Int. J. Open Problems Compt. Math., Vol. 4, No. 3, September 2011
ISSN 1998-6262;
www.i-csrs.org

Restriction Method for Approximating Square Roots

Iqbal M. Batiha

Mathematics Teacher, Ministry of Education
Zip code 21156, Irbid, Jordan
e-mail: iqbalbatiha22@yahoo.com

Abstract

The aim of this paper is to present a new method called "Restriction Method for Approximating Square Roots", that helps students to find an approximate value for any square root of positive Rational Number with an easy and simple way. Also, we prove this method and we give some examples that enhance our method.

Keywords: *Full-box, Successive Full-boxes.*

1 Introduction

Many methods were presented to find approximate values for square roots of positive Rational Numbers. These methods were built based on Functions, Sequences, Series or Derivatives…etc; which they are basic assumptions to other methods. Our method of finding approximate values for square roots is very simple and it can be applied easily by students in preliminary stages.

2 Restriction Method for Approximating Square Roots

In The beginning of this section, we introduce some definitions that will be used to present our method and its proof.

Definition 2.1 *Let* $a \in N$ *, then* a^2 *is called a Full-box since* $a \;\; = a * a$.
e.g. 64 is a Full-box for 8 since $64 = 8 * 8$.

Definition 2.2 a^2, b^2 *are two Successive Full-boxes* $\Leftrightarrow \exists\ a,b$ *such that:* $a,b \in N$ *and* $b = a+1$ *respectively.*
e.g. $9, 16$ are two Successive Full-boxes for $3, 4$ respectively.

Corollary 2.3 (New result): *Let* $x \in R^+$ and $a,b \in N$ such that $b = a+1$ and $a^2 < x < b^2$, then:

$$\sqrt{x} \approx \frac{4x + 4ab + 1}{4(a+b)}$$

Proof:
Since $a^2 < x < b^2 \Rightarrow a < \sqrt{x} < b$
So we can assume that:

$$\sqrt{x} \approx \frac{a+b}{2} \qquad (1)$$

is the first approximation for $\sqrt{x}$.
Thus;

$$x \approx \frac{(a+b)^2}{4}$$

$$2x \approx \frac{(a+b)^2}{2}$$

$$\frac{a+b}{2} \approx \frac{2x}{a+b}$$

So, from (1), we get:

$$\sqrt{x} \approx \frac{2x}{a+b} \qquad (2)$$

And since:

$$b - a = 1$$

$$(b-a)^2 = 1$$

$$b^2 - 2ab + a^2 = 1$$

$$b^2 + a^2 = 2ab + 1$$

$$b^2 + 2ab + a^2 = 4ab + 1$$

$$(b+a)^2 = 4ab + 1$$

$$b + a = \frac{4ab+1}{b+a}$$

$$\frac{a+b}{2} = \frac{4ab+1}{2(a+b)}$$

So, from (1), we get:

$$\sqrt{x} \approx \frac{4ab+1}{2(a+b)} \qquad (3)$$

Now, by applying (2), (3) on the following Rational Rule:

$$\left(\textit{If } \frac{A}{B} = r \textit{ and } \frac{C}{D} = r \textit{ then } \frac{A+C}{B+D} = r, \textit{ where } A, B, C, D, r \in R \right)$$

We get:

$$\sqrt{x} \approx \frac{4ab+1+2x}{2(a+b)+a+b}$$

And by applying (2) again on the previous Rational Rule, we get:

$$\sqrt{x} \approx \frac{4ab+1+2x+2x}{3a+3b+a+b}$$

So;

$$\sqrt{x} \approx \frac{4x+4ab+1}{4(a+b)}$$

This formula finds an approximate value for $\sqrt{x}$, Where $a, b \in N$, $x \in R^{+}$ such that $b = a+1$ and $a^2 < x < b^2$.

Example 2.4 Use Restriction Method to find an approximate value for $\sqrt{19}$.

Sol.:

$16 < 19 < 25$

$4 < \sqrt{19} < 5$

$$\sqrt{x} \approx \frac{4x+4ab+1}{4(a+b)}$$

$$\sqrt{19} \approx \frac{4(19)+4(4)(5)+1}{4(4+5)} = 4.3611$$

(4.3611) is approximately close to the calculated value (4.3589) for $\sqrt{19}$ rounded to four decimal places, and the absolute value of the error between the approximate value and the calculated value is (0.0022), also the square of the approximate value is (19.0192).

Example 2.5 Use Restriction Method to find an approximate value for $\sqrt{600}$.

Sol.:

$576 < 600 < 625$

$24 < \sqrt{600} < 25$

$$\sqrt{x} \approx \frac{4x+4ab+1}{4(a+b)}$$

$$\sqrt{600} \approx \frac{4(600)+4(24)(25)+1}{4(24+25)} = 24.4949$$

(24.4949) is approximately close to the calculated value (24.4949) for $\sqrt{600}$ rounded to four decimal places, and the absolute value of the error between the approximate value and the calculated value is (0.0000), also the square of the approximate value is (600).

3 Using Restriction Method to approximate some square roots

The following table shows some examples that demonstrate the Restriction Method to approximate some of square roots for random numbers.

Table 1: Approximating some square roots for random chosen numbers using Restriction Method.

$\sqrt{x}$	The approximate value using Restriction Method	The calculated value	The absolute value of the error	The square of the approximate value
$\sqrt{3}$	1.75	1.7321	0.0179	3.0625
$\sqrt{12}$	3.4643	3.4641	0.0002	12.0014
$\sqrt{33}$	5.75	5.7446	0.0054	33.0625
$\sqrt{50}$	7.0833	7.0711	0.0122	50.1731
$\sqrt{120}$	10.9643	10.9545	0.0098	120.2159
$\sqrt{133}$	11.5326	11.5326	0.0000	133
$\sqrt{175}$	13.2315	13.2288	0.0027	175.0726
$\sqrt{230}$	15.1694	15.1658	0.0036	230.1107
$\sqrt{890}$	29.8347	29.8329	0.0018	890.1093
$\sqrt{1260}$	35.4965	35.4965	0.0000	1260

Note 3.1 From the table above, we can notice that if x is close to $\frac{a^2+b^2}{2}$, then the error value will be decreased, and if x is close to a^2 or b^2, then the error value will be increased.

Fig. 1 below shows the accuracy variation for $\sqrt{x}$ value.

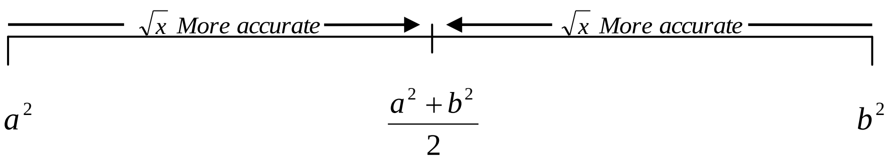

Fig. 1

4 Comparing Restriction Method with Others

In this section, we will compare our method with some other methods that show the results which are very close to our method results for some square roots, these methods are (Bisection Method, Newton-Raphson Method, Binomial Series, Linear Approximation and Taylor's Polynomial) respectively.

The following table shows the efficiency of each method, it also shows the third approximation for the square root at most when we apply the Bisection Method and Newton-Raphson Method too, and we use the first three terms of a Binomial Series and Taylor's Polynomial. Moreover we round all of these results to the fourth decimal places.

Table 2: Approximating some square roots for random chosen numbers using some methods and Restriction Method.

$\sqrt{x}$	The calculated value	The approximate value (Bisection Method)	The absolute value for the error (Bisection Method)	The approximate value (Newton-Raphson Method)	The absolute value for the error (Newton-Raphson)	The approximate value (Binomial Series)	The absolute value for the error (Binomial Series)	The approximate value (Linear Approximation)	The absolute value for the error (Linear Approximation)	The approximate value (Taylor's Polynomial)	The absolute value for the error (Taylor's Polynomial)	The approximate value (Restriction Method)	The absolute value for the error (Restriction Method)
$\sqrt{8}$	2.8284	2.875	0.0466	2.8284	0.0	2.8287	0.0003	2.8333	0.0049	2.8241	0.0043	2.85	0.0216
$\sqrt{40}$	6.3246	6.375	0.0504	6.3246	0.0	6.3241	0.0005	6.3333	0.0087	6.3148	0.0098	6.3269	0.0023
$\sqrt{87}$	9.3274	9.375	0.0476	9.3274	0.0	9.3272	0.0002	9.3333	0.0059	9.3209	0.0065	9.3289	0.0015
$\sqrt{133}$	11.5326	11.5	0.0326	11.5326	0.0	11.5329	0.0003	11.5417	0.0091	11.5242	0.0084	11.5326	0.0
$\sqrt{219}$	14.7986	14.75	0.0486	14.7986	0.0	14.7987	0.0001	14.8	0.0014	14.7973	0.0013	14.8017	0.0031
$\sqrt{890}$	29.8329	29.875	0.0421	29.8329	0.0	29.8329	0.0	29.8333	0.0004	29.8324	0.0005	29.8347	0.0018
$\sqrt{1100}$	33.1662	33.125	0.0412	33.1662	0.0	33.1662	0.0	33.1667	0.0005	33.1658	0.0004	33.1679	0.0017

From the previous table we conclude that all of the approximate values obtained by every method are close to the calculated values.

5 Conclusion

In this paper, we present a new method called *Restriction Method for Approximating Square Roots* of positive Rational Numbers. The number to find its approximating square root is restricted between two Full Boxes, then we find the square roots of them, and by applying the formula below of Restriction Method we find an approximate value to the square root of entire number.

$$\sqrt{x} \approx \frac{4x+4ab+1}{4(a+b)}$$

Where $a,b \in N$, $b = a+1$ and $x \in R^{+}$ such that: $a^2 < x < b^2$.

The Restriction Method leads to very close values from the calculated values for the square roots, and it is very easy and simple to be applied by students in the preliminary stages; because it does not contain any of complicated assumptions.

6 Open Problem

Applying Restriction Method on the following function:

$$f(x) = \begin{cases} \sqrt[n]{x}, \quad x \in R^{+}, \;\; n \;\; is \;\; even \;\; number \\ \\ \sqrt[n]{x}, \quad x \in R, \;\; n \;\; is \;\; odd \;\; number \end{cases} \quad , \;\; such \;\; that \;\; n > 2 .$$

References

[1] Howard Anton, *CALCULUS*, Anton Textbooks, Inc., (1995).

[2] John Todd, *Basic Numerical Mathematics*, ACADEMIC Press, (1980).

[3] Nashat I. Al-Obaidi, *Numerical Analysis*, Massira Press, (2011).

[4] Richard L. Burden and J. Douglas Faires, *Numerical Analysis*, PWS Publishers, (1985).

[5] Salas and Hille's, *CALCULUS*, John Wiley & Sons, Inc., (1995).

[6] Yaseen A. Al-Shbool, *Numerical Techniques*, Arabic community library, (2004).