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Local Fractional Sumudu Variational Iteration Method for Solving Partial Differential Equations with Local Fractional Derivative

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Abstract: ***In this paper, we combine the Sumudu transform method with the Adomian decomposition method in the sense of local fractional derivative, for solving linear and nonlinear local fractional partial differential equations. This method is called the Local Fractional Sumudu Variational Iteration Method (LFSVIM). The LFSVIM can easily be applied to many problems and is capable of reducing the size of computational work to find non-differentiable solutions to local fractional partial differential equations. Some illustrative examples are given, revealing the effectiveness and convenience of the method.***

Keywords: ***local fractional calculus, local fractional derivative operator, local fractional sumudu variational iteration method, local fractional partial differential equations.***

1 Introduction

It is known that some of the methods, such as Adomian decomposition method (ADM) [1], homotopy perturbation method (HPM) [2], variational iteration method (VIM) [3], Fourier transform method [4], Fourier series method [5], Laplace transform method [6], and Sumudu transform method [7] are used for solving differential equations, and then extended it to solve differential equations of fractional orders. Recently, there appeared a large part of scientific researchs concerning local fractional differential equations or local fractional partial differential equations, adopted in its entirety on the above mentioned methods to solve this new types of equations. For example, among these research we find, local fractional Adomian decomposition method ([8],[9]), local fractional homotopy perturbation method ([10],[11]), local fractional homotopy perturbation Sumudu transform method [12], local fractional variational iteration method ([13],[14]), local fractional variational iteration transform method ([15]-[17]), local fractional Fourier series method ([18]-[20]), Laplace transform series expansion method [21], local fractional Sumudu transform method ([22],[23]) and local fractional Sumudu transform series expansion method ([24],[25]).

The basic motivation of present study is to combine two powerful methods, the first method is "variational iteration method (VIM)", the second method is called "Sumudu transform method" in the sense of local fractional derivative, thus, we get the modified method local fractional Sumudu variational iteration method (LFSVIM), then we apply this modified method to solve some examples related with local fractional partial differential equations.

The present paper has been organized as follows: In Section 2 some basic definitions and properties of the local fractional calculus and local fractional Sumudu transform method. In section 3 We present an analysis of the proposed method. In section 4 We give three examples show how to apply this modified method (LFSVIM). Finally, the conclusion follows.

2 Local Fractional Calculus

In this section, we present the basic theory of local fractional calculus and we focus specifically on the definitions of the following concepts: local fractional derivative with some results, local fractional integral with some results, and some important results concerning the local fractional Sumudu transform method (see [26], [27]).

2.1 Local fractional derivative

Definition 2.1.4 The local fractional derivative of f(x) of order α at $x=x_0$ is defined by ([26], [27])

$$f^{(\alpha)}(x)=\left.\frac{d^{\alpha}f}{dx^{\alpha}}\right|_{x=x_0}=\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}}, \tag{5}$$

where

$$\Delta^{\alpha}(f(x)-f(x_0))=\Gamma(1+\alpha)\big(f(x)-f(x_0)\big). \tag{6}$$

For any x∈(a, b), there exists

$$f^{(\alpha)}(x)=D_x^{(\alpha)}f(x),$$

denoted by

$$f(x)\in D_x^{\alpha}(a,b).$$

Local fractional derivative of high order is written in the form

$$f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)} D_x^{(\alpha)} \ldots D_x^{(\alpha)}}^{ktimes}, \tag{7}$$

and local fractional partial derivative of high order

$$\frac{\partial^{k\alpha} f(x)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{ktimes}. \tag{8}$$

2.2 Local fractional integral

Definition 2.2.1 The local fractional integral of $f(x)$ of order α in the interval [a, b] is defined as ([26], [27])

$$\begin{aligned} {}_aI_b^{(\alpha)} f(t) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{9}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \Delta t_2, \ldots\}$ and $[t_j, t_{j+1}]$, $t_0 = a$, $t_N = b$ is a partition of the interval [a, b].

For any x∈(a, b), there exists,

$${}_aI_x^{(\alpha)} f(x),$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a,b).$$

2.3 Some important results

Definition 2.3.1 In fractal space, the Mittage Leffler function, sine function and cosine function are defined as

$$E_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, 0 < \alpha \le 1, \tag{10}$$

$$\sin_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, 0 < \alpha \le 1, \tag{11}$$

$$\cos_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}, 0 < \alpha \le 1. \tag{12}$$

The properties of local fractional derivatives and integral of non-differentiable functions are given by

(13) $$\frac{d^{\alpha}}{dx^{\alpha}}\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}=\frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)},$$

(14) $$\frac{d^{\alpha}}{dx^{\alpha}}E_{\alpha}(x_{\alpha})=E_{\alpha}(x^{\alpha}),$$

(15) $$\frac{d^{\alpha}}{dx^{\alpha}}\sin_{\alpha}(x_{\alpha})=\cos_{\alpha}(x^{\alpha}),$$

(16) $$\frac{d^{\alpha}}{dx^{\alpha}}\cos_{\alpha}(x_{\alpha})=-\sin_{\alpha}(x^{\alpha}),$$

(17) $${}_{0}I_{x}^{(\alpha)}\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}=\frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}.$$

2.4 Local fractional Sumudu transform

We present here the definition of local fractional Sumudu transform method (denoted in this paper by S_α) and some properties concerning this transformation (for mor see [28]).

If there is a new transform operator S_α:f(x)→F(u), namely

(18) $$S_{\alpha}\left\{\sum_{K=0}^{\infty}a_{k}x^{k\alpha}\right\}=\Gamma(1+k\alpha)u^{k\alpha}.$$

As typical examples, we have

(19) $$S_{\alpha}\left\{E_{\alpha}(i^{\alpha}x^{\alpha})\right\}=\sum_{k=0}^{\infty}\Gamma(1+k\alpha)i^{k\alpha}u^{k\alpha}.$$

(20) $$S_{\alpha}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\right\}=u^{\alpha}.$$

Definition 2.4.1 The local fractional Sumudu transform of f(x) of order α is defined as

(21) $$S_{\alpha}\left\{f(x)\right\}=F_{\alpha}(u)$$
$$=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}E_{\alpha}(-u^{\alpha}x^{\alpha})\frac{f(x)}{u^{\alpha}}(dx)^{\alpha},0<\alpha\leq 1.$$

Following (21), its inverse formula is defined as

(22) $$S_{\alpha}^{-1}\left\{F_{\alpha}(u)\right\}=f(x),0<\alpha\leq 1.$$

Theorem 2.4.2 (linearity). If $S_\alpha\{f(x)\}=F_\alpha(u)$ and $S_\alpha\{g(x)\}=G_\alpha(u)$, then one has

(23) $$S_\alpha\{f(x)+g(x)\}=F_\alpha(u)+G_\alpha(u).$$

Theorem 2.4.3 (local fractional Laplace-Sumudu duality). If $L_\alpha\{f(x)\}={}^{L}f_s^{\alpha}(s)$ and $S_\alpha\{f(x)\}=F_\alpha(u)$, then one has

(24) $$S_\alpha\{f(x)\}=\frac{1}{u^\alpha}L_\alpha\left\{f(\frac{1}{x})\right\}.$$

(25) $$L_\alpha\{f(x)\}=\frac{S_\alpha\left\{f(\frac{1}{s})\right\}}{s^\alpha}.$$

Theorem 2.4.4 (local fractional Sumudu transform of local fractional derivative).

If $S_\alpha\{f(x)\}=F_\alpha(u)$, then one has

(26) $$S_\alpha\left\{\frac{d^\alpha f(x)}{dx^\alpha}\right\}=\frac{F_\alpha(u)-f(0)}{u^\alpha}.$$

As the direct result of (26), we have the following results. If $S_\alpha\{f(x)\}=F_\alpha(u)$ then we have

(27) $$S_\alpha\left\{\frac{d^{n\alpha} f(x)}{dx^{n\alpha}}\right\}=\frac{1}{u^{n\alpha}}\left[F_\alpha(u)-\sum_{k=0}^{n-1}u^{k\alpha}f^{(k\alpha)}(0)\right].$$

When n =2, from (27), we get

(28) $$S_\alpha\left\{\frac{d^{2\alpha} f(x)}{dx^{2\alpha}}\right\}=\frac{1}{u^{2\alpha}}\left[F_\alpha(u)-f(0)-u^\alpha f^{(\alpha)}(0)\right]$$

Theorem 2.4.5 (local fractional Sumudu transform of local fractional integral). If $S_\alpha\{f(x)\}=F_\alpha(u)$, then one has

(29) $$S_\alpha\left\{{}_0I_x^{(\alpha)}f(x)\right\}=u^\alpha F_\alpha(u).$$

Theorem 2.4.6 (local fractional convolution). If $S_\alpha\{f(x)\}=F_\alpha(u)$ and $S_\alpha\{g(x)\}=G_\alpha(u)$, then one has

(30) $$S_\alpha\{f(x)*g(x)\}=u^\alpha F_\alpha(u)G_\alpha(u),$$

where

(31) $$f(x)*g(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(t)g(x-t)(dt)^\alpha.$$

3 Local Fractional Sumudu Variational Iteration Method

Let us consider the following nonlinear operator with local fractional derivative :

(32) $$L_\alpha W(x,t)+N_\alpha W(x,t)+R_\alpha W(x,t)=k(x,t),$$

where $L_\alpha=(\partial^{2\alpha}/\partial^{2\alpha}t)$ denotes linear local fractional derivative operator of order 2α, R_α denotes linear local fractional derivative operator of order less than L_α, N_α denotes nonlinear local fractional operator, and k(x,t) is the non-differentiable source term.

Taking the local fractional Sumudu transform (denoted in this paper by S_α) on both sides of Eq.(32), we get:

(33) $$\begin{aligned}&S_\alpha[L_\alpha W(x,t)]+S_\alpha[N_\alpha W(x,t)+R_\alpha W(x,t)]\\&=S_\alpha[k(x,t)].\end{aligned}$$

Using the property of the local fractional Sumudu transform, we have:

(34) $$\begin{aligned}&S_\alpha[W(x,t)]=W(x,0)+W_t^{(\alpha)}(x,0)u^\alpha+u^{2\alpha}S_\alpha[k(x,t)]\\&-u^{2\alpha}S_\alpha[N_\alpha W(x,t)+R_\alpha W(x,t)].\end{aligned}$$

Taking the inverse local fractional Sumudu transform on both sides of Eq.(34), gives:

(35) $$\begin{aligned}&W(x,t)=W(x,0)+W_t^{(\alpha)}(x,0)\frac{t^\alpha}{\Gamma(1+\alpha)}\\&+S_\alpha^{-1}(u^{2\alpha}S_\alpha[k(x,t)-N_\alpha W(x,t)+R_\alpha W(x,t)]).\end{aligned}$$

Applying $(\partial^\alpha/\partial t^\alpha)$ on both sides of Eq.(35), we have:

(36) $$\begin{aligned}&\frac{\partial^\alpha W(x,t)}{\partial t^\alpha}-W_t^{(\alpha)}(x,0)\\&+S_\alpha^{-1}(u^{2\alpha}S_\alpha[N_\alpha W(x,t)+R_\alpha W(x,t)-k(x,t)])=0.\end{aligned}$$

The correction functional of the variational iteration method (29), is given by:

(37) $$W_{n+1}=W_n-{}_0I_t^{(\alpha)}\left[\begin{aligned}&\frac{\partial^\alpha W_n}{\partial t^\alpha}-W_t^{(\alpha)}(x,0)\\&+S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[NW_n+RW_n-k(x,t)\right]\right)\end{aligned}\right].$$

The solution is calculated by the following limit:

(38) $$W(x,t)=\lim_{x\to\infty}W_n(x,t).$$

4 Application

In this section, we will implement the proposed method local fractional Sumudu variational iteration method (LFSVIM) for solving some local fractional partial differential equations.

Example 4.1 First, we consider the following local fractional partial differential equation:

$$\frac{\partial^{2\alpha}W(x,t)}{\partial t^{2\alpha}}+\frac{\partial^{2\alpha}W(x,t)}{\partial x^{2\alpha}}-3W(x,t)=0, t>0, x\in R, \tag{39}$$

subject to the initial conditions:

$$W(x,0)=\sin_{\alpha}(x^{\alpha}), \frac{\partial^{\alpha}W(x,0)}{\partial t^{\alpha}}=2\sin_{\alpha}(x^{\alpha}). \tag{40}$$

From (37) and (39), the formula of successive approximations is given by:

$$W_{n+1}=W_n-{}_0I_t^{(\alpha)}\left[\begin{array}{l}\dfrac{\partial^{2\alpha}W_n}{\partial\tau^{2\alpha}}-\dfrac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}\\ +S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\dfrac{\partial^{2\alpha}W_n}{\partial x^{2\alpha}}-3W_n\right]\right)\end{array}\right]. \tag{41}$$

According to the successive formula (41) and the initial conditions (40), we obtain:

$$W_0=\sin_{\alpha}(x^{\alpha})+2\sin_{\alpha}(x^{\alpha})\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{42}$$

$$W_1=W_0-{}_0I_t^{(\alpha)}\left[\begin{array}{l}\dfrac{\partial^{2\alpha}W_0}{\partial\tau^{2\alpha}}-\dfrac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}\\ +S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\dfrac{\partial^{2\alpha}W_0}{\partial x^{2\alpha}}-3W_0\right]\right)\end{array}\right], \tag{43}$$

$$W_2=W_1-{}_0I_t^{(\alpha)}\left[\begin{array}{l}\dfrac{\partial^{2\alpha}W_1}{\partial\tau^{2\alpha}}-\dfrac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}\\ +S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\dfrac{\partial^{2\alpha}W_1}{\partial x^{2\alpha}}-3W_1\right]\right)\end{array}\right], \tag{44}$$

(45) $$W_3 = W_2 - {}_0I_t^{(\alpha)}\left[\begin{array}{l}\dfrac{\partial^{2\alpha}W_2}{\partial\tau^{2\alpha}} - \dfrac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}} \\ + S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[\dfrac{\partial^{2\alpha}W_2}{\partial x^{2\alpha}} - 3W_2\right]\right)\end{array}\right],$$

$$\vdots$$

and so on.

From the formulas (42)-(45), the first terms of local fractional Sumudu variational iteration method is given by:

$$W_0(x,t) = \sin_\alpha(x^\alpha)\left(1 + \frac{2t^\alpha}{\Gamma(1+\alpha)}\right),$$

$$W_1(x,t) = \sin_\alpha(x^\alpha)\left(1 + \frac{2t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\right),$$

$$W_2(x,t) = \sin_\alpha(x^\alpha)\left(1 + \frac{2t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\right.$$
$$\left. + \frac{16t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{32t^{5\alpha}}{\Gamma(1+5\alpha)}\right),$$

$$W_3(x,t) = \sin_\alpha(x^\alpha)\left(1 + \frac{2t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\right.$$
$$\left. + \frac{16t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{32t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{64t^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{128t^{7\alpha}}{\Gamma(1+7\alpha)}\right),$$

$$\vdots$$

and so on.

Then, the non-differentiable solution of (38), has the form:

$$W(x,t) = \sin_\alpha(x^\alpha)\lim_{n\to\infty}\left(1 + \frac{2t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\right.$$
$$\left. + \frac{16t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{32t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{64t^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{128t^{7\alpha}}{\Gamma(1+7\alpha)} + \cdots + \frac{(2t^\alpha)^n}{\Gamma(1+n\alpha)}\right)$$
$$= \sin_\alpha(x^\alpha)E_\alpha(2t^\alpha).$$

Example 4.1 Second, we consider the following local fractional partial differential equation:

$$\frac{\partial^{2\alpha}W(x,t)}{\partial t^{2\alpha}}+\frac{\partial^{3\alpha}W(x,t)}{\partial x^{3\alpha}}=1, t>0, x\in R \tag{43}$$

subject to the initial conditions:

$$W(x,0)=E_{\alpha}((2x)^{\alpha}), \frac{\partial^{\alpha}W(x,0)}{\partial t^{\alpha}}=0. \tag{44}$$

From (37) and (43), the formula of successive approximations is given by:

$$W_{n+1}=W_n-{}_0I_t^{(\alpha)}\left[\begin{array}{l}\frac{\partial^{2\alpha}W_n}{\partial\tau^{2\alpha}}-\frac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}\\+S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\frac{\partial^{3\alpha}W_n}{\partial x^{3\alpha}}-1\right]\right)\end{array}\right]. \tag{45}$$

According to the successive formula (45) and the initial conditions (44), we obtain:

$$W_0(x,t)=E_{\alpha}((2x)^{\alpha}),$$

$$W_1=W_0-{}_0I_t^{(\alpha)}\left[\frac{\partial^{2\alpha}W_0}{\partial\tau^{2\alpha}}-\frac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}+S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\frac{\partial^{3\alpha}W_0}{\partial x^{3\alpha}}-1\right]\right)\right],$$

$$W_2=W_0-{}_0I_t^{(\alpha)}\left[\frac{\partial^{2\alpha}W_1}{\partial\tau^{2\alpha}}-\frac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}+S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\frac{\partial^{3\alpha}W_1}{\partial x^{3\alpha}}-1\right]\right)\right],$$

$$W_3=W_2-{}_0I_t^{(\alpha)}\left[\frac{\partial^{2\alpha}W_2}{\partial\tau^{2\alpha}}-\frac{\partial^{\alpha}W(x,0)}{\partial\tau^{\alpha}}+S_{\alpha}^{-1}\left(u^{2\alpha}S_{\alpha}\left[\frac{\partial^{3\alpha}W_2}{\partial x^{3\alpha}}-1\right]\right)\right],$$

$$\vdots$$

Then, the first terms of local fractional sumudu variational iteration method has the form

$$W_0(x,t)=E_{\alpha}((2x)^{\alpha}),$$

$$W_1(x,t)=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+E_{\alpha}((2x)^{\alpha})\left(1-2^{2\alpha}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right),$$

$$W_2(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + E_\alpha((2x)^\alpha)\Bigg(1 - 2^{2\alpha}\,\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$+ 2^{4\alpha}\,\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - 2^{6\alpha}\,\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}\Bigg),$$

$$W_3(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + E_\alpha((2x)^\alpha)\Bigg(1 - 2^{2\alpha}\,\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + 2^{4\alpha}\,\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

$$- 2^{6\alpha}\,\frac{t^{6\alpha}}{\Gamma(1+6\alpha)} + 2^{8\alpha}\,\frac{t^{8\alpha}}{\Gamma(1+8\alpha)} - 2^{10\alpha}\,\frac{t^{10\alpha}}{\Gamma(1+10\alpha)}\Bigg),$$

$$\vdots$$

and so on.

Then, the non-differentiable solution of Eq.(39), is given by:

$$W(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + E_\alpha((2x)^\alpha)\lim_{n\to\infty}\Bigg(1 - \frac{(2t)^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(2t)^{4\alpha}}{\Gamma(1+4\alpha)}$$

$$- \frac{(2t)^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{(2t)^{8\alpha}}{\Gamma(1+8\alpha)} - \frac{(2t)^{10\alpha}}{\Gamma(1+10\alpha)} + \cdots + (-1)^n\,\frac{(2t)^{2n\alpha}}{\Gamma(1+2n\alpha)}\Bigg),$$

$$= \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + E((2x)^\alpha)\cos_\alpha((2t)^\alpha).$$

Example 4.1 Finally, we consider the following Nonlinear local fractional partial differential equation:

(46) $$W_{tt}^{(2\alpha)}(x,t) - W_{xx}^{(2\alpha)}(x,t) + (W(x,t))^2 + W(x,t) = 0,$$

where t>0, x∈R and subject to the initial conditions

(47) $$W(x,0) = \sin_\alpha(x^\alpha), W_t^{(\alpha)}(x,0) = 0.$$

From (37) and (46), the formula of successive approximations is given by:

$$W_{n+1} = W_n - {}_0I_t^{(\alpha)}\begin{bmatrix} W_{n,\tau}^{(\alpha)} - W_\tau^{(\alpha)}(x,0) \\ + S_\alpha^{-1}\Big(u^{2\alpha}S_\alpha\big[-W_{n,xx}^{(2\alpha)} + (W_n)^2 + W_n\big]\Big)\end{bmatrix}.$$

Consequently, one can derive the approximations of the first three terms:

(48) $$W_0 = \sin_\alpha(x^\alpha),$$

(49) $$W_1 = W_0 - {}_0I_t^{(\alpha)}\begin{bmatrix} W_{0,\tau}^{(\alpha)} - W_\tau^{(\alpha)}(x,0) \\ + S_\alpha^{-1}\Big(u^{2\alpha}S_\alpha\big[-W_{0,xx}^{(2\alpha)} + (W_0)^2 + W_0\big]\Big)\end{bmatrix}.$$

$$W_2 = W_1 - {}_0I_t^{(\alpha)} \begin{bmatrix} W_{1,\tau}^{(\alpha)} - W_\tau^{(\alpha)}(x,0) \\ + S_\alpha^{-1}\left(u^{2\alpha} S_\alpha\left[-W_{1,xx}^{(2\alpha)} + (W_1)^2 + W_1\right]\right) \end{bmatrix}. \tag{50}$$

$$W_3 = W_2 - {}_0I_t^{(\alpha)} \begin{bmatrix} W_{2,\tau}^{(\alpha)} - W_\tau^{(\alpha)}(x,0) \\ + S_\alpha^{-1}\left(u^{2\alpha} S_\alpha\left[-W_{2,xx}^{(2\alpha)} + (W_2)^2 + W_2\right]\right) \end{bmatrix}, \tag{51}$$

$$\vdots$$

According to the formulas (48)-(51), the first terms of local fractional Sumudu variational iteration method, has the form:

$$W_0(x,t) = \sin_\alpha(x^\alpha),$$

$$W_1(x,t) = \sin_\alpha(x^\alpha)\left(1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right),$$

$$W_2(x,t) = \sin_\alpha(x^\alpha)\left(1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{t^{6\alpha}}{\Gamma(1+6\alpha)}\right),$$

$$W_3(x,t) = \sin_\alpha(x^\alpha)\left(1 - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{t^{8\alpha}}{\Gamma(1+8\alpha)} - \frac{t^{10\alpha}}{\Gamma(1+10\alpha)}\right),$$

$$\vdots$$

and so on.

Then, the non-differentiable solution of Eq.(46), is given by:

$$W(x,t) = \lim_{n\to\infty}\left(\sin_\alpha(x^\alpha)\sum_{k=0}^{n}(-1)^k \frac{t^{2k\alpha}}{\Gamma(1+2k\alpha)}\right) = \sin_\alpha(x^\alpha)\cos_\alpha(t^\alpha).$$

4 Conclusion

In this article, we have seen that the coupling of variational iteration method (VIM) and the Sumudu transform method in the sense of local fractional derivative, proved very effective to solve linear and nonlinear local fractional partial differential equations. The local fractional Sumudu variational iteration method (LFSVIM) is suitable for such problems and is very user friendly. The advantage of this method is its ability to combine two powerful methods for obtaining exact or approximate solutions for linear and nonlinear local fractional partial differential equations. That's why we say that modified LFSVIM is an alternative analytical method for solving linear and nonlinear local fractional partial differential equations.

6 Open Problem

In this article, we have combine two methods, namely variational iteration method with Sumudu transform method in the sence of local fractional derivative, for solving local fractional partial differential equations. The results proved that this method is effective in solving this type of equations.
The question is: does the combination of the variational iteration method and the natural transform method yield the same previous results?

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