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Inelastic Admissible Curves in the Pseudo – Galilean Space G_3^1

Alper Osman Öğrenmiş, Mustafa Yeneroğlu and Mihriban Külahcı

Fırat University, Faculty of Science, Department of Mathematics
23119 Elazığ / TURKEY
e-mail: ogrenmisalper@gmail.com, myeneroglu23@yahoo.com,
mihribankulahci@gmail.com

Abstract

In this manuscript, we define inelastic flow of curves in Pseudo - Galilean space $G_3^{\ 1}$. Some conditions are given for an inelastic curve flow as a partial differential equation involving the curvature and torsion.

Keywords: *Pseudo - Galilean Space, Inelastic Curve.*

1 Introduction

The terms of elastic and inelastic mostly come up in physics. There are elastic and inelastic collisions in physics. In elastic collision, both the kinetic energy and momentum are conserved. In inelastic collision, the kinetic energy is not conserved in the collision. However the momentum is conserved.

Curves are a natural shape that many users often wish to use in many different areas such as mathematicians, physicists and engineers.

Recently, the study of the motion of inelastic curves has arisen in a number of diverse engineering applications.

The flow of a curve is said to be inelastic if the arc-length is preserved. Physically, inelastic curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications. In [4,5] Kwon et all. study inelastic flows of curves and developable surface in R^3. Moreover in [6] Latifi et all study inextensible flows of curves in Minkowski 3-space.

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. [10]

Recently, some characterizations of curves are given in different spaces. Many interesting results in Non-Euclidean spaces have been obtained by many mathematicians. This subject have been studied by many researcher [9,11].

Differential geometry of the Pseudo - Galilean space G_3^1 has been largely developed in [1,2,3,7,8]

In this paper, we derive inelastic flows of curves in Pseudo-Galilean space G_3^1. Some conditions for an inelastic curve flow are expressed as a partial differential equation involving the curvature and torsion. We use some idea from [4,5,6] in this paper.

2 Inelastic Admissible Curve Flows in Pseudo - Galilean Space

Let r be a spatial curve given first by

$$r(t) = (x(t), y(t), z(t)) \tag{1}$$

where $x(t), y(t), z(t) \in C^3$ (the set of three-times continuously differentiable functions) and t run through a real interval [2].

Definition 2.1 *A curve r given by (1) is admissible if*

$$\dot{x}(t) \neq 0. \tag{2}$$

Then the curve r can be given by

$$r(x) = (x, y(x), z(x)) \tag{3}$$

and we assume in addition that, in [2]

$$y''^2(x) - z''^2(x) \neq 0. \tag{4}$$

Definition 2.2 *For an admissible curve given by (1) the parameter of arc-length is defined by*

$$ds = |\dot{x}(t)dt| = |dx|. \tag{5}$$

For simplicity we assume $dx = ds$ and $x = s$ as the arc-length of the curve r. From now on, we will denote the derivation by s by upper prime $'$ [2].
The vector $\mathbf{t}(s) = r'(s)$ is called the tangential unit vector of an admissible curve r in a point P(s). Further, we define the so called osculating plane of r spanned by the vectors $r'(s)$ and $r''(s)$ in the same point. The absolute point of the osculating plane is

$$\mathrm{H}(0:0:\mathrm{y}''(\mathrm{s}):\mathrm{z}''(\mathrm{s})) \tag{6}$$

We have assumed in (4) that H is not light-like. H is a point at infinity of a line which direction vector is $r''(\mathrm{s})$. Then the unit vector

$$\mathbf{n}(\mathrm{s}) = \frac{r''(\mathrm{s})}{\sqrt{|\mathrm{y}''^2(s) - \mathrm{z}''^2(s)|}} \tag{7}$$

so called the principal normal vector of the curve r in the point P.
Now the vector

$$\mathbf{b}(s) = \frac{(0, \varepsilon z''(s), \varepsilon y''(s))}{\sqrt{|y''^2(s) - z''^2(s)|}} \tag{8}$$

is orthogonal in pseudo-Galilean sense to the osculating plane and we call it the binormal vector of the given curve in the point P. Here $\varepsilon = +1$ or -1 is chosen by the criterion $det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$. That means

$$\sqrt{|y''^2(s) - z''^2(s)|} = \varepsilon(y''^2(s) - z''^2(s)) \tag{9}$$

By the above construction the following can be summarized [2].

Definition 2.3 *In each point of an admissible curve in G_3^1 the associated orthonormal (in pseudo-Galilean sense) trihedron* $\{\mathbf{t}(\mathrm{s}), \mathbf{n}(\mathrm{s}), \mathbf{b}(\mathrm{s})\}$ *can be defined. This trihedron is called pseudo-Galilean Frenet trihedron [2].*

If a curve is parameterized by the arc-length i.e. given by (3), then the tangent vector is non-isotropic and has the form of

$$\mathbf{t}(s) = r'(s) = (1, y'(s), z'(s)). \tag{10}$$

Now we have

$$\mathbf{t}'(s) = r''(s) = (0, y''(s), z''(s)). \tag{11}$$

According to the classical analogy we write (7) in the form

$$r''(s) = \kappa(s)\mathbf{n}(s) \tag{12}$$

and so the curvature of an admissible curve r can be defined as follows

$$\kappa(s) = \sqrt{|y''^2(s) - z''^2(s)|}. \tag{13}$$

Remark 2.1 *In [2], for the pseudo-Galilean Frenet trihedron of an admissible curve r given by (3) the following derivative Frenet formulas are true.*

$$\begin{aligned} \mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) &= \tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) &= \tau(s)\mathbf{n}(s) \end{aligned} \tag{14}$$

*where **t**(s) is a space-like, **n**(s) is a space-like and **b**(s) is a time-like vector, κ(s) is the pseudo-Galilean curvature given by (13) and τ(s) is the pseudo-Galilean torsion of r defined by*

$$\tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}. \tag{15}$$

The formula (15) can be written as

$$\tau(s) = \frac{det(r'(s), r''(s), r'''(s))}{\kappa^2(s)} \tag{16}$$

Throughout this paper, we assume that $F\colon [0,l] \times [0,t_\infty] \to G_3{}^1$ is a one parameter family of smooth curves in Pseudo - Galilean space $G_3{}^1$, where l is the arc-length of the initial curve. Let u be the curve parameterization variable, $0 \le u \le l$.
The arc-length of F is given by

$$s(u) = \int_0^u \left|\frac{\partial F}{\partial u}\right| du$$

where $|\frac{\partial F}{\partial u}| = |\langle\frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\rangle|^{1/2}$.
Defining $v = |\frac{\partial F}{\partial u}|$, the operator $\frac{\partial}{\partial u}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

while the arc-length parameter is $ds = v.du$. See [4], for a review of curve theory. Any flow of F can be represented as

$$\frac{\partial F}{\partial t} = f\mathbf{t} + g\mathbf{n} + h\mathbf{b} \tag{17}$$

Letting the arc-length variation be in the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by condition $\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}du=0$ for all $u\in[0,l]$.

Definition 2.4 *A curve evolution $F(u,t)$ and its flow $\frac{\partial F}{\partial t}$ in Pseudo - Galilean space G_3^1 are said to be inelastic if* $\frac{\partial}{\partial t}\left|\frac{\partial F}{\partial u}\right|=0$.

Some conditions for inelastic flow in Pseudo - Galilean space G_3^1 are given by the following theorem.

Theorem 2.1 *Let $\frac{\partial F}{\partial t}=f\mathbf{t}+g\mathbf{n}+h\mathbf{b}$ be a smooth flow of the curve F in Pseudo - Galilean space $G_3{}^1$. Let the flow is inelastic then f is constant.*

Proof. *According to definition of F, we have $v^2=\langle\frac{\partial F}{\partial u},\frac{\partial F}{\partial u}\rangle$. $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since u and t are independent coordinates. So we have*

$$\begin{aligned}2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\langle\frac{\partial F}{\partial u},\frac{\partial F}{\partial u}\rangle\\ &= 2\langle\frac{\partial F}{\partial u},\frac{\partial}{\partial u}(\frac{\partial F}{\partial t})\rangle\\ &= 2\langle\frac{\partial F}{\partial u},\frac{\partial}{\partial u}(f\boldsymbol{t}+g\boldsymbol{n}+h\boldsymbol{b})\rangle\\ &= 2v\langle\boldsymbol{t},\frac{\partial f}{\partial u}\boldsymbol{t}+fv\kappa\boldsymbol{n}+\frac{\partial g}{\partial u}\boldsymbol{n}+gv\tau\boldsymbol{b}\\ &\quad+\frac{\partial h}{\partial u}\boldsymbol{b}+hv\tau\boldsymbol{n}\rangle\\ &= 2v\frac{\partial f}{\partial u}.\end{aligned}$$

Thus we get

$$\frac{\partial v}{\partial t}=\frac{\partial f}{\partial u}$$
$$\frac{\partial f}{\partial s}=0$$
$$f=const.$$

We now restrict ourselves to arc-length parameterized curves. That is, $v=1$, and the local coordinate u corresponds to the curve arc-length s. We require the following lemma.

Lemma 2.1

$$\frac{\partial \mathbf{t}}{\partial t} = \left(f\kappa + \frac{\partial g}{\partial s} + h\tau\right)\mathbf{n} + \left(g\tau + \frac{\partial h}{\partial s}\right)\mathbf{b}$$

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(f\kappa + \frac{\partial g}{\partial s} + h\tau\right)\mathbf{t} + \psi\mathbf{b},$$

$$\frac{\partial \mathbf{b}}{\partial t} = -\left(g\tau + \frac{\partial h}{\partial s}\right)\mathbf{t} - \psi\mathbf{n}$$

where $\psi = \langle \frac{\partial \mathbf{n}}{\partial t}, \mathbf{b}\rangle$.

Proof. *Using Eq. (14) and Theorem 2.1. , we calculate*

$$\begin{aligned}\frac{\partial \mathbf{t}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial s}\right) = \frac{\partial}{\partial s}(f\mathbf{t} + g\mathbf{n} + h\mathbf{b}) \\ &= \frac{\partial f}{\partial s}\mathbf{t} + f\kappa\mathbf{n} + \frac{\partial g}{\partial s}\mathbf{n} + g\tau\mathbf{b} + \frac{\partial h}{\partial s}\mathbf{b} + h\tau\mathbf{n} \\ &= \left(f\kappa + \frac{\partial g}{\partial s} + h\tau\right)\mathbf{n} + \left(g\tau + \frac{\partial h}{\partial s}\right)\mathbf{b}.\end{aligned}$$

Now differentiate the Frenet frame by t*:*

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\langle \mathbf{t}, \mathbf{n}\rangle = \langle \frac{\partial \mathbf{t}}{\partial t}, \mathbf{n}\rangle + \langle \mathbf{t}, \frac{\partial \mathbf{n}}{\partial t}\rangle \\ &= \left(f\kappa + \frac{\partial g}{\partial s} + h\tau\right) + \langle \mathbf{t}, \frac{\partial \mathbf{n}}{\partial t}\rangle \\ 0 &= \frac{\partial}{\partial t}\langle \mathbf{t}, \mathbf{b}\rangle = \langle \frac{\partial \mathbf{t}}{\partial t}, \mathbf{b}\rangle + \langle \mathbf{t}, \frac{\partial \mathbf{b}}{\partial t}\rangle \\ &= \left(g\tau + \frac{\partial h}{\partial s}\right) + \langle \mathbf{t}, \frac{\partial \mathbf{b}}{\partial t}\rangle \\ 0 &= \frac{\partial}{\partial t}\langle \mathbf{n}, \mathbf{b}\rangle = \langle \frac{\partial \mathbf{n}}{\partial t}, \mathbf{b}\rangle + \langle \mathbf{n}, \frac{\partial \mathbf{b}}{\partial t}\rangle \\ &= \psi + \langle \mathbf{n}, \frac{\partial \mathbf{b}}{\partial t}\rangle.\end{aligned}$$

From the above we obtain

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(f\kappa + \frac{\partial g}{\partial s} + h\tau\right)\mathbf{t} + \psi\mathbf{b}$$

and

$$\frac{\partial \mathbf{b}}{\partial t} = -\left(g\tau + \frac{\partial h}{\partial s}\right)\mathbf{t} - \psi\mathbf{n},$$

since $\langle \frac{\partial \mathbf{n}}{\partial t}, \mathbf{n}\rangle = \langle \frac{\partial \mathbf{b}}{\partial t}, \mathbf{b}\rangle = 0$.

Theorem 2.2 *Suppose the curve flow* $\frac{\partial F}{\partial t} = f\mathbf{t} + g\mathbf{n} + h\mathbf{b}$ *is inelastic. Then the following system of partial differential equations holds:*

$$\begin{aligned}
\frac{\partial \kappa}{\partial t} &= \frac{\partial}{\partial s}(f\kappa) + \frac{\partial^2 g}{\partial s^2} + \frac{\partial}{\partial s}(h\tau) + g\tau^2 + \tau\frac{\partial h}{\partial s},\\
\frac{\partial \tau}{\partial t} &= \kappa(g\tau + \frac{\partial h}{\partial s}) + \frac{\partial \psi}{\partial s},\\
\kappa\psi &= \tau(f\kappa + \frac{\partial g}{\partial s} + h\tau) + \frac{\partial}{\partial s}(g\tau) + \frac{\partial^2 h}{\partial s^2}.
\end{aligned}$$

Proof. *Noting that* $\frac{\partial}{\partial s}\frac{\partial \mathbf{t}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{t}}{\partial s}$,

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{t}}{\partial t} &= \frac{\partial}{\partial s}[(f\kappa + \frac{\partial g}{\partial s} + h\tau)\mathbf{n} + (g\tau + \frac{\partial h}{\partial s})\mathbf{b}]\\
&= [\frac{\partial}{\partial s}(f\kappa) + \frac{\partial^2 g}{\partial s^2} + \frac{\partial}{\partial s}(h\tau)]\mathbf{n} + (f\kappa + \frac{\partial g}{\partial s} + h\tau)(\tau\mathbf{b})\\
&\quad + [\frac{\partial}{\partial s}(g\tau) + (\frac{\partial^2 h}{\partial s^2}]\mathbf{b} + (g\tau + \frac{\partial h}{\partial s})(\tau\mathbf{n}),
\end{aligned}$$

while

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \mathbf{t}}{\partial s} &= \frac{\partial}{\partial t}(\kappa\mathbf{n})\\
&= \mathbf{n}(\frac{\partial \kappa}{\partial t}) + \kappa\frac{\partial \mathbf{n}}{\partial t}\\
&= \mathbf{n}\frac{\partial \kappa}{\partial t} + \kappa[-(f\kappa + \frac{\partial g}{\partial s} + h\tau)\mathbf{t} + \psi\mathbf{b}].
\end{aligned}$$

Hence we see that

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial s}(f\kappa) + \frac{\partial^2 g}{\partial s^2} + \frac{\partial}{\partial s}(h\tau) + g\tau^2 + \tau\frac{\partial h}{\partial s},$$

and

$$\kappa\psi = \tau(f\kappa + \frac{\partial g}{\partial s} + h\tau) + \frac{\partial}{\partial s}(g\tau) + \frac{\partial^2 h}{\partial s^2}.$$

Since $\frac{\partial}{\partial s}\frac{\partial \mathbf{b}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{b}}{\partial s}$, *we have*

$$\frac{\partial}{\partial s}\frac{\partial \mathbf{b}}{\partial t} = \frac{\partial}{\partial s}[-(g\tau+\frac{\partial h}{\partial s})\mathbf{t}-\psi\mathbf{n}]$$
$$= -[\frac{\partial^2 h}{\partial s^2}+\frac{\partial}{\partial s}(g\tau)]\mathbf{t}-(g\tau+\frac{\partial h}{\partial s})\kappa\mathbf{n}-\frac{\partial\psi}{\partial s}\mathbf{n}-\psi\tau\mathbf{b}$$

while

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{b}}{\partial s} = \frac{\partial}{\partial t}(\tau\mathbf{n})$$
$$= [\frac{\partial\tau}{\partial t}\mathbf{n}+\tau[-(f\kappa+\frac{\partial g}{\partial s}+h\tau)\mathbf{t}+\psi\mathbf{b}]].$$

Thus we get

$$\frac{\partial\tau}{\partial t} = \kappa(g\tau+\frac{\partial h}{\partial s})+\frac{\partial\psi}{\partial s}.$$

3 Open Problem

The aim of this study is to derive and study the inelastic flows of curves some spaces. We propose to study various conditions for an inelastic curve flow are expressed as a partial diferential equation involving the curvature and torsion. In this manuscript, we define inelastic flow of curves in Pseudo -Galilean space G_3^1. But we didn't give conditions in some isotropic space. Therefore it is left as an open problem.

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