Restriction Method for Approximating Square Roots

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Abstract

The aim of this paper is to present a new method called "Restriction Method for Approximating Square Roots", that helps students to find an approximate value for any square root of positive Rational Number with an easy and simple way. Also, we prove this method and we give some examples that enhance our method.

Keywords: Full-box, Successive Full-boxes.

1 Introduction

Many methods were presented to find approximate values for square roots of positive Rational Numbers. These methods were built based on Functions, Sequences, Series or Derivatives…etc; which they are basic assumptions to other methods. Our method of finding approximate values for square roots is very simple and it can be applied easily by students in preliminary stages.

2 Restriction Method for Approximating Square Roots

In The beginning of this section, we introduce some definitions that will be used to present our method and its proof.

Definition 2.1 Let \( a \in N \), then \( a^2 \) is called a Full-box since \( a = a \ast a \).
e.g. 64 is a Full-box for 8 since \( 64 = 8 \ast 8 \).
Definition 2.2 \( a^2, b^2 \) are two Successive Full-boxes \( \iff \exists \ a, b \) such that:

\( a, b \in N \) and \( b = a + 1 \) respectively.

e.g. 9, 16 are two Successive Full-boxes for 3, 4 respectively.

Corollary 2.3 (New result): Let \( x \in R^+ \) and \( a, b \in N \) such that \( b = a + 1 \) and

\( a^2 < x < b^2 \), then:

\[
\sqrt{x} \approx \frac{4x + 4ab + 1}{4(a + b)}
\]

Proof:

Since \( a^2 < x < b^2 \Rightarrow a < \sqrt{x} < b \)

So we can assume that:

\[
\sqrt{x} \approx \frac{a + b}{2}
\]

Thus: 

\[
\begin{align*}
x & \approx \frac{(a + b)^2}{4} \\
2x & \approx \frac{(a + b)^2}{2} \\
\frac{a + b}{2} & \approx \frac{2x}{a + b}
\end{align*}
\]

So, from (1), we get:

\[
\sqrt{x} \approx \frac{2x}{a + b}
\]

And since:

\[
\begin{align*}
b - a &= 1 \\
(b - a)^2 &= 1 \\
b^2 - 2ab + a^2 &= 1 \\
b^2 + a^2 &= 2ab + 1 \\
b^2 + 2ab + a^2 &= 4ab + 1 \\
(b + a)^2 &= 4ab + 1 \\
b + a &= \frac{4ab + 1}{b + a} \\
a + b &= \frac{4ab + 1}{2(a + b)}
\end{align*}
\]

So, from (1), we get:

\[
\sqrt{x} \approx \frac{4ab + 1}{2(a + b)}
\]
Now, by applying (2), (3) on the following Rational Rule:

\[ \text{If } \frac{A}{B} = r \text{ and } \frac{C}{D} = r \text{ then } \frac{A+C}{B+D} = r, \text{ where } A, B, C, D, r \in R \]

We get:

\[ \sqrt{x} \approx \frac{4ab + 1 + 2x}{2(a + b) + a + b} \]

And by applying (2) again on the previous Rational Rule, we get:

\[ \sqrt{x} \approx \frac{4ab + 1 + 2x + 2x}{3a + 3b + a + b} \]

So:

\[ \sqrt{x} \approx \frac{4x + 4ab + 1}{4(a + b)} \]

This formula finds an approximate value for \( \sqrt{x} \), Where \( a, b \in N, x \in R^+ \) such that \( b = a + 1 \) and \( a^2 < x < b^2 \).

**Example 2.4** Use Restriction Method to find an approximate value for \( \sqrt{19} \).

Sol.:

\[ 16 < 19 < 25 \]
\[ 4 < \sqrt{19} < 5 \]
\[ \sqrt{19} \approx \frac{4x + 4ab + 1}{4(a + b)} \]
\[ \sqrt{19} \approx \frac{4(19) + 4(4)(5) + 1}{4(4 + 5)} = 4.3611 \]

(4.3611) is approximately close to the calculated value (4.3589) for \( \sqrt{19} \) rounded to four decimal places, and the absolute value of the error between the approximate value and the calculated value is (0.0022), also the square of the approximate value is (19.0192).

**Example 2.5** Use Restriction Method to find an approximate value for \( \sqrt{600} \).

Sol.:

\[ 576 < 600 < 625 \]
\[ 24 < \sqrt{600} < 25 \]
\[ \sqrt{600} \approx \frac{4x + 4ab + 1}{4(a + b)} \]
\[ \sqrt{600} \approx \frac{4(600) + 4(24)(25) + 1}{4(24 + 25)} = 24.4949 \]

(24.4949) is approximately close to the calculated value (24.4949) for \( \sqrt{600} \) rounded to four decimal places, and the absolute value of the error between the
approximate value and the calculated value is (0.0000), also the square of the approximate value is (600).

3 Using Restriction Method to approximate some square roots

The following table shows some examples that demonstrate the Restriction Method to approximate some of square roots for random numbers.

<table>
<thead>
<tr>
<th>( \sqrt{x} )</th>
<th>The approximate value using Restriction Method</th>
<th>The calculated value</th>
<th>The absolute value of the error</th>
<th>The square of the approximate value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{3} )</td>
<td>1.75</td>
<td>1.7321</td>
<td>0.0179</td>
<td>3.0625</td>
</tr>
<tr>
<td>( \sqrt{12} )</td>
<td>3.4643</td>
<td>3.4641</td>
<td>0.0002</td>
<td>12.0014</td>
</tr>
<tr>
<td>( \sqrt{33} )</td>
<td>5.75</td>
<td>5.7446</td>
<td>0.0054</td>
<td>33.0625</td>
</tr>
<tr>
<td>( \sqrt{50} )</td>
<td>7.0833</td>
<td>7.0711</td>
<td>0.0122</td>
<td>50.1731</td>
</tr>
<tr>
<td>( \sqrt{120} )</td>
<td>10.9643</td>
<td>10.9545</td>
<td>0.0098</td>
<td>120.2159</td>
</tr>
<tr>
<td>( \sqrt{133} )</td>
<td>11.5326</td>
<td>11.5326</td>
<td>0.0000</td>
<td>133</td>
</tr>
<tr>
<td>( \sqrt{175} )</td>
<td>13.2315</td>
<td>13.2288</td>
<td>0.0027</td>
<td>175.0726</td>
</tr>
<tr>
<td>( \sqrt{230} )</td>
<td>15.1694</td>
<td>15.1658</td>
<td>0.0036</td>
<td>230.1107</td>
</tr>
<tr>
<td>( \sqrt{890} )</td>
<td>29.8347</td>
<td>29.8329</td>
<td>0.0018</td>
<td>890.1093</td>
</tr>
<tr>
<td>( \sqrt{1260} )</td>
<td>35.4965</td>
<td>35.4965</td>
<td>0.0000</td>
<td>1260</td>
</tr>
</tbody>
</table>

Note 3.1 From the table above, we can notice that if \( x \) is close to \( \frac{a^2 + b^2}{2} \), then the error value will be decreased, and if \( x \) is close to \( a^2 \) or \( b^2 \), then the error value will be increased.

Fig. 1 below shows the accuracy variation for \( \sqrt{x} \) value.

\[
\frac{a^2}{2} \quad \frac{a^2 + b^2}{2} \quad b^2
\]

Fig. 1
4 Comparing Restriction Method with Others

In this section, we will compare our method with some other methods that show the results which are very close to our method results for some square roots, these methods are (Bisection Method, Newton-Raphson Method, Binomial Series, Linear Approximation and Taylor’s Polynomial) respectively.

The following table shows the efficiency of each method, it also shows the third approximation for the square root at most when we apply the Bisection Method and Newton-Raphson Method too, and we use the first three terms of a Binomial Series and Taylor’s Polynomial. Moreover we round all of these results to the fourth decimal places.

Table 2: Approximating some square roots for random chosen numbers using some methods and Restriction Method.

<table>
<thead>
<tr>
<th>$\sqrt{x}$</th>
<th>The calculated value</th>
<th>The approximate value (Bisection Method)</th>
<th>The absolute value for the error (Bisection Method)</th>
<th>The approximate value (Newton-Raphson Method)</th>
<th>The absolute value for the error (Newton-Raphson)</th>
<th>The approximate value (Binomial Series)</th>
<th>The absolute value for the error (Binomial Series)</th>
<th>The approximate value (Linear Approximation)</th>
<th>The absolute value for the error (Linear Approximation)</th>
<th>The approximate value (Taylor’s Polynomial)</th>
<th>The absolute value for the error (Taylor’s Polynomial)</th>
<th>The approximate value (Restriction Method)</th>
<th>The absolute value for the error (Restriction Method)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{8}$</td>
<td>2.8264</td>
<td>2.8264</td>
<td>0.0004</td>
<td>2.8264</td>
<td>0.0004</td>
<td>2.8264</td>
<td>0.0004</td>
<td>2.8264</td>
<td>0.0004</td>
<td>2.8264</td>
<td>0.0004</td>
<td>2.8264</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\sqrt{40}$</td>
<td>6.3246</td>
<td>6.3246</td>
<td>0.0004</td>
<td>6.3246</td>
<td>0.0004</td>
<td>6.3246</td>
<td>0.0004</td>
<td>6.3246</td>
<td>0.0004</td>
<td>6.3246</td>
<td>0.0004</td>
<td>6.3246</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\sqrt{87}$</td>
<td>9.3274</td>
<td>9.3274</td>
<td>0.0004</td>
<td>9.3274</td>
<td>0.0004</td>
<td>9.3274</td>
<td>0.0004</td>
<td>9.3274</td>
<td>0.0004</td>
<td>9.3274</td>
<td>0.0004</td>
<td>9.3274</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\sqrt{133}$</td>
<td>11.5326</td>
<td>11.5326</td>
<td>0.0004</td>
<td>11.5326</td>
<td>0.0004</td>
<td>11.5326</td>
<td>0.0004</td>
<td>11.5326</td>
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<td>11.5326</td>
<td>0.0004</td>
<td>11.5326</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\sqrt{219}$</td>
<td>14.7986</td>
<td>14.7986</td>
<td>0.0004</td>
<td>14.7986</td>
<td>0.0004</td>
<td>14.7986</td>
<td>0.0004</td>
<td>14.7986</td>
<td>0.0004</td>
<td>14.7986</td>
<td>0.0004</td>
<td>14.7986</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\sqrt{890}$</td>
<td>29.8329</td>
<td>29.8329</td>
<td>0.0004</td>
<td>29.8329</td>
<td>0.0004</td>
<td>29.8329</td>
<td>0.0004</td>
<td>29.8329</td>
<td>0.0004</td>
<td>29.8329</td>
<td>0.0004</td>
<td>29.8329</td>
<td>0.0004</td>
</tr>
<tr>
<td>$\sqrt{1100}$</td>
<td>33.1625</td>
<td>33.1625</td>
<td>0.0004</td>
<td>33.1625</td>
<td>0.0004</td>
<td>33.1625</td>
<td>0.0004</td>
<td>33.1625</td>
<td>0.0004</td>
<td>33.1625</td>
<td>0.0004</td>
<td>33.1625</td>
<td>0.0004</td>
</tr>
</tbody>
</table>
From the previous table we conclude that all of the approximate values obtained by every method are close to the calculated values.

5 Conclusion

In this paper, we present a new method called Restriction Method for Approximating Square Roots of positive Rational Numbers. The number to find its approximating square root is restricted between two Full Boxes, then we find the square roots of them, and by applying the formula below of Restriction Method we find an approximate value to the square root of entire number.

\[ \sqrt{x} \approx \frac{4x + 4ab + 1}{4(a + b)} \]

Where \( a, b \in N, \ b = a + 1 \) and \( x \in R^+ \) such that: \( a^2 < x < b^2 \).

The Restriction Method leads to very close values from the calculated values for the square roots, and it is very easy and simple to be applied by students in the preliminary stages; because it does not contain any of complicated assumptions.

6 Open Problem

Applying Restriction Method on the following function:

\[
\begin{cases} 
\sqrt[n]{x}, & x \in R^+, \ n \ is \ even \ number \\
\sqrt[n]{x}, & x \in R, \ n \ is \ odd \ number 
\end{cases}
\]

References