Int. J. Open Problems Compt. Math., Vol. 10, No. 3, November 2017
ISSN 1998-6262;
www.i-csrs.org

Addendum to
"On finite groups with perfect subgroup order subsets" [IJOPCM, vol. 7 (2014), no. 1, 41-46]

Marius Tărnăuceanu

Faculty of Mathematics, "Al. I. Cuza" University, Iaşi, Romania
e-mail: tarnauc@uaic.ro

Abstract

In this short note we give an example which disproves Conjecture 8 of [1]. An interesting characterization of the cyclic group $\mathbb{Z}_6$ is also obtained.

Keywords: *finite groups, number of subgroups.*
2010 Mathematics Subject Classification: Primary 20D60; Secondary 20D30.

1 Introduction

A finite group G is said to be a *PSOS-group* if for every subgroup H of G the cardinality of the set $\{K \le G \,|\, |K| = |H|\}$ divides $|G|$. In [1] we proved that $G \times G$ is not a PSOS-group if $|G|$ is odd (see Theorem 7) and conjectured that this is also happen for $|G|$ even (see Conjecture 8). We are now able to disprove this conjecture.

Example 1.1. Since the group

$$\mathbb{Z}_6 \times \mathbb{Z}_6 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3)$$

is a direct product of groups of coprime orders, we infer that its subgroups are of type $H \times K$, where $H \le \mathbb{Z}_2 \times \mathbb{Z}_2$ and $K \le \mathbb{Z}_3 \times \mathbb{Z}_3$. Then the subgroup lattice of $\mathbb{Z}_6 \times \mathbb{Z}_6$ consists of the following 30 subgroups:

- one subgroup of order 1;

- 3 subgroups of order 2;
- 4 subgroups of order 3;
- one subgroup of order 4;
- one subgroup of order 9;
- 12 subgroups of order 6;
- 4 subgroups of order 12;
- 3 subgroups of order 18;
- one subgroup of order 36.

Obviously, all these numbers divide $36 = |\mathbb{Z}_6 \times \mathbb{Z}_6|$ and so $\mathbb{Z}_6 \times \mathbb{Z}_6$ is a PSOS-group.

Moreover, we are able to give the following characterization of $\mathbb{Z}_6$.

Theorem 1.2. *$\mathbb{Z}_6$ is the unique group G of order pq, where p and q are primes, such that $G \times G$ is a PSOS-group.*

Clearly, all finite cyclic groups are PSOS-groups. The above example shows that the class of abelian PSOS-groups also contains several non-cyclic groups. This make the problem of determining all abelian PSOS-groups (see Section 3 of [1]) more interesting.

2 Proof of Theorem 1.2

Let G be a group of order pq, where p and q are primes, such that $G \times G$ is a PSOS-group. Then $p \neq q$ by Theorem 1 of [1]. Also, one of the numbers p and q must be 2 by Theorem 7 of [1]. Assume that $p = 2$. It is well-known that there are two types of groups of order $2q$ and therefore we distinguish the following two cases.

Case 1. G is not abelian
Then the number $n_2(G)$ of subgroups of order 2 in G is q. It follows that

$$n_2(G \times G) = 2n_2(G) + n_2(G)^2 = 2q + q^2,$$

implying

$$2q + q^2 \mid 4q^2,$$

a contradiction.

Case 2. G is abelian

Then

$$G \cong \mathbb{Z}_{2q} \cong \mathbb{Z}_2 \times \mathbb{Z}_q.$$

It is easy to see that the number of subgroups of order $2q$ in $G \times G$ is $3(q+1)$ and therefore

$$3(q+1) \mid 4q^2.$$

This leads to $3 \mid q$, i.e. $q = 3$. Hence $G \cong \mathbb{Z}_6$, completing the proof. ■

3 Open Problem

Finally, we note that another interesting problem concerning PSOS-groups is to find all positive integers n such that $\mathbb{Z}_n \times \mathbb{Z}_n$ is a PSOS-group.

References

[1] M. Tărnăuceanu, *On finite groups with perfect subgroup order subsets*, Int. J. Open Problems Compt. Math., vol. **7** (2014), no. 1, 41–46.