Palindrome words on the Arnoux-Rauzy sequences
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Abstract

A palindrome is a word, or a sequence of characters which reads the same backward as forward. In this article we study palindrome words on the Arnoux Rauzy substitutions sequences.

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1   Introduction

Combinatorics on words is a fairly new area of discrete mathematics which began in the beginning of the $20'$th century. Among the first publications were Axel Thue's avoid-ability results published in 1906 and 1912, [5, 6]. In these papers Thue showed, the existance of an infinite binary cube-free word and an infinite ternary square-free word.

Research in combinatorics on words has been active and systematic since the 1950's, much later than the beginning of the study of the area. One important year for this area of discrete mathematics is 1983, when the first book of the field was published. It was the book Combinatorics on words by M. Lothaire [3], which was a presentation of the research done so far.

Combinatorics on words has many connections to other areas of mathematics. The connections to automata theory and formal languages are clear because in both fields one operates with words. A set of binary infinites words, so called Sturmian words, have several equivalent Definitions and these provide an interesting link between combinatorics, number theory and dynamical systems.

Sturmian sequences are the infinite sequences of two letters with factor complexity $n + 1$. They are a classical object of symbolic dynamics. These sequences correspond to natural codings of irrational rotations on the circle.

These sequences admit an S-adic Definition: They are the sequences that belong to the symbolic dynamical systems generated by iterationg the two substitutions $\sigma_1 : 1 \to 1, 2 \to 21$, and $\sigma_2 : 1 \to 12, 2 \to 2$; [4].

Arnoux-Rauzy sequences were introduced in [5] in order to generalize Sturmian sequences to three-letter alphabets. They can be defined in an S-adic way by iterating the Arnoux-Rauzy substitutions σ_i, $(i = 1, 2, 3)$ defined as:

$$\sigma_i : \begin{cases} j \to j & \text{if} & j = i \\ j \to ji & \text{if} & j \neq i \end{cases}$$

where each of the substitution σ_i occurs infinitely often in the iteration. These sequences have factor complexity $2n+1$. It was conjectured that Arnoux-Rauzy sequences correspond to natural codings of translations on the two-dimensional torus as in the sturmian case. Arnoux-Rauzy sequences are also studied in combinatorics word. Indeed they belong to the family of episturmian words (see [1]). For more details about the Arnoux-Rauzy sequences see [**?**].

In this article, we are interested in the sequences of the Arnous-Rauzy substitutions. We focus on the study of palindromic words on the sequences $\sigma_1 \sigma_2^k \sigma_3$ for all integers k.

Theorem 1.1 *Let τ be a finite product of the Arnoux-Rauzy substitutions σ_1, σ_2, σ_3, where $\tau = \sigma_i \sigma_j^k \sigma_s$, such that $i, j, s \in \{1, 2, 3\}$ and $i \neq j \neq s$. Then $\tau(1)$ is a palindrome word.*

This paper is organized as follows: In section 2, we give the Definitions of these objects and we introduce the combinatorics of words. In section 3, we introduce the k-bonacci substitutions and their properties and we characterize palindrome words on the product of these substitutions. Finally we give some examples.

2   Preliminaries

In this section we introduce basic notions and definitions of combinatorics on words. Let $\mathcal{A}$ be a finite set called the alphabet. The elements of the alphabet are called letters. A word is a sequence of letters and it can be of finite or infinite length. In fact, infinite words can be one-way infinite or two-ways infinite, meaning that the word has either a starting point and is infinite in one direction or that it is infinite in both directions without any specific starting point. In this paper we are interested on finite and one-way infinite words. Finite words are of the form $w = a_1 a_2 a_3 \ldots a_n$ and infinite ones of the form $w = a_1 a_2 a_3 \ldots$ where $a_i \in \mathcal{A}$. The word that does not contain any letters is called the empty word and denoted by ε.

The set of all finite and infinite non-empty words over $\mathcal{A}$ is denoted respectively by $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$. This set can be viewed as a free semigroup with respect to the product operation defined by concatenation. The empty word is an identity element.

If $w \in \mathcal{A}^*$, there exists a unique integer $r \geq 0$ and unique letters $a_1, a_2, \ldots a_r \in \mathcal{A}$ such that $w = a_1 a_2 \ldots a_r$, the number r is called the length of w and denoted by $|w|$. If $w \in \mathcal{A}^*$ and $a \in \mathcal{A}$, then $|w|_a$ denotes the numbers of occurences of the letter a in the word w so that

$$|w| = \sum_{a \in \mathcal{A}} |w|_a.$$

If $w = a_1 a_2 \ldots a_r$, with $a_1, a_2, \ldots a_r \in \mathcal{A}$, then the reversal of w is the word

$$w^* = a_r \ldots a_2 a_1.$$

A word w is a palindrome if $w = w^*$.

A substitution over the alphabet $\mathcal{A}$ is an endomorphism of the free monoid $\mathcal{A}^*$ such that the image of each letter of $\mathcal{A}$ is a nonempty word.
A substitution σ is primitive if there exists an integer k such that, for each pair $(a, b) \in \mathcal{A}^2$, $|\sigma^k(a)|_b > 0$. A substitution σ naturally extends to the set $\mathcal{A}^{\mathbb{N}}$ of infinite sequences by setting

$$\sigma(a_0 a_1 a_2 \ldots) = \sigma(a_0)\sigma(a_1)\sigma(a_2)\ldots$$

In this paper we are interested in the Arnoux-Rauzy substitutions defined on the alphabet of three letters as follows:

$$\sigma_1 : \begin{cases} a \to a \\ b \to ab \\ c \to ac \end{cases} \qquad \sigma_2 : \begin{cases} a \to ba \\ b \to b \\ c \to bc \end{cases} \qquad \sigma_3 : \begin{cases} a \to ca \\ b \to cb \\ c \to c \end{cases}$$

Arnoux-Rauzy sequences were introduced in [5] to generalize Sturmian sequences to $3-$letters alphabets. They are infinite sequences of three symbols obtained by iterating the Arnoux-Rauzy substitutions $\sigma_1, \sigma_2, \sigma_3$ where each σ_i occurs infinitely often. These sequences have factor complexity $2n + 1$.

3   Main results

In this paper we are interested in palindrome words obtained by Arnoux-Rauzy sequences. We prove that $\sigma_i \sigma_j \sigma_s^k(i)$ and $\sigma_i \sigma_j^k \sigma_s(i)$ for $i, j, k \in \{1, 2, 3\}$ and $k \in \mathbb{N}^*$ such that $i \neq j \neq s$. First, we prove the result for $\sigma_1 \sigma_2^k \sigma_3(1)$ and we then generalize the result by the same method. We give some Definition and lemma to be used:

Definition 3.1 *For $k \geq 1$, we define the word z_k:*

$$z_k = \sigma_1(2^k)1.$$

Lemma 3.2 *The word z_k is a palindrome.*

proof

$$
\begin{aligned}
z_k &= \sigma_1(2^k)1 \\
&= \sigma_1(2)\sigma_1(2)\sigma_1(2)...\sigma_1(2)\sigma_1(2)\sigma_1(2)1 \\
&= (12)(12)(12)...(12)(12)(12)1 \\
&= 121212...1212121 \\
&= 1(21)(21)(21)...(21)(21)(21) \\
&= 1\sigma_2^*(1)\sigma_2^*(1)\sigma_2^*(1)...\sigma_2^*(1)\sigma_2^*(1)\sigma_2^*(1) \quad \text{because} \quad \sigma_i(j) = \sigma_j^*(i) \\
&= 1\sigma_2^*(1^k) \\
&= z_k^*
\end{aligned}
$$

Definition 3.3 *for $k \geq 2$, we define the word F_k:*

$$
\begin{cases}
F_1 = 1213121 \\
F_k = z_k 3 z_k
\end{cases}
$$

remark It is clear that F_k is a palindrome word.

We will prove that $\sigma_1\sigma_2^k\sigma_3(1) = F_k$. We need some lemma.

Lemma 3.4 *For all $k \geq 1$, we have these equalities:*

1. $\sigma_2^k(1) = 2^k1$

2. $\sigma_2^k(3) = 2^k3$

proof

Reasoning recursively on k :

1. For $k = 1$

$$\sigma_2(1) = 21.$$

Assume that the equality holds true up to the order k, we show that it holds true for the order $(k+1)$.

$\sigma_2^{k+1}(1) = \sigma_2\sigma_2^k(1) = \sigma_2(2^k1) = \sigma_2(2^k)\sigma_2(1) = 2^k21 = 2^{k+1}1$

2. The same raisonning.

Lemma 3.5 *For all $k \geq 1$, we have:*

$$\sigma_1 \sigma_2^k \sigma_3(1) = F_k.$$

proof
Reasoning recursively on k :
The case $k = 1$ is obvious, since $F_1 = 1213121$, and $\sigma_1 \sigma_2 \sigma_3(1) = 1213121$.

Assume that the equality holds up to the order k, show that it holds to the order $(k+1)$, suppose $F_k = \sigma_1 \sigma_2^k \sigma_3(1)$, then

$$
\begin{aligned}
\sigma_1 \sigma_2^{k+1} \sigma_3(1) &= \sigma_1 \sigma_2^{k+1} \sigma_3(1) \\
&= \sigma_1 \sigma_2^{k+1}(31) \\
&= \sigma_1 \sigma_2^{k+1}(3) \sigma_2^{k+1}(1) \\
&= \sigma_1((2^{k+1}3)(2^{k+1}1)) \\
&= \sigma_1(2^{k+1}3)\sigma_1(2^{k+1}1) \\
&= \sigma_1(2^{k+1})\sigma_1(3)\sigma_1(2^{k+1})\sigma_1(1) \\
&= \sigma_1(2^{k+1})13\sigma_1(2^{k+1})1 \\
&= z_{k+1}3z_{k+1}
\end{aligned}
$$

Theorem 3.1 *Let σ_1, σ_2 and σ_3 the three Arnoux-Rauzy substitutions. Then the sequence $\sigma_1 \sigma_2^k \sigma_3(1)$ is a palindrome word.*

proof Since $\sigma_1 \sigma_2^k \sigma_3(1) = F_k$ then this sequence is a palindrome.
Now we will proof the result for more general sequences. Let

$$\tau_k = \sigma_i \sigma_j^k \sigma_s$$

for $i, k, s \in \{1, 2, 3\}$ and $i \neq j \neq s$.

Definition 3.6 *For $k \geq 1$, we define the word z_k:*

$$z_k = \sigma_i(j^k)i.$$

Lemma 3.7 *The word z_k is a palindrome.*

proof

$$
\begin{aligned}
z_k &= \sigma_i(j^k)i \\
&= \sigma_i(j)\sigma_i(j)\sigma_i(j)\ldots\sigma_i(j)\sigma_i(j)\sigma_i(j)i \\
&= (ij)(ij)(ij)\ldots(ij)(ij)(ij)i \\
&= ijijij\ldots ijijiji \\
&= i(ji)(ji)(ji)\ldots(ji)(ji)(ji) \\
&= i\sigma_j^*(i)\sigma_j^*(i)\sigma_j^*(i)\ldots\sigma_j^*(i)\sigma_j^*(i)\sigma_j^*(i) \quad \text{since}\ \ \sigma_i(j)=\sigma_j^*(i) \\
&= i\sigma_j^*(i^k) \\
&= z_k^*
\end{aligned}
$$

The word z_k is a palindrome.

Definition 3.8 *For $k \geq 1$, we define the sequence F_k:*

$$
F_k = z_k s z_k.
$$

remark The word F_k is a palindrome.

    remark

Now we will proof that $\tau_k(i) = F_k$. We begin by proving some lemma.

Lemma 3.9 *For all $k \geq 1$,*

1. $\sigma_j^k(i) = j^k i$

proof

    Reasoning recursively on k :

For $k = 1$ $\sigma_j(i) = ji$ True.

Assumed that the equality holds up to the order k, show that it holds to the order $(k+1)$.

    $\sigma_j^{k+1}(i) = \sigma_j\sigma_j^k(i) = \sigma_j(j^k i) = \sigma_j(j^k)\sigma_j(i) = j^k ji = j^{k+1} i$

Lemma 3.10 *For all $k \geq 1$,*

$$
\tau_k(i) = F_k.
$$

proof

    Reasoning recursively on k :

For $k = 1$

$$
F_1 = \tau_1(i)
$$

Let $F_1 = ijisiji$, et $\tau_1(i) = ijisiji$ True. It is assumed that the equality holds up to the order k, show that it holds to the order $(k+1)$.

suppose $F_k = \tau_k(i)$ then,

$$
\begin{aligned}
\tau_{k+1}(i) &= \sigma_i \sigma_j^{k+1} \sigma_s(i) \\
&= \sigma_i \sigma_j^{k+1}(si) \\
&= \sigma_i \sigma_j^{k+1}(s) \sigma_j^{k+1}(i) \\
&= \sigma_i((j^{k+1}s)(j^{k+1}i)) \\
&= \sigma_i(j^{k+1}s)\sigma_i(j^{k+1}i) \\
&= \sigma_i(j^{k+1})\sigma_i(s)\sigma_i(j^{k+1})\sigma_i(i) \\
&= \sigma_i(j^{k+1})is\sigma_i(j^{k+1})i \\
&= z_{k+1}sz_{k+1}
\end{aligned}
$$

Theorem 3.2 *Let σ_1, σ_2 and σ_3 the three Arnoux-Rauzy substitutions. Then the sequence $\sigma_i\sigma_j^k\sigma_s(1)$ is a palindrome word for $i,j,s \in \{1,2,3\}$ and $i \neq j \neq s$.*

4   Open Problem

The open problem here is to generalise the main results of this paper. Is that $\sigma_i^{k_1}\sigma_j^{k_2}\sigma_s^{k_3}(1)$ a palindrome word? Can we obtain the same result with other type of substitutions?

References

[1] A. Glen, J. Justin, *Episturmian words*, Theor. Inform. Appl. **43** (2009), no. 3, 403–442.

[2] M. Hbaib, *Beta-expansion with Pisot bases over $F_q(x^{-1})$*. Bull. Korean Math. Soc. **49**, No. 1, 127–133 (2012).

[3] M. Lothaire, *Combinatorics on words.* Addison-Wesley, 1983.

[4] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, edited by V. Berthé, S. Ferenczi, C. Mauduit, et al.), Lecture Notes in Mathematics **1794**, Springer, Berlin, 2002

[5] A. Thue. Uber unendliche Zeichenreihen, Videnskapsselskapets Skrifter, *I. Mat.-naturv. Klasse* (1906) 1–22.

[6] A. Thue. Uber die gegenseitige Lage gleicher Teile gewisser Zzichenreihen. Videnskapsselskapets Skrifter, *I. Mat.-naturv. Klasse* (1912) 1–67.