrt{2}/2} \frac{dt}{1-t^2+\sqrt{1-2t^2}}.$$

Applying the formula[see [2]]

$$\frac{1-2x-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} T_n x^n,$$

we easily obtain

$$\frac{2}{1-2x+\sqrt{1-4x}} = \sum_{n=0}^{\infty} T_n x^n.$$

So, we have

$$\frac{1}{1-t^2+\sqrt{1-2t^2}}=\sum_{n=0}^{\infty}\frac{T_n}{2^{n+1}}t^{2n}.$$

Hence, we have

$$\begin{aligned} I &= \tfrac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\tfrac{T_n}{2^{n+1}}\int_0^{\sqrt{2}/2}t^{2n}dt \\ &= \sum_{n=0}^{\infty}\tfrac{T_n}{(2n+1)4^{n+1}}. \end{aligned}$$

This complete the proof.

**Corollary 2.2** *The sequence $T_n$ first appeared in a letter dated September 4, 1751, Euler proposed to Goldbach. This sequence $T_n$ is closely related to famous Catalan sequence. In detail, the reader may see reference [[2]].*

**Theorem 2.3** *The following identity holds true:*

$$I=\frac{\pi}{2\Gamma^2\left(3/4\right)}\sum_{n=0}^{\infty}(-1)^n(n+1)\sum_{k=0}^{\infty}(-1)^k\binom{n/4}{k}\frac{2^{2k+1}}{(4k+3)[3\cdot 7\cdots(4k-1)]^2}. \tag{2.2}$$

**Proof.** Using series representation

$$\frac{1}{\left(1+x\right)^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

and substitutions $2x=t, t=\pi-u$, we have

$$\begin{aligned} I &= \sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^{\pi/2}\sin x\left(\sin 2x\right)^{n/2}dx \\ &= \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^{\pi}\sin\left(\tfrac{t}{2}\right)\left(\sin t\right)^{n/2}dt \\ &= \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^{\pi}\cos\left(\tfrac{u}{2}\right)\left(1-\cos^2 u\right)^{n/4}du \\ &= \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^{\pi}\cos\left(\tfrac{u}{2}\right)\sum_{k=0}^{\infty}(-1)^k\binom{n/4}{k}\cos^{2k}u\,du \\ &= \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)\sum_{k=0}^{\infty}(-1)^k\binom{n/4}{k}\int_0^{\pi}\cos\left(\tfrac{u}{2}\right)\cos^{2k}u\,du. \end{aligned}$$

Using the formula(See the reference [1])

$$\int_0^{\pi/2}\cos^{p-1}x\cos(bx)dx=\frac{\pi}{2^p}\frac{\Gamma(p)}{\Gamma\left(\frac{p+b+1}{2}\right)\Gamma\left(\frac{p-b+1}{2}\right)} \tag{2.3}$$

with $p = 2k+1, b = \frac{1}{2}$, we have

$$\begin{aligned} I &= \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)\sum_{k=0}^{\infty}(-1)^k\binom{n/4}{k}\tfrac{\pi}{2^{2k+1}}\tfrac{\Gamma(2k+1)}{\Gamma(k+5/4)\Gamma(k+3/4)} \\ &= \tfrac{\pi}{2\Gamma^2(3/4)}\sum_{n=0}^{\infty}(-1)^n(n+1)\sum_{k=0}^{\infty}(-1)^k\binom{n/4}{k}\tfrac{2^{2k+1}}{(4k+3)[3\cdot 7\cdots(4k-1)]^2}. \end{aligned}$$

This complete the proof.

# 3 Open Problem

**Open problem 3.1** Let $p, q > 0$ be two integers. Compute

$$I(p,q) = \int_0^{\pi/n} \frac{\sin x}{\left(1+\sqrt[p]{\sin qx}\right)^p}dx. \tag{3.1}$$

**Open problem 3.2** Compute

$$T = \int_0^{\pi/4} \frac{\tan x}{\left(1+\sqrt{\tan 2x}\right)^2}dx. \tag{3.2}$$

**ACKNOWLEDGEMENTS.** This research is supported by NSFC11401041, PhD research capital of Binzhou University under grant 2013Y02, Science and Technology Project of Shandong Province under grant J16li52.

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