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On The Stability of a Four Species Syn Eco-System with Commensal Prey-Predator Pair with Prey-Predator Pair of Hosts-V

[Predator (S_2) Washed Out States]

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Abstract

The present paper is devoted to an investigation on a Four Species (S_1, S_2, S_3, S_4) Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts [Predator (S_2) washed out states]. The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are Commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 is Prey for S_4 and S_4 is Predator for S_3. The pair (S_1, S_2) may be referred as 1st level Prey-Predator and the pair (S_3, S_4) the 2nd level Prey-Predator. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of three of these sixteen equilibrium points : Predator (S_2) Washed Out States are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories are illustrated.

Keywords: Commensal, Eco-System, Equillibrium point, Host, Neutrally Stable, Prey, Predator, Quasi-linearization, Stable, Trajectories.

1. Introduction

Research in the area of theoretical Ecology was initiated in 1925 by Lotka [9] and in 1931 by Volterra [14]. Since then many Mathematicians and Ecologists contributed to the growth of this area of knowledge reported in the treatises of May [10], Smith [12], Kushing [7], Kapur [6] etc. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so on.
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Srinivas [13] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later Lakshminarayan and Pattabhi Ramacharyulu [8] studied Prey-Predator ecological models with partial cover for the Prey and alternate food for the Predator. Recently, Archana Reddy [1] and Bhaskara Rama Sharma [2] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Further Phani Kumar and Pattabhi Ramacharyulu [11] studied Three Species Ecosystem Consisting of a Prey, Predator and a Host Commensal to the Prey. The present authors Hari Prasad and Pattabhi Ramacharyulu [3,4,5] discussed on the stability of a four species: A Prey-Predator-Host-Commensal Syn Eco-System.

A Schematic Sketch of the system under investigation is shown here under Fig.1.

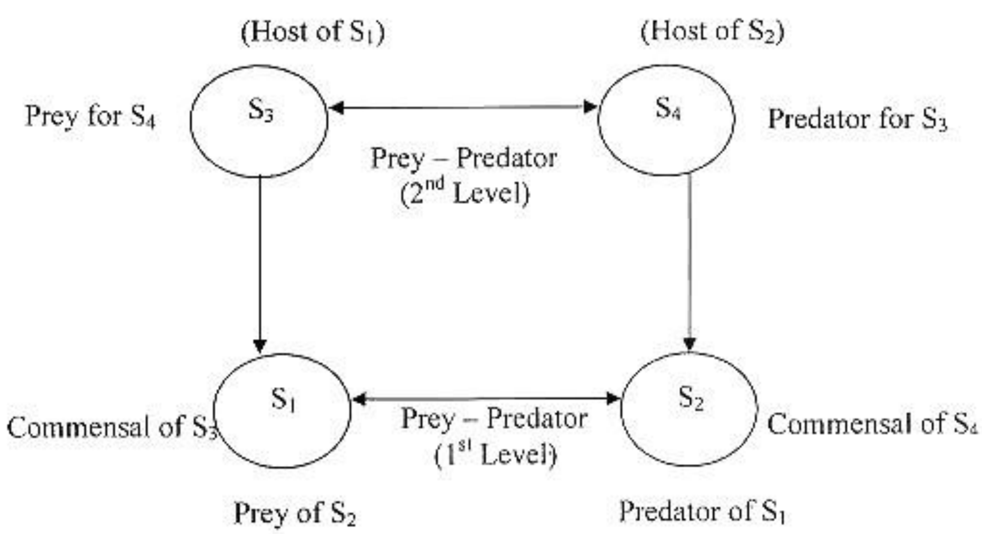


Fig. 1 Schematic Sketch of the Syn Eco - System

2. Basic Equations:

The model equations for a four species syn eco-system is given by the following system of first order non-linear ordinary differential equations employing the following notation.

Notation :

S_1	:	Prey for S_2 and commensal for S_3.
S_2	:	Predator surviving upon S_1 and commensal for S_4.
S_3	:	Host for the commensal (S_1) and Prey for S_4.
S_4	:	Host of the commensal (S_2) and Predator surviving upon S_4.
$N_i(t)$	:	The Population strength of S_i at time t, i = 1, 2, 3, 4
t	:	Time instant
a_i	:	Natural growth rate of S_i, i = 1, 2, 3, 4
a_{ii}	:	Self inhibition coefficient of S_i, i = 1, 2, 3, 4
a_{12},a_{21}	:	Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{34},a_{43}	:	Interaction (Prey-Predator) coefficients of S_3 due to S_4 and S_4 due to S_3
a_{13}, a_{24}	:	Coefficients for commensal for S_1 due to the Host S_3 and S_2 due to the Host S_4
$K_i = \dfrac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, i = 1, 2, 3, 4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}, a_{34}, a_{43} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^{\,2} - a_{12}N_1N_2 + a_{13}N_1N_3 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^{\,2} + a_{21}N_1N_2 + a_{24}N_2N_4 \tag{2.2}$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^{\,2} - a_{34}N_3N_4 \tag{2.3}$$

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^{\,2} + a_{43}N_3N_4 \tag{2.4}$$

3. Equilibrium States:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1,2,3,4 \qquad (3.1)$$

as given in the following Table-1.

Table – 1

S.No.	Equilibrium State	Equilibrium Point
1	Fully Washed out state	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
2	Only the Host (S_4)of S_2 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
3	Only the Host (S_3)of S_1 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
4	Only the Predator (S_2) survives	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$
5*	**Only the Prey (S_1) survives**	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
6	Prey (S_1) and Predator (S_2) washed out	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{\alpha}{\beta}, \overline{N_4} = \frac{\gamma}{\beta}$ where $\alpha = a_3a_{44} - a_4a_{34},\ \beta = a_{33}a_{44} + a_{34}a_{43} > 0$ $\gamma = a_3a_{43} + a_4a_{33} > 0$
7	Prey (S_1) and Host (S_3) of S_1 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{\delta_1}{a_{22}a_{44}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$ where $\delta_1 = a_2a_{44} + a_4a_{24} > 0$
8	Prey (S_1) and Host (S_4) of S_2 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
9*	**Predator (S_2) and Host (S_3) of S_1 washed out**	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
10*	**Predator (S_2) and Host (S_4) of S_2 washed out**	$\overline{N_1} = \frac{\delta_2}{a_{11}a_{33}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$ where $\delta_2 = a_1a_{33} + a_3a_{13} > 0$
11	Prey (S_1) and Predator (S_2)survives	$\overline{N_1} = \frac{\alpha_1}{\beta_1}, \overline{N_2} = \frac{\gamma_1}{\beta_1}, \overline{N_3} = 0, \overline{N_4} = 0$ where $\alpha_1 = a_1a_{22} - a_2a_{12},\ \beta_1 = a_{11}a_{22} + a_{12}a_{21} > 0$ $\gamma_1 = a_1a_{21} + a_2a_{11} > 0$
12	Only the Prey (S_1) washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2\beta + a_{24}\gamma}{a_{22}\beta}, \overline{N_3} = \frac{\alpha}{\beta}, \overline{N_4} = \frac{\gamma}{\beta}$

13*	**Only the predator (S_2) washed out**	$\overline{N_1}=\dfrac{a_1\beta+a_{13}\alpha}{a_{11}\beta},\overline{N_2}=0,\overline{N_3}=\dfrac{\alpha}{\beta},\overline{N_4}=\dfrac{\gamma}{\beta}$
14	Only the Host (S_3) of S_1 washed out	$\overline{N_1}=\dfrac{a_1a_{22}a_{44}-a_{12}\delta_1}{a_{44}\beta_1},\overline{N_2}=\dfrac{a_1a_{21}a_{44}+a_{11}\delta_1}{a_{44}\beta_1},$ $\overline{N_3}=0,\overline{N_4}=\dfrac{a_4}{a_{44}}$
15	Only the Host (S_4) of S_2 washed out	$\overline{N_1}=\dfrac{a_{22}\delta_2-a_2a_{12}a_{33}}{a_{33}\beta_1},\overline{N_2}=\dfrac{a_{21}\delta_2+a_2a_{11}a_{33}}{a_{33}\beta_1},$ $\overline{N_3}=\dfrac{a_3}{a_{33}},\overline{N_4}=0$
16	The co-existent state (or) Normal steady state	$\overline{N_1}=\dfrac{a_{22}\alpha_2-a_{12}\gamma_2}{\beta_1},\overline{N_2}=\dfrac{a_{11}\gamma_2+a_{21}\alpha_2}{\beta_1},$ $\overline{N_3}=\dfrac{\alpha}{\beta},\overline{N_4}=\dfrac{\gamma}{\beta}$ *where* $\alpha_2=a_1+a_{13}\dfrac{\alpha}{\beta},\ \gamma_2=a_2+a_{24}\dfrac{\gamma}{\beta}>0$

The present paper deals with the Predator (S_2) washed out states only (Sr. Nos. 5, 9, 10, 13 marked * in the above Table -1). The stability of the other equilibrium states will be presented in the forth coming communications.

4. Stability of the Equilibrium States:

Let $N=\left(N_1,N_2,N_3,N_4\right)=\bar{N}+U$ (4.1)

where $U=\left(u_1,u_2,u_3,u_4\right)$ is a perturbation over the equilibrium state $\bar{N}=\left(\bar{N}_1,\bar{N}_2,\bar{N}_3,\bar{N}_4\right)$.

The basic equations (2.1), (2.2), (2.3), (2.4) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt}=AU \tag{4.2}$$

where

$$A=\begin{bmatrix} a_1-2a_{11}\bar{N}_1-a_{12}\bar{N}_2+a_{13}\bar{N}_3 & -a_{12}\bar{N}_1 & a_{13}\bar{N}_1 & 0 \\ a_{21}\bar{N}_1 & a_2-2a_{22}\bar{N}_2+a_{21}\bar{N}_1+a_{24}\bar{N}_4 & 0 & a_{24}\bar{N}_2 \\ 0 & 0 & a_3-2a_{33}\bar{N}_3-a_{34}\bar{N}_3 & -a_{34}\bar{N}_3 \\ 0 & 0 & a_{34}\bar{N}_4 & a_4-2a_{44}\bar{N}_4+a_{43}\bar{N}_3 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is $\det\left[A-\lambda I\right]=0$ (4.4)

The equilibrium state is stable, if both the roots of the equation (4.4) are negative in case they are real or have negative real parts in case they are complex.

5. Stability of the Predator (S_2) washed out equilibrium states: (Sl. Nos. 5, 9, 10, 13 marked * in Table.1)

The equilibrium state $\overline{N}_1 = \dfrac{a_1}{a_{11}}, \overline{N}_2 = 0, \overline{N}_3 = 0, \overline{N}_4 = 0$ (Sl. No.5) was already discussed in the paper "On the stability of a four species Syn Eco-system with Commensal Prey-Predator pair with Prey-Predator pair of Hosts-III" communicated to 'Japanese Journal of Mathematics'.

5.1. Equilibrium point $\overline{N}_1 = \dfrac{a_1}{a_{11}}, \overline{N}_2 = 0, \overline{N}_3 = 0, \overline{N}_4 = \dfrac{a_4}{a_{44}}$:

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = -a_1u_1 - a_{12}k_1u_2 + a_{13}k_1u_3,\ \frac{du_2}{dt} = l_2u_2 \tag{5.1.1}$$

$$\frac{du_3}{dt} = l_3u_3,\ \frac{du_4}{dt} = a_{43}k_4u_3 - a_4u_4 \tag{5.1.2}$$

Here $l_2 = a_2 + a_{21}k_1 + a_{24}k_4 > 0,\ l_3 = a_3 - a_{34}k_4$ (5.1.3)

The characteristic equation for which is

$$(\lambda + a_1)(\lambda - l_2)(\lambda - l_3)(\lambda + a_4) = 0 \tag{5.1.4}$$

The characteristic roots are $-a_1, l_2, l_3, -a_4$. Since one of these four roots l_2 is positive, hence the state is **unstable**. The equations (5.1.1), (5.1.2) yield the solutions

$$u_1 = (u_{10} + P - Q)\ e^{-a_1t} - Pe^{l_2t} + Qe^{l_3t},\ u_2 = u_{20}e^{l_2t} \tag{5.1.5}$$

$$u_3 = u_{30}e^{l_3t},\ u_4 = \mathrm{Re}^{l_3t} + (u_{40} - R)\,e^{-a_4t} \tag{5.1.6}$$

Here $P = \dfrac{a_{12}K_1u_{20}}{a_1 + l_2}, Q = \dfrac{a_{13}k_1u_{30}}{a_1 + l_3}, R = \dfrac{a_{43}k_4u_{30}}{a_4 + l_3}$ (5.1.7)

and $u_{10}, u_{20}, u_{30}, u_{40}$ are the initial values of u_1, u_2, u_3, u_4 respectively.

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations. And the solution curves are illustrated in Figures (2) to (6) and the conclusions are presented here.

Case (A): When $l_3 < 0$ (i.e., $a_3 < a_{34}\, k_4$)

Case (i): If $u_{10} < u_{30} < u_{40} < u_{20}$ and $l_3 < a_1 < l_2 < a_4$

In this case the natural birth rates of the Host (S_3) of S_1, Host (S_4) of S_2, Prey (S_1) and the Predator (S_2) are in ascending order. Initially the Host (S_4) of S_2, Host (S_3) of S_1 dominates over the Prey (S_1) till the time instant t^*_{14}, t^*_{13} respectively and thereafter the dominance is reversed. t^*_{14} and t^*_{13} may be called the dominance times of S_4 over S_1 and S_3 over S_1 respectively.

Case (ii): If $u_{20} < u_{10} < u_{40} < u_{30}$ and $l_2 < a_1 < a_4 < l_3$

In this case the natural birth rates of the Host (S_4) of S_2, Host (S_3) of S_1, Predator (S_2) and the Prey (S_1) are in ascending order. Initially the Host (S_3) of S_1 dominates over the Prey (S_1), Predator (S_2) till the time instant t^*_{13}, t^*_{23} respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Prey (S_1), Predator (S_2) till the time instant t^*_{14}, t^*_{24} respectively and the dominance gets reversed thereafter.

Case (B): When $l_3 > 0$ (i.e., $a_3 > a_{34}\, k_4$)

Case (i): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $l_2 < l_3 < a_1 < a_4$

In this case the natural birth rates of the Predator (S_2), Host (S_3) of S_1, Prey (S_1) and the Host (S_4) of S_2 are in ascending order. Initially the Predator (S_2) dominates over the Prey (S_1), Host (S_4) of S_2, Host (S_3) of S_1 till the time instant $t^*_{12}, t^*_{42}, t^*_{32}$ respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{41} and the dominance gets reversed thereafter.

Case (ii): If $u_{40} < u_{10} < u_{20} < u_{30}$ and $a_1 < a_4 < l_2 < l_3$

In this case the natural birth rates of the Prey (S_1), Host (S_4) of S_2, Predator (S_2) and the Host (S_3) of S_1 are in ascending order. Initially the Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{41} and thereafter the dominance is reversed.

Case (C): When $l_3 = 0$ (i.e, $a_3 - a_{34}k_4 = 0$)

In this case (5.1.1), (5.1.2) becomes

$$u_1 = (u_{10} + P)\, e^{-a_1 t} + P\left(1 - e^{-a_1 t}\right) - Pe^{l_2 t},\ u_2 = u_{20} e^{l_2 t} \tag{5.1.8}$$

$$u_3 = u_{30},\ u_4 = u_{40} e^{-a_4 t} + R\left(1 - e^{-a_4 t}\right) \tag{5.1.9}$$

5.1.A Trajectories of Perturbations:

The trajectories in the $u_2 - u_3$ plane given by $y_1^{l_3} = y_2^{l_2}$ (5.1.10)

and are shown in Fig.7 and the trajectories in the other planes are

$$x = P_1 y_1^{\frac{-a_1}{l_2}} - \overline{P} y_1 + \overline{Q} y_1^{\frac{l_3}{l_2}},\ x = P_1 y_2^{\frac{-a_1}{l_3}} - \overline{P} y_2^{\frac{l_2}{l_3}} + \overline{Q} y_2 \tag{5.1.11}$$

$$y_3 = \overline{R} y_1^{\frac{l_3}{l_2}} + R_1 y_1^{\frac{-a_4}{l_2}},\ \ y_3 = \overline{R} y_2 + R_1 y_2^{\frac{-a_4}{l_3}} \tag{5.1.12}$$

where $P_1 = 1 + \dfrac{P - Q}{u_{10}},\ \overline{P} = \dfrac{P}{u_{10}},\ \overline{Q} = \dfrac{Q}{u_{10}},\ \overline{R} = \dfrac{R}{u_{40}},\ R_1 = 1 - \dfrac{R}{u_{10}}$ (5.1.13)

and $x = \dfrac{u_1}{u_{10}},\ y_1 = \dfrac{u_2}{u_{20}},\ \ y_2 = \dfrac{u_3}{u_{30}},\ y_3 = \dfrac{u_4}{u_{40}}$ (5.1.14)

5.2. Equilibrium point $\overline{N}_1 = \dfrac{\delta_2}{a_{11}\, a_{33}},\ \overline{N}_2 = 0,\ \overline{N}_3 = \dfrac{a_3}{a_{33}},\ \overline{N}_4 = 0$:

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = -r_1 u_1 - \frac{a_{12}\delta_2}{a_{11}a_{33}}\, u_2 + \frac{a_{13}\delta_2}{a_{11}a_{33}}\, u_3,\ \frac{du_2}{dt} = r_2 u_2 \tag{5.2.1}$$

$$\frac{du_3}{dt} = -a_3 u_3 - a_{34}k_3 u_4,\quad \frac{du_4}{dt} = r_4 u_4 \tag{5.2.2}$$

Here $\quad r_1 = a_1 + a_{13}k_3 > 0,\ \ r_2 = a_2 + \dfrac{a_{21}\delta_2}{a_{11}a_{33}} > 0,\ \ r_4 = a_4 + a_{43}k_3 > 0$ (5.2.3)

The characteristic equation for which is

$$\left(\lambda + r_1\right)\left(\lambda - r_2\right)\left(\lambda + a_3\right)\left(\lambda - r_4\right) = 0 \tag{5.2.4}$$

The characteristic roots are $-r_1, r_2, -a_3, r_4$. Since two of these four roots r_2, r_4 are positive, hence the state is **unstable**. The equations (5.2.1), (5.2.2) yield the solutions

$$u_1 = \left[u_{10} - \left(H + I + J\right)\right]e^{-r_1 t} + He^{r_2 t} + Ie^{-a_3 t} + Je^{r_4 t}, \quad u_2 = u_{20}e^{r_2 t} \tag{5.2.5}$$

$$u_3 = \left(u_{30} - K\right)e^{-a_3 t} - Ke^{r_4 t}, \; u_4 = u_{40}e^{r_4 t} \tag{5.2.6}$$

$$H = \frac{a_{12}\delta_2 u_{20}}{a_{11}a_{33}\left(r_1 - r_2\right)}, \; I = \frac{a_{13}\delta_2\left(K - u_{30}\right)}{a_{11}a_{33}\left(r_1 + a_3\right)}, J = \frac{a_{13}\delta_2 K}{a_{11}a_{33}\left(r_1 - r_4\right)}, \; K = \frac{a_{34}k_3 u_{40}}{a_3 + r_4} \tag{5.2.7}$$

The solution carves are illustrated in figures (8), (9) and the conclusions are presented here.

Case (i): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $r_2 < r_4 < r_1 < a_3$

In this case the natural birth rates of the Predator $\left(S_2\right)$, Host $\left(S_4\right)$ of S_2, Prey $\left(S_1\right)$ and the Host $\left(S_3\right)$ of S_1 are in ascending order. Initially the Predator $\left(S_2\right)$ and its Host $\left(S_4\right)$ dominates over the Prey $\left(S_1\right)$ till the time instant t_{12}^* and t_{14}^* respectively and thereafter the dominance is reversed.

Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_3 < r_2 < r_1 < r_4$

In this case the natural birth rates of the Host $\left(S_3\right)$ of S_1, Predator $\left(S_2\right)$, Prey $\left(S_1\right)$ and the Host $\left(S_4\right)$ of S_2 are in ascending order. Initially the Prey $\left(S_1\right)$ dominates over the Host $\left(S_4\right)$ of S_2 till the time instant t_{41}^* and therafter the dominance is reversed also the Host $\left(S_3\right)$ of S_1 dominates over the Predator $\left(S_2\right)$ till the time instant t_{23}^* and the dominance gets reversed thereafter.

5.2.A Trajectoreis of perturbations:

The trajectories in the $u_2 - u_4$ plane given by $y_1^{r_4} = y_3^{r_2}$ (5.2.8)

and are shown in Fig.10 and the trajectories in the other planes are

$$x = H_1\, y_1^{\frac{-r_1}{r_2}} + \overline{H}y_1 + \overline{I}y_1^{\frac{-a_3}{r_1}} + \overline{J}\; y_1^{\frac{r_4}{r_1}}, \; y_2 = \overline{K}_1 y_1^{\frac{-a_3}{r_2}} - \overline{K}\; y_1^{\frac{r_4}{r_2}} \tag{5.2.9}$$

$$x = H_1\, y_3^{\frac{-r_1}{r_4}} + \overline{H}y_3^{\frac{r_2}{r_4}} + \overline{I}y_1^{\frac{-a_3}{r_4}} + \overline{J}\; y_3, \; y_2 = \overline{K}_1 y_3^{\frac{-a_3}{r_4}} - \overline{K}\; y_3 \tag{5.1.10}$$

where $H_1 = 1 - \dfrac{H+I+J}{u_{10}}, \bar{H} = \dfrac{H}{u_{10}}, \bar{I} = \dfrac{I}{u_{10}}$ (5.1.11)

$$\bar{J} = \frac{J}{u_{10}}, \bar{K}_1 = 1 - \frac{K}{u_{30}}, \bar{K} = \frac{K}{u_{30}} \tag{5.1.12}$$

5.3. Equilibrium point $\bar{N}_1 = \dfrac{a_1\beta + a_{13}\alpha}{a_{11}\beta}, \bar{N}_2 = 0, \bar{N}_3 = \dfrac{\alpha}{\beta}, \bar{N}_4 = \dfrac{\gamma}{\beta}$:

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = -m_1u_1 - a_{12}\left(k_1 + \frac{a_{13}\alpha}{a_{11}\beta}\right)u_2 + a_{13}\left(k_1 + \frac{a_{13}\alpha}{a_{11}\beta}\right)u_3, \frac{du_2}{dt} = m_2u_2 \tag{5.3.1}$$

$$\frac{du_3}{dt} = m_3u_3 - \frac{a_{34}\alpha}{\beta}u_4, \frac{du_4}{dt} = \frac{a_{43}\gamma}{\beta}u_3 + m_4u_4 \tag{5.3.2}$$

Here $m_1 = a_1 + a_{13}\dfrac{\alpha}{\beta}, m_2 = a_2 + a_{21}\left(k_1 + \dfrac{a_{13}\alpha}{a_{11}\beta}\right) + a_{24}\dfrac{\gamma}{\beta}$ (5.3.3)

$$m_3 = a_3 - \left(2a_{33}\frac{\alpha}{\beta} + a_{34}\frac{\gamma}{\beta}\right), m_4 = \left(a_4 + a_{43}\frac{\alpha}{\beta}\right) - 2a_{44}\frac{\gamma}{\beta} \tag{5.3.4}$$

The characteristic equation for which is

$$(\lambda + m_1)(\lambda - m_2)\left[\lambda^2 - (m_3 + m_4)\lambda + m_3m_4 + \frac{\alpha\gamma}{\beta^2}\right] = 0 \tag{5.3.5}$$

Two of the four roots $-m_1, m_2$. Let λ_1, λ_2 be the zero's of the quadratic polynomial on the L.H.S of the equation (5.3.5)

Case (A) : When $m_1 > 0, m_2 < 0$ and

(i): $(m_3 - m_4)^2 > 4\dfrac{\alpha\gamma}{\beta^2}$, the roots λ_1, λ_2 noted to be negative. Hence the state is **stable** and the equations (5.3.1), (5.3.2) yield the solutions

$$u_1 = [u_{10} - (\mu_1 + \mu_2 + \mu_3)]e^{-m_1t} + \mu_1e^{m_2t} + \mu_2e^{\lambda_1t} + \mu_3e^{\lambda_2t}, u_2 = u_{20}e^{m_2t} \tag{5.3.6}$$

$$u_3 = \left[\frac{\beta(m_3-\lambda_2)u_{30}-a_{34}\alpha u_{40}}{\beta(\lambda_1-\lambda_2)}\right]e^{\lambda_1 t} + \left[\frac{\beta(m_3-\lambda_1)u_{30}-a_{34}\alpha u_{40}}{\beta(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t} \tag{5.3.7}$$

$$u_4 = \left[\frac{\beta(m_3-\lambda_2)u_{30}-a_{34}\alpha u_{40}}{a_{34}\alpha(\lambda_1-\lambda_2)}\right](m_3-\lambda_1)e^{\lambda_1 t} + \left[\frac{\beta(m_3-\lambda_1)u_{30}-a_{34}\alpha u_{40}}{a_{34}\alpha(\lambda_2-\lambda_1)}\right](m_3-\lambda_2)e^{\lambda_2 t} \tag{5.3.8}$$

Here $$\mu_1 = \left(k_1+\frac{a_{13}\alpha}{a_{11}\beta}\right)\frac{a_{12}u_{30}}{m_2-m_1},\ \mu_2 = a_{13}\left(k_1+\frac{a_{13}\alpha}{a_{11}\beta}\right)\left[\frac{\beta(m_3-\lambda_2)u_{30}-a_{34}\alpha u_{40}}{\beta(\lambda_1+m_1)(\lambda_1-\lambda_2)}\right] \tag{5.3.9}$$

$$\mu_3 = a_{13}\left(k_1+\frac{a_{13}\alpha}{a_{11}\beta}\right)\left[\frac{\beta(m_3-\lambda_1)u_{30}-a_{34}\alpha u_{40}}{b(\lambda_2+m_1)(\lambda_2-\lambda_1)}\right] \tag{5.3.10}$$

If $u_{10} < u_{30} < u_{40} < u_{20}$ and $a_3 < m_1 < m_2 < a_4$

In this case the natural birth rates of the Host (S_3) of S_1, Prey (S_1), Predator (S_2) and the Host (S_4) of S_2 are in ascending order. Initially the Host (S_3) of S_1 dominates over the Prey (S_1) till the time instant t_{13}^* and thereafter the dominance is reversed. Also the Predator (S_2) dominates its Host (S_4) till the time instant t_{42}^* and the dominance gets reversed thereafter. This is illustrated in Fig. 11

(ii): $(m_3-m_4)^2 < \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1, λ_2 are complex. Hence the state is **stable** and this is illustrated in Fig. 12

(iii): $(m_3-m_4)^2 > \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1 is positive and λ_2 is negative. Hence the state is **unstable**

If $u_{20} < u_{30} < u_{10} < u_{40}$ and $m_1 < a_4 < m_2 < a_3$

In this case the natural birth rates of the Prey (S_1), Predator (S_2), Host (S_4) of S_2 and the Host (S_3) of S_1 are in ascending order. Initially the Prey (S_1) dominates over the Predator (S_2), Host (S_3) of S_1 till the time instant t_{21}^*, t_{31}^* respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t_{34}^* and the dominance gets reversed thereafter. This is illustrated in Fig. 13.

Case (B): When $m_1 > 0, m_2 = 0$ and

(i): $(m_3-m_4)^2 > \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1, λ_2 noted to be negative. Hence the state is **neutrally stable**.

In this case equation (5.3.6) becomes

$$u_1 = [u_{10} - (\mu_2 + \mu_3)]e^{-m_1 t} + \mu_1 \left(1 - e^{-m_1 t}\right) + \mu_2\, e^{\lambda_1 t} + \mu_3 e^{\lambda_2 t},\ u_2 = u_{20} \qquad (5.3.11)$$

If $u_{30} < u_{10} < u_{40} < u_{20}$ and $m_1 < a_3 < a_4 < m_2$

n this case the natural birth rates of the Prey (S_1), Host (S_3) of S_1, Host (S_4) of S_2 and the Predator (S_2) are in ascending order. Initially the Prey (S_1) dominates its Host (S_3) till the time instant t^*_{31} and thereafter the dominance is reversed. Further we notice that u_1 is asymptotic to u_1^* which is evident from the equation (5.3.11). Hence the state is **neutrally stable** and this is illustrated in Fig.14

If $u_{40} < u_{30} < u_{20} < u_{10}$ and $m_1 < a_4 < m_2 < a_3$

In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Host (S_3) of S_1, Host (S_4) of S_2 till the time instant $t^*_{21}, t^*_{31}, t^*_{41}$ respectively and thereafter the dominance is reversed. Further we notice that u_1 is asymptotic to u_1^* which is evident from the equation (5.3.11). Hence the state is **neutrally stable** and this is illustrated in Fig.15.

(ii): $(m_3 - m_4) < \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1, λ_2 are complex. Hence the state is **neutrally stable** and this is illustrated in Fig.16.

(iii): $(m_3 - m_4)^2 > \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1 is positive and λ_2 is negative. Hence the state is **unstable.**

If $u_{20} < u_{30} < u_{10} < u_{40}$ and $m_2 < m_1 < a_3 < a_4$

In this case the natural birth rates of Predator (S_2), Prey (S_1), Host (S_3) of S_1 and the Host (S_4) of S_2 are in ascending order. Initially the Prey (S_1) dominates its Host (S_3) till the time instant t^*_{31} and thereafter the dominance is reversed. This is illustrated in Fig.17.

Case (C): When $m_1 > 0, m_2 > 0$ and

(i): The roots λ_1, λ_2 noted to be negative hence the state is **unstable**.

If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_3 < a_4 < m_1 < m_2$

In this case the natural birth rates of Host (S_3) of S_1, Host (S_4) of S_1, Prey (S_1) and the Predator (S_2) are in ascending order. Initially the Prey (S_1) dominates over the predator (S_2) till the time instant t^*_{21} and thereafter the dominance is reversed. This is illustrated in Fig. 18.

(ii): $(m_3 - m_4)^2 > \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1 is positive and λ_2 is negative. Hence the state is **unstable.**

(iii): $(m_3 - m_4)^2 < \frac{4\alpha\gamma}{\beta^2}$, the roots λ_1, λ_2 are complex therefore the state is **unstable.**

Case (D): When $m_1 < 0, m_2 < 0$ and

(i): The roots λ_1, λ_2 noted to be negative. Hence the state is **unstable**.

If $u_{40} < u_{10} < u_{30} < u_{20}$ and $m_1 < a_4 < a_3 < m_2$

In this case the natural birth rates of Host (S_4) of S_2, Host (S_3) of S_1, Predator (S_2) and the Prey (S_1) are in ascending order. Initially the Predator (S_2), Host (S_3) of S_1 dominates over the Prey (S_1) till the time instant t^*_{12}, t^*_{13} respectively and thereafter the dominance is reversed. This is illustrated in Fig. 19.

(ii): The roots λ_1, λ_2 are complex (or) λ_1 is positive, λ_2 is negative. Therefore the state is **unstable.**

Case (E): When $m_1 = 0, m_2 < 0$ and λ_1, λ_2 noted to be negative (or) complex hence the state is **neutrally stable.**

Case (F): When $m_1 = 0, m_2 = 0$ and λ_1, λ_2 noted to be negative or complex the state is **neutrally stable.**

5.3.A Trajectories of Perturabations:

The trajectories in the $u_3 - u_4$ plane given by

$$\left[u_4^{(1-a)(v_2-v_1)}\right] d = \frac{(u_3 - v_1 u_4)^{av_1+m_4}}{(u_3 - v_2 u_4)^{av_2+m_4}} \tag{3.3.12}$$

where v_1 and v_2 are roots of quadratic equation $av^2 + bv + c = 0$ (3.3.13)

$$a = \frac{a_{43}\gamma}{\beta}, b = m_4 - m_3, \ c = \frac{a_{34}\alpha}{\beta} \tag{3.3.14}$$

and d is an arbitary constant This is illustrated in Fig. 20 and the trajectories in the other planes are

$$x = \frac{A}{u_{10}} y_1^{\frac{m_1}{m_2}} + \frac{\mu_1}{u_{10}} y_1 + \frac{\mu_2}{u_{10}} y_1^{\frac{\lambda_1}{m_2}} + \frac{\mu_3}{u_{10}} y_1^{\frac{\lambda_2}{m_2}},\ y_2 = A_1 y_1^{\frac{\lambda_1}{m_2}} + A_2 y_1^{\frac{\lambda_2}{m_2}},\ y_2 = B_1 y_1^{\frac{\lambda_1}{m_2}} + B_2 y_1^{\frac{\lambda_2}{m_2}} \quad (3.3.15)$$

where $A = u_{10} - (\mu_1 + \mu_2 + \mu_3),\ A_1 = \frac{\beta(m_3 - \lambda_2) u_{30} - a_{34}\alpha u_{40}}{\beta(\lambda_1 - \lambda_2) u_{30}},\ A_2 = \frac{\beta(m_3 - \lambda_1) u_{30} - a_{34}\alpha u_{40}}{\beta(\lambda_2 - \lambda_1) u_{30}}$ (3.3.16)

$$B_1 = (m_3 - \lambda_1) \frac{\beta(m_3 - \lambda_2) u_{30} - a_{34}\alpha u_{40}}{a_{34}\alpha(\lambda_1 - \lambda_2) u_{40}},\ B_2 = (m_3 - \lambda_2) \frac{\beta(m_3 - \lambda_1) u_{30} - a_{34}\alpha u_{40}}{a_{34}\alpha(\lambda_2 - \lambda_1) u_{40}} \quad (3.3.17)$$

6. Perturbation Graphs:

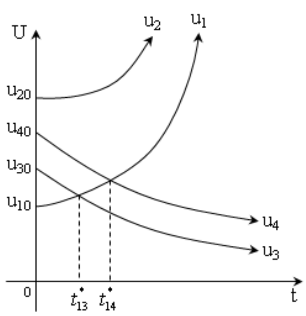


Fig – 2

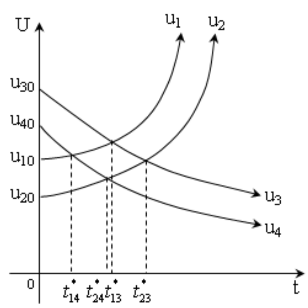


Fig – 3

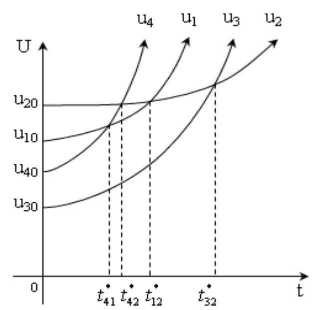


Fig - 4

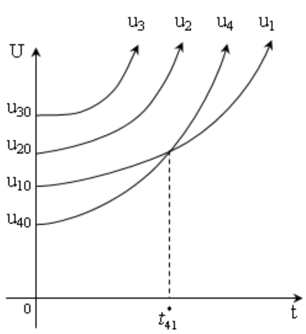


Fig - 5

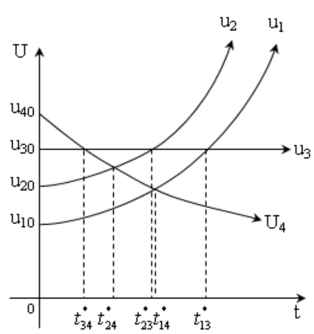


Fig - 6

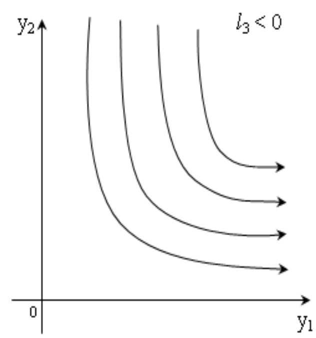


Fig – 7

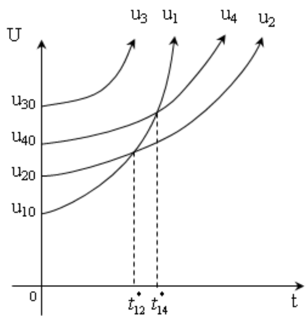


Fig - 8

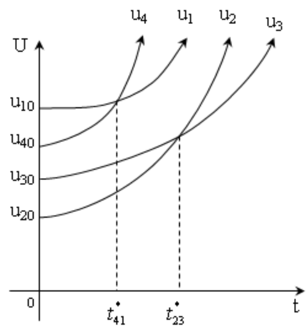


Fig - 9

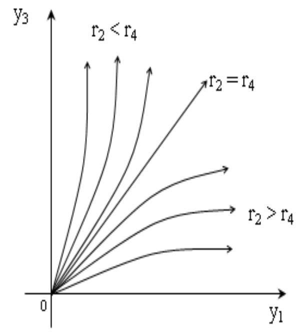


Fig – 10

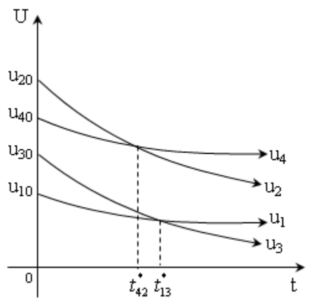


Fig – 11

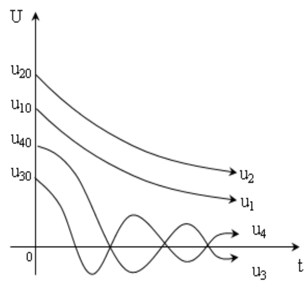


Fig – 12

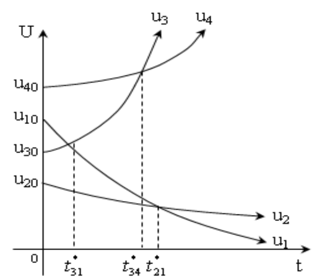


Fig – 13

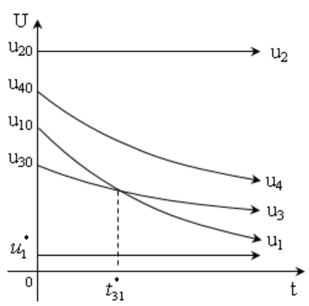


Fig – 14

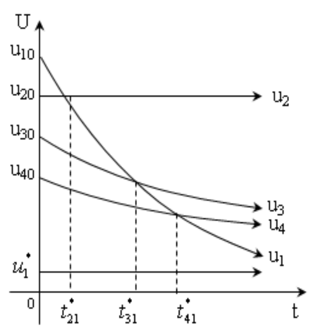


Fig - 15

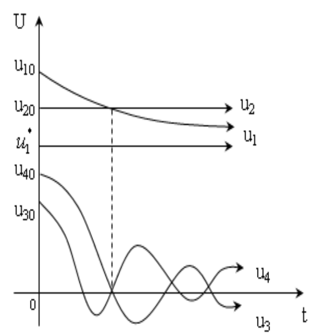
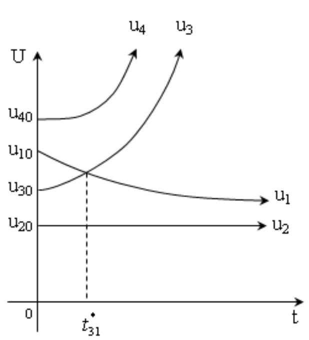
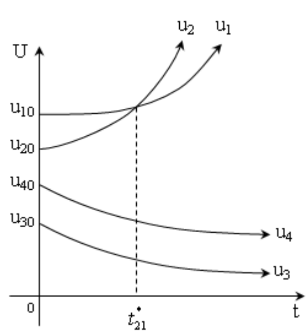


Fig – 16





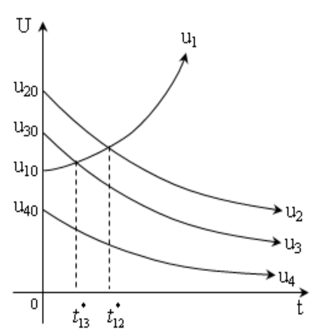


Fig - 17 Fig - 18 Fig – 19

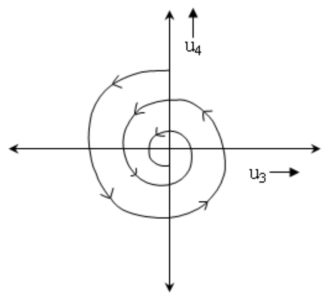


Fig – 20

7. Open Problem

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between four species (S_1, S_2, S_3, S_4) with the population relations.

S_1 a Prey to S_2 and Commensal to S_3, S_2 is a Predator living on S_1 and Commensal to S_4, S_3 a Host to S_1, S_4 a Host to S_2 and S_3 a Prey to S_4, S_4 a Predator to S_3.

The present paper deals with the study on stability of Predator (S_2) washed out states only of the above problem. The stability of the other equilibrium states is to be investigated and the perturbation curves, the trajectories of perturbations of the other equilibrium states are to be studied. The numerical solutions for the growth rate equations can also computed employing Runge Kutta fourth order method.

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