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Some commutativity criteria for rings with involution

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Abstract

In this paper we establish some commutativity criteria for a ring with involution $(R, *)$ in which derivations satisfy certain algebraic identities. Some related results characterizing commutativity of prime rings have been discussed. Furthermore, we provide examples to show that the conditions imposed in the hypotheses of our results are necessary.

Keywords: *Prime ring, involution, commutativity, derivation.*

1 Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. R is 2-torsion free if whenever $2x = 0$, with $x \in R$ implies $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive map $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$). An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$

and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denote by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$.
A derivation on R is an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Several authors have investigated the relationship between the commutativity of the ring R and certain special types of maps on R. Long ago Herstein [8] proved that if a prime ring R of characteristic different from two admits a derivation d such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, then R is commutative. Motivated by this result Bell and Daif [6], obtained the same result by considering the identity $d[x, y] = 0$ for all x, y in a non zero ideal of R. Later Daif and Bell [7] established commutativity of semiprime ring satisfying $d[x, y] = [x, y]$ for all x, y in a non zero ideal of R, and d a derivation of R. Further, in the year 1997 M. Hongan [9] established commutativity of 2-torsion free semiprime ring R which admits a derivation d satisfying $d[x, y] + [x, y] \in Z(R)$ for all $x, y \in I$ or $d[x, y] - [x, y] \in Z(R)$ for all $x, y \in I$, where I is an ideal of R. In the present paper we prove that if $(R, *)$ is a 2-torsion free prime ring with involution of the second kind and d be a derivation of R, then the following properties are equivalent: (i) $d([x, x^*]) \pm [x, x^*] \in Z(R)$, (ii) $d \neq 0$ and $d([x, x^*]) \in Z(R)$, (iii) $d(x \circ x^*) \pm x \circ x^* \in Z(R)$, (iv) $d \neq 0$ and $d(x \circ x^*) \in Z(R)$, (v) $d([x, x^*]) \pm x \circ x^* \in Z(R)$, (vi) $d(x \circ x^*) \pm [x, x^*] \in Z(R)$, (vii) R is commutative.

2 Commutativity criteria involving derivations

Fact : Let $(R, *)$ be a ring with involution. If R is prime and $S(R) \cap Z(R) \neq (0)$, then $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ implies that $d(z) = 0$ for all $z \in Z(R)$.
Indeed, if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, replacing h by k^2 where $k \in Z(R) \cap S(R)$, then we have $d(k)k = 0$ for all $k \in Z(R) \cap S(R)$, so $d(k) = 0$ for all $k \in Z(R) \cap S(R)$. As conclusion, we get $d(z) = 0$ for all $z \in Z(R)$.

Lemma 2.1 ([12], Lemma 1) *Let R be a prime ring with involution of the second kind. Then $*$ is centralizing if and only if R is commutative.*

Lemma 2.2 ([12], Lemma 2) *Let R be a prime ring with involution of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

In ([4], Theorem 2.2) S. Ali et al. proved that if $(R, *)$ is a 2-torsion free prime ring with involution equipped with a nonzero derivation d such that $d([x, x^*]) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative. In the following result we prove the commutativity of R in a more general situation.

Theorem 2.3 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and let d be a derivation of R. The following assertions are equivalent:*
(1) $d([x, x^*]) + [x, x^*] \in Z(R)$ *for all* $x \in R$;
(2) $d([x, x^*]) - [x, x^*] \in Z(R)$ *for all* $x \in R$;
(3) *R is commutative.*
Moreover, if $d \neq 0$, then $d([x, x^*]) \in Z(R)$ implies that R is commutative.
It is obvious that (3) implies both of (1) and (2). So we need to prove that (1) and (2) implies (3).
If $d = 0$, then $[x, x^*] \in Z(R)$, so our theorem follows from Lemma 2.1, therefore we can suppose $d \neq 0$.
(1) $\Rightarrow$ (3) Assuming that

$$d([x, x^*]) + [x, x^*] \in Z(R) \quad \text{for all } \ x \in R. \tag{1}$$

Linearizing (1) we find that

$$d([x, y^*]) + d([y, x^*]) + [x, y^*] + [y, x^*] \in Z(R) \quad \text{for all } \ x, y \in R, \tag{2}$$

and thus

$$[d([x, y]), r] + [d([y^*, x^*]), r] + [[x, y], r] + [[y^*, x^*], r] = 0 \quad \text{for all } \ r, x, y \in R. \tag{3}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)$ and using (3), we obtain

$$([[x, y], r] + [[y^*, x^*], r])d(h) = 0 \quad \text{for all } \ r, x, y \in R. \tag{4}$$

Taking $y = x^*$, we get

$$[[x, x^*], r]d(h) = 0 \quad \text{for all } \ r, x \in R. \tag{5}$$

Since R is prime, then $[[x, x^*], r] = 0$, in the this case Lemma 2.1, forces that R is commutative, or $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ and from Fact, we have

$$d(z) = 0 \quad \text{for all } \ z \in Z(R). \tag{6}$$

Substituting zy for y in (3), where $z \in Z(R)$, yields

$$([d([x, y]), r] + [[x, y], r])z + ([d([y^*, x^*]), r] + [[y^*, x^*], r])z^* = 0 \quad \text{for all } \ r, x, y \in R. \tag{7}$$

Let $z \in Z(R) \cap H(R) \backslash \{0\}$, equation (7) gives

$$[d([x, y]), r] + [[x, y], r] + [d([y^*, x^*]), r] + [[y^*, x^*], r] = 0 \quad \text{for all } \ r, x, y \in R. \tag{8}$$

On the other hand let $z \in Z(R) \cap S(R)$ we obtain

$$[d([x, y]), r] + [[x, y], r] - [d([y^*, x^*]), r] - [[y^*, x^*], r] = 0 \quad \text{for all } \ r, x, y \in R. \tag{9}$$

Using (8) together with (9), we conclude that $d([x,y]) + [x,y] \in Z(R)$ for all $x, y \in R$ and ([9], Corollary 1) implies that R is commutative.
(2) $\Rightarrow$ (3) Suppose that

$$d([x,x^*]) - [x,x^*] \in Z(R) \quad \text{for all } \ x \in R. \tag{10}$$

Linearizing the last equation, we find that

$$d([x,y^*]) + d([y,x^*]) - [x,y^*] - [y,x^*] \in Z(R) \quad \text{for all } \ x,y \in R, \tag{11}$$

and thus

$$[d([x,y]),r] + [d([y^*,x^*]),r] - [[x,y],r] - [[y^*,x^*],r] = 0 \quad \text{for all } \ r,x,y \in R, \tag{12}$$

Replacing y by yh in (12) where $h \in Z(R) \cap H(R)$, we get

$$([[x,y],r] + [[y^*,x^*],r])d(h) = 0 \quad \text{for all } \ r,x,y \in R. \tag{13}$$

This equation is the same as (4), then reasoning as above we have R is commutative.
Now, assume that $d \neq 0$ and $d([x,x^*]) \in Z(R)$ for all $x \in R$, we replace x by $x+y$, to get

$$d([x,y^*]) + d([y,x^*]) \in Z(R) \quad \text{for all } \ x,y \in R. \tag{14}$$

Accordingly, we get

$$[d([x,y]),r] + [d([y^*,x^*]),r] = 0 \quad \text{for all } \ r,x,y \in R. \tag{15}$$

Replacing in the last equation y by yh, where $h \in Z(R) \cap H(R)$, yields

$$([[x,y],r] + [[y^*,x^*],r])d(h) = 0 \quad \text{for all } \ r,x,y \in R, \tag{16}$$

Taking $y = x^*$, we find

$$[[x,x^*],r]d(h) = 0 \quad \text{for all } \ r,x \in R, \tag{17}$$

The primeness of R implies that $[x,x^*] \in Z(R)$ and Lemma 2.1 assures the commutativity of R or $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ in this case the Fact implies that

$$d(z) = 0 \quad \text{for all } \ z \in Z(R). \tag{18}$$

Substituting yz for y in (14) where $z \in Z(R)$, we obtain

$$d([x,y^*])z^* + d([y,x^*])z \in Z(R) \tag{19}$$

for all $x, y \in R$. For $z \in Z(R) \cap H(R)\backslash\{0\}$, (19) becomes

$$d([x,y]) + d([y^*,x^*]) \in Z(R) \quad \text{for all } \ x,y \in R \tag{20}$$

and for $z \in Z(R) \cap S(R)$, we obtain

$$d([x,y]) - d([y^*,x^*]) \in Z(R) \quad \text{for all } \ x,y \in R. \tag{21}$$

Using (20) together with (21), we get $d([x,y]) \in Z(R)$ for all $x, y \in R$. Hence ([13], Lemma 4) implies that R is commutative.

Proposition 2.4 *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and let d be a derivation of R. Then the following assertions are equivalent:*
(1) $d(xx^*) \pm xx^* \in Z(R)$ for all $x \in R$;
(2) $d(xx^*) \pm x^*x \in Z(R)$ for all $x \in R$;
(3) $d([x,y]) \pm [x,y] \in Z(R)$ for all $x, y \in R$;
(4) R is commutative.
Moreover, if $d \neq 0$, then $d(xx^*) \in Z(R)$ (resp. $d([x,y]) \in Z(R)$) implies that R is commutative.

In [4], Theorem 2.3, S. Ali et al. proved that if R be a prime ring with involution $*$ with char$(R) \neq 2$, d is a nonzero derivation of R such that $d(x \circ x^*) = 0$ for all $x \in R$ and $S(R) \cap Z(R) \neq (0)$. Then R is commutative. In the following result we give a more generalization of this result.

Theorem 2.5 *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and let d be a derivation of R. Then the following assertions are equivalent:*
(1) $d(x \circ x^*) + x \circ x^* \in Z(R)$ for all $x \in R$;
(2) $d(x \circ x^*) - x \circ x^* \in Z(R)$ for all $x \in R$;
(3) R is commutative.
Moreover, if $d \neq 0$, then $d(x \circ x^*) \in Z(R)$ implies that R is commutative.

It is clear that (3) implies (1) and (2), so we need to prove that (1) and (2) implies (3).
Remark that if $d = 0$, then $x \circ x^* \in Z(R)$, so our result follows from Lemma 2.2.
Therefore, we may assume that $d \neq 0$.
(1) $\Rightarrow$ (3) We are given that

$$d(x \circ x^*) + x \circ x^* \in Z(R) \quad \text{for all } \ x \in R. \tag{22}$$

A linearization of (22) gives

$$d(x \circ y^*) + d(y \circ x^*) + x \circ y^* + y \circ x^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{23}$$

Accordingly, we get

$$d(x \circ y) + d(y^* \circ x^*) + x \circ y + y^* \circ x^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{24}$$

Replacing y by yh, where $h \in Z(R) \cap H(R)$ and using (24), we obtain

$$(x \circ y + y^* \circ x^*)d(h) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{25}$$

As R is prime, then either $d(h) = 0$ or $x \circ y + y^* \circ x^* \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, using Fact , we get

$$d(z) = 0 \quad \text{for all } \ z \in Z(R). \tag{26}$$

Substituting yz for y in (24), where $z \in Z(R) \cap S(R)$, that gives

$$\{(d(x \circ y) + x \circ y) - (d(y^* \circ x^*) + y^* \circ x^*)\}z \in Z(R) \quad \text{for all } \ x, y \in R. \tag{27}$$

That is

$$d(x \circ y) + x \circ y - d(y^* \circ x^*) - y^* \circ x^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{28}$$

Using (24) together with (28), we find that $d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in R$, so by ([13], Theorem 8), we conclude that R is commutative.
If $x \circ y + y^* \circ x^* \in Z(R)$ for all $x, y \ \in R$, we replace y by x^* we get $x \circ x^* \in Z(R)$ for all $x \in R$, thus Lemma 2.2 yields that R is commutative.
(2) $\Rightarrow$ (3) We suppose that:

$$d(x \circ x^*) - x \circ x^* \in Z(R) \quad \text{for all } \ x \in R. \tag{29}$$

Linearizing (29), we obtain

$$d(x \circ y^*) + d(y \circ x^*) - x \circ y^* - y \circ x^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{30}$$

Accordingly, we get

$$d(x \circ y) + d(y^* \circ x^*) - x \circ y - y^* \circ x^* \in Z(R) \quad \text{for all } \ x, y \in R. \tag{31}$$

Replacing y by yh where $h \in Z(R) \cap H(R)$, and using (31) we have

$$(x \circ y + y^* \circ x^*)d(h) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{32}$$

This equation is the same as (25), then proving as above we conclude that R is commutative.
If $d \neq 0$ and $d(x \circ x^*) \in Z(R)$ for all $x \in R$, we replace x by $x + y$, yields that

$$d(x \circ y^*) + d(y \circ x^*) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{33}$$

Accordingly, we get

$$[d(x \circ y), r] + [d(y^* \circ x^*), r] = 0 \quad \text{for all} \;\; r, x, y \in R. \tag{34}$$

Replacing in the last equation y by yh where $h \in Z(R) \cap H(R)$ and using this, we arrive at

$$([x \circ y, r] + [y^* \circ x^*, r])d(h) = 0 \quad \text{for all} \;\; r, x, y \in R, \tag{35}$$

Taking $y = x^*$, we obtain

$$[x \circ x^*, r]Rd(h) = 0 \quad \text{for all} \;\; r, x \in R. \tag{36}$$

The primeness of R together with (36) forces that $x \circ x^* \in Z(R)$ for all $x \in R$, in this case by Lemma 2.1, we conclude that R is commutative or $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, from Fact, we have

$$d(z) = 0 \quad \text{for all} \;\; z \in Z(R). \tag{37}$$

Substituting z for y in (34), where $z \in Z(R)$, (34) gives

$$[d(x), r]z + [d(x^*), r]z^* = 0 \quad \text{for all} \;\; r, x \in R. \tag{38}$$

Taking $z \in Z(R) \cap H(R) \backslash \{0\}$, equation (38) gives

$$[d(x), r] + [d(x^*), r] = 0 \quad \text{for all} \;\; r, x \in R, \tag{39}$$

and taking again $z \in Z(R) \cap S(R)$, we obtain

$$[d(x), r] - [d(x^*), r] = 0 \quad \text{for all} \;\; r, x \in R. \tag{40}$$

Combining (39) with(40), we conclude that $[d(x), x] = 0$ for all $x \in R$. Applying Posner's Theorem we conclude that R is commutative.

Proposition 2.6 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and let d be a derivation of R. Then the following assertions are equivalent:*
(1) *$d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in R$;*
(2) *$d(x \circ y) - x \circ y \in Z(R)$ for all $x, y \in R$;*
(3) *R is commutative.*
Moreover, if $d \neq 0$, then $d(x \circ y) \in Z(R)$ implies that R is commutative.

Theorem 2.7 *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind. Let d be a derivation of R, then the following assertions are equivalent:*
(1) $d([x,x^*]) + x \circ x^* \in Z(R)$ *for all* $x \in R$;
(2) $d([x,x^*]) - x \circ x^* \in Z(R)$ *for all* $x \in R$;
(3) $d(x \circ x^*) + [x,x^*] \in Z(R)$ *for all* $x \in R$;
(4) $d(x \circ x^*) - [x,x^*] \in Z(R)$ *for all* $x \in R$;
(5) R *is commutative.*

For the nontrivial sense, to prove that (1) $\Rightarrow$ (5) assume that

$$d([x,x^*]) + x \circ x^* \in Z(R) \quad \text{for all } \ x \in R. \tag{41}$$

Replacing x by x^*, we get

$$-d([x,x^*]) + x \circ x^* \in Z(R) \quad \text{for all } \ x \in R. \tag{42}$$

Adding (41) with (42), we obtain $x \circ x^* \in Z(R)$ for all $x \in R$ and by Lemma 2.2 we have R is commutative.
Similarly, one can prove the other implications.

Proposition 2.8 *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and let d be a derivation of R. Then the following assertions are equivalent:*
(1) $d([x,y]) + x \circ y \in Z(R)$ *for all* $x,y \in R$;
(2) $d([x,y]) - x \circ y \in Z(R)$ *for all* $x,y \in R$;
(3) $d(x \circ y) + [x,y] \in Z(R)$ *for all* $x,y \in R$;
(4) $d(x \circ y) - [x,y] \in Z(R)$ *for all* $x,y \in R$;
(5) R *is commutative.*

The following example proves that the condition " * is of the second kind" is necessary in our theorems.

Example 1.
Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in Z \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
It is straightforward to check that R is a prime ring and $*$ is an involution of the first kind.
Moreover, if we set $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$, then d is a nonzero derivation satisfying the conditions of Theorems 2.3, 2.5 and 2.7; but R is not commutative.

The following example proves that the primeness hypothesis in Theorems 2.3, 2.5 and 2.7 is not superfluous.

Example 2.
Let $R_1 = \mathbf{Q}[X] \times T$ where T is a noncommutative 2-torsion free ring and set $d(P, t) = (P', 0)$. It is obvious that R_1 is a noncommutative ring and d is a derivation of R_1 such that $[d(r), s] = 0$ for all $r, s \in R_1$.
Consider $\mathcal{R} = R_1 \times R_1^0$ provided with the involution of the second kind $*_{ex}$ given by $*_{ex}(x, y) = (y, x)$ and define $D : \mathcal{R} \longrightarrow \mathcal{R}$ by $D(x, y) = (d(x), 0)$. It is easy to verify that D is derivation of $\mathcal{R}$ which satisfies the conditions of Theorem 2.3, 2.5 and 2.7; however $\mathcal{R}$ is a noncommutative ring.

3 Open Problems

To end this paper we introduce the following open questions, where m and n are two positives integers :
(i) Does the condition $d([x, x^*]^n) - ([x, x^*])^m \in Z(R)$ for all $x \in R$, implies that R is commutative.
(ii) Does the condition $d((x \circ x^*)^n) - (x \circ x^*)^m \in Z(R)$ for all $x \in R$, implies that R is commutative.
(iii) Does the condition $d([x, x^*]^n) - (x \circ x^*)^m \in Z(R)$ for all $x \in R$, implies that R is commutative.
(iv) Does the condition $d((x \circ x^*)^n) - ([x, x^*])^m \in Z(R)$ for all $x \in R$, implies that R is commutative.

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