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# Nonparametric Estimation of the Diffusion Coefficient Under $\tilde{\rho}$-Mixing

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**Abstract**

*Obtained is the uniform almost sure convergence for a kernel estimate of the variance function in the diffusion model for a $\widetilde{\rho}-$mixing process when the data belong to a sequence of compact sets which increases to* R.

**Keywords:** *Diffusion coefficient, kernel estimate, $\widetilde{\rho}$-mixing.*

# 1 Introduction

The problem of estimating the diffusion coefficient is subject to several investigations when using diffusion process for modeling financial data. Consider a diffusion $(X_t)$ defined as the solution of the stochastic differential equation:

$$dX_t = \sigma(X_t)dW_t \ , \quad t \in \mathrm{R}^+ \tag{1}$$

where $(W_t;\ t \in \mathrm{R}^+)$ is a standard Brownian motion, $\sigma$ is a Lipschitz and unknown function of class $\mathcal{C}^1$, strictly positive. Under Lipschitz conditions about $\sigma$, there exists for any given initial $X_0$ independent of $(W_t;\ t \in \mathrm{R}^+)$, a unique solution with probability one, to equation (1) and this solution is a measurable Markov process (Wong [17], Prop. 4.1, $P_5$ and $P_6$, p. 150).

Also, under suitable conditions ( Banon [4]), the unique solution of equation (1) must have a stationary transitional density, say $f_{X_t|X_0=.}$, satisfying the forward equation of Kolmogorov:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x) f_{X_t|X_0=}(x)\Big) = \frac{\partial}{\partial t} f_{X_t|X_0=}(x)$$

the limit of $f_{X_t|X_0=.}$ being a density, say $f$, as $t$ goes to infinity.

For simplicity, suppose that the initial distribution of $X_0$ has a density $f$, so that $(X_t)$ is a stationary process and study the estimation of $\sigma^2(x)$ for each $x \in E$, where $E$ is the nonempty set $\{x \in \mathrm{R} \quad / \quad f(x) > 0\}$. In some practical instances, for the estimation of the volatility in mathematical finance models of Black and Scholes type, the diffusion term $\sigma$ is of major interest.

The above problem has been considered by Donhal [8] when the diffusion coefficient depends on a parameter $\theta$, and by Genon-Catalot and Jacod [9] in the multidimensional case. Genon-Catalot *et al.* [10] used the wavelets method to estimate $\sigma^2$, Arfi [2] considered the estimation of $\sigma^2$ under the ergodic condition and Arfi & Lecoutre [3] studied the estimation of $\sigma^2$ when the observed process is strong mixing and established an almost-sure convergence. Hoffman [12] estimated the diffusion coefficient from a 1-dimensional diffusion process sampled at time. Abdolsadeh [1] considered the least squares method to estimate the diffusion coefficient.

In this paper; a study of the nonparametric kernel type estimate of the diffusion coefficient is conducted and the uniform almost sure consistency under $\widetilde{\rho}-$ mixing condition (see Bradley [5] ) is obtained.

Let $\Delta$ be positive and fixed and $n \in \mathrm{N}$; the Markov observation $(X_{i\Delta},\ 1 \leq i \leq n)$ permits to write:

$$X_{i\Delta+\Delta} - X_{i\Delta} = \sigma_\Delta(X_{i\Delta})\varepsilon_{i\Delta+\Delta}$$

where

$$\sigma^2_\Delta(X_t) = V(X_{t+\Delta}|X_t)$$

is supposed to exist and defines a discrete version of $\sigma^2$, $(\varepsilon_t)$ being a stationary Gaussian process such that:

$$E(\varepsilon_{t+\Delta}|X_s;\ s \leq t) = 0 \quad \text{and} \quad E(\varepsilon^2_{t+\Delta}|X_s;\ s \leq t) = 1$$

First, consider an estimator of $\sigma^2_\Delta$ based on the discrete observation $(X_{i\Delta},\ 1 \leq i \leq n)$:

$$S_{\Delta,n}(x) = \frac{\sum_{i=1}^n K\left(\dfrac{x - X_{i\Delta}}{h_n}\right)(X_{i\Delta+\Delta} - X_{i\Delta})^2}{\sum_{i=1}^n K\left(\dfrac{x - X_{i\Delta}}{h_n}\right)} \qquad \forall x \in E$$

where $(h_n)$ is a positive sequence of real numbers such that $h_n \to 0$ and $nh_n \longrightarrow \infty$ when $n \to \infty$, $K$ is a Parzen-Rosenblatt kernel, that is a bounded function satisfying $\int_{\mathrm{R}} K(x)dx = 1$ and $\lim_{|x|\to\infty} |x|K(x) = 0$; moreover it will be assumed to be strictly positive and with bounded variation.

The uniform almost sure convergence of $S_{\Delta,n}$ to $\sigma^2_\Delta$ is established under $\widetilde{\rho}-$mixing hypothesis when the data belong to a sequence of compact sets which increases to R when $n \to \infty$. Then, using the fact that:

$$\sigma^2(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} E\{(X_{t+\Delta} - X_t)^2 | X_t = x\}$$

$\frac{S_{\Delta,n}}{\Delta}$ is shown to be a consistent estimate of $\sigma^2$ if $\Delta = \Delta(n)$ such that $N = n\Delta \to \infty$, which is a necessary condition for $Nh_n \to \infty$ and the use of the $\widetilde{\rho}-$mixing assumption.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(X_i,\ i \geq 1)$ be a sequence of random variables; then write $\mathcal{F}_2 = \sigma(X_i, i \in S \subset N)$. Given the $\sigma$-algebras $\mathcal{B}$, $\mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup \{\mathrm{corr(X,Y)},\ \mathrm{X} \in \mathrm{L}_2(\mathcal{B}),\ \mathrm{Y} \in \mathrm{L}_2(\mathcal{R})\},$$

where $\mathrm{corr}(X,Y) = (EXY - EXEY)/\sqrt{\mathrm{var(X)var(Y)}}$. Bradley [5] introduced the following coefficients of dependence:

$$\widetilde{\rho}(k) = \sup \{\rho(\mathcal{F}_S, \mathcal{F}_T)\}, \quad k \geq 0,$$

where the supermum is taken over all finite subsets $S,\ T \subset N$ such that $\mathrm{dist}(S,T) \geq k$. Obviously, $0 \leq \widetilde{\rho}(k+1) \leq \widetilde{\rho}(k) \leq 1$, $k \geq 0$, and $\widetilde{\rho}(0) = 1$.

## 1.1 Definition

*A random variable sequence* $(X_i, i \geq 1)$ *is said to be a* $\widetilde{\rho}-$*mixing sequence if there exists* $k \in N$ *such that* $\widetilde{\rho}(k) < 1$.

Without loss of generality, assume that $(X_{i\Delta}, i \geq 1)$ is such that $\widetilde{\rho}(1) < 1$.

Such an approach has been subject to several investigations and a number of distinguished papers is devoted to this topic. There are among others Bradley ([5] , [6]) for the central limit theorem, Bryc and Smolenski [7] for moment inequalities and almost sure convergence, Peligrad &Gut [14] for almost-sure results , Shixin [15] who studied the almost sure convergence and obtained some new results, Guang-Hui [11] and Meng-Hu [13] both of them sudied the strong law of large numbers under different conditions, Sung [16] for complete convergence for weighed sums.

# 2 General assumptions

**H1.** The process $(X_{i\Delta})$, $i \in N$, is strictly stationary and $\widetilde{\rho}-$mixing.
**H2.** The $(X_{i\Delta})$ have a continuous and bounded density $f$ in R.
**H3.** The initial random variable $X_0$ is of second order: $EX_0^2 < \infty$.
**H4.** The function $\sigma(.)$ is Borel measurable on R, satisfying, for $x$, $y \in$R, $\sigma(x) \geq \sigma_0 > 0$, the uniform Lipschitz condition:

$$|\sigma(x) - \sigma(y)| \leq c_1|x - y|$$

and the linear growth condition:

$$\sigma(x) \leq c_2\sqrt{1 + x^2}$$

where $c_1$ and $c_2$ are two positive constants.

**H5.** $\exists\ a > 0, \forall x \in C_n,\ \ f(x) \geq n^{-a}$ for $n \geq 1$
where $C_n = \{x : ||x|| \leq\ c_n\}$ , $c_n \to \infty$.
**H6.** The density $f$ is twice differentiable and its derivatives are bounded.

# 3 Results

## 3.1 Theorem

*Suppose that the assumptions* H1 - H6 *hold and further assume that the function $\sigma_\Delta$ is Lipschitz and bounded, the sequence $h_n$ is such that $h_n = o(n^{-a})$.*
*If the kernel $K$ is Lipschitz, even with $\int z^2K(z)dz < \infty$ then:*

$$\sup_{x\in C_n} |S_{\Delta,n}(x) - \sigma^2_\Delta(x)| \longrightarrow 0,\ \ a.s.\,,\ n \to \infty.$$

## 3.2 Corollary

*Under the assumptions of the Theorem , with the new choice of $h_n$ and $\Delta = \Delta(n)$ such as:*

$$\Delta \to 0,\ \ n\Delta \to \infty,\ \ \frac{h_n}{\Delta} = o(1),\ \ \lim_{n\to\infty} n\Delta h_n = \infty,$$

*then :*

$$\sup_{x\in C_n} \left|\frac{S_{\Delta,n}(x)}{\Delta} - \sigma^2(x)\right| \longrightarrow 0,\ \ a.s.\ \ n \to \infty.$$

### 3.2.1 Remarks

1) The sequence $h_n$ in the Theorem can be chosen such as: $h_n = h_0 n^{-\tau/2}$ for $0 < \tau < 1/2$ with $h_0$ being a positive constant.

2) Similarly, in Corollary one can choose $h_n = h_0 n^{-\tau/2}$, $\Delta = \delta_0 n^{-\tau/4}$ with $h_0$ and $\delta_0$ being some positive constants.

3) The construction of the estimator requires a choice of $K$ and $h_n$. If the choice of $K$ does not much influence the asymptotic behavior of $S_{\Delta,n}$, on the contrary the choice of $h_n$ turns to be crucial for the estimator's accuracy. One can employ a cross-validation or plug-in method.

4) If $X_0$ is independent of $(W_t, t \in \mathrm{R}^+)$, the condition $\sigma(x) \leq c_2\sqrt{1+x^2}$ implies that $(X_t)$ is stationary ( Wong [17]).

5) Assumptions H1 and H2 are satisfied in the case of an Ornstein-Uhlenbeck process if $X_0$ follows a centered normal law.

# 4 Preliminary results

The following decomposition is used:

$$S_{\Delta,n}(x) - \sigma_\Delta^2(x) = \frac{1}{f(x)}\left\{[g_n(x) - \sigma_\Delta^2(x)f(x)] - S_{\Delta,n}(x)[f_n(x) - f(x)\right\}$$

with

$$g_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x - X_{i\Delta}}{h_n}\right)(X_{i\Delta+\Delta} - X_{i\Delta})^2$$

$$f_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x - X_{i\Delta}}{h_n}\right).$$

Then,

$$\sup_{x\in C_n}\left|S_{\Delta,n}(x) - \sigma_\Delta^2(x)\right| \leq \frac{1}{\inf\limits_{x\in C_n} f(x)}\left\{\sup_{x\in C_n}\left|g_n(x) - \sigma_\Delta^2(x)f(x)\right| + \sup_{x\in C_n}|S_{\Delta,n}(x)|\,|f_n(x) - f(x)|\right\}.$$

Now, if $\sup_{x\in C_n}|S_{\Delta,n}(x)| \leq M_n$ *a.s.* where $M_n = n^{\xi}$ with $\xi \in ]0, 1/2[$

$$\sup_{x\in C_n}\left|S_{\Delta,n}(x) - \sigma_\Delta^2(x)\right| \leq n^a\left\{\sup_{x\in C_n}\left|g_n(x) - \sigma_\Delta^2(x)f(x)\right| + M_n\sup_{x\in C_n}|f_n(x) - f(x)|\right\}.$$

## 4.1 Lemma 1

*Under the assumptions of the Theorem :*

$$n^a\sup_{x\in C_n}|g_n(x) - \sigma_\Delta^2(x)f(x)| \xrightarrow{a.s.} 0, \quad n \to \infty.$$

*Proof.* Split into a stochastic part $[g_n(x) - Eg_n(x)]$ and a deterministic part $[Eg_n(x) - \sigma_\Delta^2(x)f(x)]$ and study each component apart.

Consider the stochastic component.

Because of the possible large values for the variables $(X_{i\Delta+\Delta} - X_{i\Delta})^2$, a truncation technique is used which consists in decomposing $g_n(x)$ to $g_n^+(x)$ and $g_n^-(x)$ where:

$$g_n^+(x) = \frac{1}{nh_n}\sum_{i=1}^n K\left(\frac{x - X_{i\Delta}}{h_n}\right)(X_{i\Delta+\Delta} - X_{i\Delta})^2 \mathrm{I}_{[(X_{i\Delta+\Delta}-X_{i\Delta})^2 \geq M_n]}$$

and $g_n^-(x) = g_n(x) - g_n^+(x)$ where $M_n = n^\xi$ for some fixed $\xi$ in $]0, 1/2[$.

Then,

$$n^a \sup_{x\in C_n} |g_n^+(x) - Eg_n^+(x)| \xrightarrow{a.s.} 0, \quad n \to \infty.$$

For the purpose, write

$$n^a \sup_{x\in C_n} |g_n^+(x) - Eg_n^+(x)| \leq E_n + F_n$$

where,

$$E_n = \frac{1}{n^{1-a}h_n} \sup_{x\in C_n}\left|\sum_{i=1}^n K\left(\frac{x - X_{i\Delta}}{h_n}\right)(X_{i\Delta+\Delta} - X_{i\Delta})^2 \mathrm{I}_{[(X_{i\Delta+\Delta}-X_{i\Delta})^2 \geq M_n]}\right|.$$

Having $(E_n \neq 0) \subset \{\exists i_0 \in [1, 2, ..., n]$ such that$(\mathrm{X_{i\Delta+\Delta}} - \mathrm{X_{i\Delta}})^2 \geq \mathrm{M_n}\}$ then:

$$(E_n \neq 0) \subset \bigcup_{i=1}^n \left\{(X_{i\Delta+\Delta} - X_{i\Delta})^2 \geq M_n\right\}$$

$$P(E_n \neq 0) \leq \sum_{i=1}^n P((X_{i\Delta+\Delta} - X_{i\Delta})^2 \geq M_n) = nP((X_{i_0\Delta+\Delta} - X_{i_0\Delta})^2 \geq M_n)$$

$$\sum_{n=1}^\infty P(E_n \neq 0) \leq \sum_{n=1}^\infty nP((X_{i_0\Delta+\Delta} - X_{i_0\Delta})^2 \geq M_n) \leq c_3 \sum_{n\geq 1} \frac{n}{M_n^\beta}$$

where $c_3$ is a positive constant and $\beta$ such that $\beta > 2/\xi$. Then $E_n \longrightarrow 0$, $a.s.$ when $n \to \infty$ and $\sup_{1\leq i\leq n}(X_{i\Delta+\Delta} - X_{i\Delta})^2 \leq M_n$.

The kernel $K$ being strictly positive, conclude that $\sup_{x\in C_n} |S_{\Delta,n}(x)| \leq M_n$ $a.s.$

Moreover,

$$F_n = \frac{1}{n^{1-a}h_n} \sup_{x \in C_n} \left| \sum_{i=1}^{n} E\left( K\left(\frac{x - X_{i\Delta}}{h_n}\right) (X_{i\Delta+\Delta} - X_{i\Delta})^2 \mathrm{I}_{[(X_{i\Delta+\Delta} - X_{i\Delta})^2 \geq M_n]} \right) \right|$$

$$F_n \leq \frac{K_1}{n^{-a}h_n} E\left( (X_{i\Delta+\Delta} - X_{i\Delta})^2 \mathrm{I}_{[(X_{i\Delta+\Delta} - X_{i\Delta})^2 \geq M_n]} \right)$$

where $K_1$ is an upperbound of $K$. The fact that $\sigma_\Delta$ is bounded and the properties of the process $(\epsilon_t)$ permit to write

$$F_n \leq \frac{K_1}{n^{-a}h_n} (E(X_{i\Delta+\Delta} - X_{i\Delta})^4)^{1/2} \left( P\left( (X_{i\Delta+\Delta} - X_{i\Delta})^2 \geq M_n \right) \right)^{1/2} \leq$$

$$c_4 n^a h_n^{-1} M_n^{-\beta/2} \longrightarrow 0, \ n \to \infty$$

with the choice $M_n = n^\xi$ for $\xi \in ]0, 1/2[$, $h_n = n^{-\tau/2}$ for $\tau \in ]0, 1/2[$ and $\beta > (2a + \tau)/\xi$ with $c_4$ being a positive constant.

Now, the following is established:

$$n^a \sup_{x \in C_n} |g_n^-(x) - Eg_n^-(x)| \xrightarrow{a.s.} 0, \quad n \to \infty.$$

For simplicity, define for fixed $i$ , $K_i(x) = K\left(h_n^{-1}(x - X_{i\Delta})\right)$,
$Y_{i\Delta} = (X_{i\Delta+\Delta} - X_{i\Delta})^2$ and write:

$$g_n^-(x) - Eg_n^-(x) = \sum_{i=1}^{n} Z_i$$

with

$$Z_i = \frac{1}{nh_n} \left\{ K_i(x) Y_{i\Delta} \mathrm{I}_{[Y_{i\Delta} \leq M_n]} - E\left( K_i(x) Y_{i\Delta} \mathrm{I}_{[Y_{i\Delta} \leq M_n]} \right) \right\}$$

then the following is obtained: $|Z_i| \leq 2K_1 M_n / (nh_n)$ ,
$E\,|Z_i| \leq 2K_1 M_n / n$ where $K_1$ is an upperbound of $K$.

Now, write:

$$\sum_{n \geq 1} P\left( n^a \left| g_n^-(x) - Eg_n^-(x) \right| > \epsilon \right) = \sum_{n \geq 1} P\left( \left| g_n^-(x) - Eg_n^-(x) \right| > n^{-a}\epsilon \right) =$$

$$\sum_{n \geq 1} P\left( \left| \sum_{i=1}^{n} Z_i \right| > n^{-a}\epsilon \right)$$

and,

$$W_{ni} = Z_i \mathrm{I}_{(|\mathrm{Z_i}| \le \mathrm{n}^\alpha)} \quad \mathrm{V_{ni}} = \mathrm{Z_i I}_{(|\mathrm{Z_i}| > \mathrm{n}^\alpha)} \text{ for } \ 1 \ \le \mathrm{i} \le \mathrm{n} \text{ and } \alpha > 1.$$

Then,

$$\left|\sum_{i=1}^{n} Z_i\right| \le \left|\sum_{i=1}^{n} (W_{ni} - EW_{ni})\right| + \left|\sum_{i=1}^{n} V_{ni}\right| + \left|\sum_{i=1}^{n} EW_{ni}\right|. \tag{2}$$

The following are to be shown:

$$\sum_{n\ge 1} P\left(n^a \left|\sum_{i=1}^{n} (W_{ni} - EW_{ni})\right| > \epsilon n^\alpha/3\right) < \infty \tag{3}$$

$$\sum_{n\ge 1} P\left(n^a \left|\sum_{i=1}^{n} V_{ni}\right| > \epsilon n^\alpha/3\right) < \infty \tag{4}$$

$$n^a \left|\sum_{i=1}^{n} EW_{ni}\right| / n^\alpha \longrightarrow 0, \ \mathrm{n} \to \infty. \tag{5}$$

Start by showing (3).

The Markov inequality leads to:
For $\epsilon_n = \epsilon n^{-a}$ write

$$\sum_{n\ge 1} P\left(n^a \left|\sum_{i=1}^{n} (W_{ni} - EW_{ni})\right| > \epsilon n^\alpha/3\right) = \sum_{n\ge 1} P\left(\left|\sum_{i=1}^{n} (W_{ni} - EW_{ni})\right| > \epsilon_n n^\alpha/3\right) \le$$

$$c_5 \sum_{n\ge 1} \sum_{i=1}^{n} E|W_{ni}|^\beta / n^{\alpha\beta} \le c_6 \sum_{n\ge 1} n^{\xi - \alpha\beta} < \infty$$

where $c_5$ and $c_6$ are two positive constants, $a > 0$ and $\beta$ such that $\beta > 2/\xi$ for $\xi \in ]0, 1/2[$.Then the Borel-Cantelli lemma concludes for (3).

To show (4) note that:

$$\left(\left|\sum_{i=1}^{n} V_{ni}\right| > \epsilon n^\alpha/3\right) \subset \bigcup_{i=1}^{n} (|Z_i| > n^\alpha)$$

hence for $\epsilon_n = \epsilon n^{-a}$,

$$\sum_{n\ge 1} P\left(\left|\sum_{i=1}^{n} V_{ni}\right| > \epsilon_n n^\alpha/3\right) \le \sum_{n\ge 1} nP\left(|Z_{i_0}| > n^\alpha\right) \le$$

$$\sum_{n\geq 1}\frac{nE|Z_{i_0}|^{\beta}}{n^{\alpha\beta}}\leq c_7\sum_{n\geq 1}n^{\xi-\alpha\beta}<\infty$$

where $c_7$ is a positive constant, $\beta$ and $a$ defined previously.

It remains to show that (5) holds.

$$n^a\left|\sum_{i=1}^{n}EW_{ni}\right|/n^{\alpha}\leq n^a\left|\sum_{i=1}^{n}EV_{ni}\right|/n^{\alpha}=n^{a-\alpha}\sum_{i=1}^{n}E|Z_i|\mathrm{I}_{(|\mathrm{Z_i}|>\mathrm{n}^{\alpha})}=$$

$$n^{a+1-\alpha}E|Z_i|\mathrm{I}_{(|\mathrm{Z_i}|>\mathrm{n}^{\alpha})}\longrightarrow 0,\mathrm{n}\rightarrow\infty$$

with the choice $\alpha>1+a,\ \alpha>1$ and $a>0$.

Next, cover $C_n$ by $\mu_n$ spheres in the shape of $\{x:||x-x_{nj}||\leq c_n\mu_n^{-1}\}$ with $1\leq j\leq\mu_n$ and $\mu_n\rightarrow\infty$, to be defined later. Then make the following decomposition:

$$\left|g_n^-(x)-Eg_n^-(x)\right|\leq\left|g_n^-(x)-g_n^-(x_{nj})\right|+\left|E[g_n^-(x)-g_n^-(x_{nj})]\right|+\left|g_n^-(x_{nj})-Eg_n^-(x_{nj})\right|.$$

The first and the second terms in the right-hand side of the inequality above are considered similarly. Having:

$$\left|g_n^-(x)-g_n^-(x_{nj})\right|\leq\frac{M_n}{nh_n}\sum_{i=1}^{n}|K_i\,(x)-K_i(x_{nj})|$$

The kernel $K$ being Lipschitz in the sense that :
$|K(u)-K(v)|\leq L_K||u-v||^k$, we obtain:

$$\left|g_n^-(x)-g_n^-(x_{nj})\right|\leq\frac{L_KM_n}{h_n^{1+k}}||x-x_{nj}||^k\leq\frac{L_KM_nc_n^k}{h_n^{1+k}\mu_n^k}=\frac{1}{Logn}$$

with the following choice for $\mu_n$:

$$\mu_n=\frac{L_K^{1/k}M_n^{1/k}c_n(Logn)^{1/k}}{h_n^{(1+1/k)}}\longrightarrow\infty.$$

Thus

$$\sup_{x\in C_n}\left|g_n^-(x)-Eg_n^-(x)\right|\leq\sup_{1\leq j\leq\mu_n}\left|g_n^-(x_{nj})-Eg_n^-(x_{nj})\right|+\frac{2}{Logn}$$

so that for all $n\geq n_1(\epsilon_n),\ \forall\epsilon_n>0$ and for $\epsilon_{n,1}=n^{-a}\epsilon_n$

$$P\left(n^a \sup_{x\in C_n}\left|\sum_{i=1}^{n} Z_i\right| > 2\epsilon_n\right) \leq \sum_{j=1}^{\mu_n} P\left(\left|g_n^-(x_{nj}) - Eg_n^-(x_{nj})\right| > \epsilon_{n,1}\right).$$

Now, using similar decomposition as in (2) $\mu_n$ times; permits to conclude that

$$n^a \sup_{x\in C_n} |g_n^-(x) - Eg_n^-(x)| \longrightarrow 0, \quad a.s. \quad n\to\infty.$$

It remains to show that:

$$n^a \sup_{x\in \mathrm{R}} |Eg_n(x) - \sigma_\Delta^2(x)f(x)| \longrightarrow 0, \quad n\to\infty.$$

Properties of the process $(\varepsilon_t)$ permit to write:

$$|Eg_n(x) - \sigma_\Delta^2(x)f(x)| \leq D_1 + D_2$$

with

$$D_1 = \frac{1}{h_n}\int_{\mathrm{R}} K\left(\frac{x-u}{h_n}\right)|\sigma_\Delta^2(u) - \sigma_\Delta^2(x)|f(u)du$$

$$D_2 = \sigma_\Delta^2(x)\left|\frac{1}{h_n}\int_{\mathrm{R}} K\left(\frac{x-u}{h_n}\right)f(u)du - f(x)\right|$$

The function $\sigma_\Delta^2$ being Lipschitz in the sense:

$$\forall\,(x,y)\ \in\ \mathrm{R}\times\mathrm{R} \qquad |\sigma_\Delta^2(x) - \sigma_\Delta^2(y)| \leq c_6|x-y|$$

where $c_6$ is a positive constant.

By change of variable, $z = (x-u)/h_n$ the following is obtained:

$$D_1 \leq \Gamma c_6 n^a h_n \int_{\mathrm{R}} |z|K(z)dz \to 0, \quad n\to\infty,$$

where $\Gamma$ is an upperbound of $f$ and where $h_n = o(n^{-a})$.

On the other hand writing $z = h_n^{-1}(x-u)$ , a Taylor expansion gives

$$D_2 \leq \sigma_\Delta^2(x)\left(n^a h_n \int zK(z)dz + 0.5 n^a h_n^2 \int z^2K(z)dz\right).$$

The fact that $\sigma_\Delta^2$ is bounded, the choice $h_n = n^{-\tau/2}$ with $a < \tau/2$ imply that $D_2$ goes to zero when $n$ approaches infinity.

Thus, the result:

$$n^a \sup_{x\in \mathrm{R}} |Eg_n(x) - \sigma_\Delta^2(x)f(x)| \longrightarrow 0, \quad n\to\infty.$$

### 4.2 Lemma 2

*Under the assumptions of the Theorem :*

$$M_n n^a \sup_{x \in C_n} |f_n(x) - f(x)| \xrightarrow{a.s.} 0, \quad n \to \infty.$$

*Proof.* This is a particular case of the Lemma 1, with $M_n n^a$ instead of $n^a$ and $(X_{i\Delta+\Delta} - X_{i\Delta})^2 = 1$.

## 5 Proof of the Theorem

It is sufficient to write:

$$\sup_{x \in C_n} |S_{\Delta,n}(x) - \sigma^2_\Delta(x) \leq$$

$$\frac{\sup_{x \in C_n} |g_n(x) - \sigma^2_\Delta(x) f(x)| + \sup_{x \in C_n} |S_{\Delta,n}(x)||f_n(x) - f(x)|}{\inf_{x \in C_n} f(x)}$$

The result follows from the Lemmas 1 and 2.

## 6 Proof of the corollary

It suffices to write:

$$\frac{S_{\Delta,n}(x)}{\Delta} - \sigma^2(x) = \frac{S_{\Delta,n}(x) - \sigma^2_\Delta(x)}{\Delta} + \left( \frac{\sigma^2_\Delta(x)}{\Delta} - \sigma^2(x) \right)$$

then,

$$\sup_{x \in C_n} \left| \frac{S_{\Delta,n}(x)}{\Delta} - \sigma^2(x) \right| \leq \sup_{x \in C_n} \left| \frac{S_{\Delta,n}(x) - \sigma^2_\Delta(x)}{\Delta} \right| + \sup_{x \in \mathrm{R}} \left| \frac{\sigma^2_\Delta(x)}{\Delta} - \sigma^2(x) \right|.$$

Similar arguments to those in the proof of the Theorem , and the conditions of the Corollary allow to conclude.

# 7 Open Problem

In absence of smoothness assumptions, the rate of convergence is not easy to control, the addition of more assumptions is relevant.

An open problem is to get the optimal rate in absence of smoothness without affecting the $\widetilde{\rho}-$mixing condition.

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