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On Existence of Solutions to Some Fractional Nonlinear Integrodifferential Equations

Mohammed M. Matar

Alazhar University-Gaza, Palestine
e-mail: mohammed_mattar@hotmail.com

Abstract

In this paper, the existence and uniqueness problems of the solution of a generalized Cauchy type fractional nonlocal integrodifferential equation are investigated. The results are obtained using Banach and Krasnoselkii fixed point theorems.

Keywords: *Fractional calculus, existence and uniqueness; integrodifferential equations; Banach and Krasnoselkii fixed point theorems; nonlocal condition.*

1 Introduction

Fractional differential equations are emerged as a new branch of applied mathematics by which many physical and engineering approaches can be modelled. The fact that fractional differential equations are considered as alternative models to nonlinear differential equations which induced extensive researches in various fields including the theoretical part (see [1]-[12] and references therein). The existence and uniqueness problems of fractional nonlinear differential and integrodifferential equations as a basic theoretical part are investigated by many authors (see [1],[2],[3],[5],[6],[7],[8], [9],[11]). In [5], the authors considered the Cauchy type problem for nonlinear fractional differential equations in weighted spaces of continuous functions. The Cauchy problems for some fractional abstract differential equations with nonlocal conditions are investigated by the authors in [1] and [11] using the Banach and Krasnoselkii fixed point theorems. The Banach fixed point theorem is used in [8], and [9] to investigate the existence problem of fractional integrodifferential equations in Banach space. Motivated by these works we study in this paper the existence of solution of generalized Cauchy problem for fractional nonlocal integrodifferential equations in Banach spaces by using fractional calculus and fixed point theorems.

2 Preliminaries

We need some basic definitions and properties of fractional calculus which will be used in this paper.

Definition 2.1 *A real function $f(t)$ is said to be in the space C_μ, $\mu \in \mathbf{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0,\infty)$, and it is said to be in the space C_μ^n iff $f^{(n)} \in C_\mu, n \in \mathbf{N}$.*

Definition 2.2 *A function $f \in C_\mu$, $\mu \geq -1$ is said to be fractional integrable of order $\alpha \geq 0$ if*

$$I^\alpha f(t) = (I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds < \infty,$$

where $\alpha > 0$, and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Next, we introduce the Caputo fractional derivative.

Definition 2.3 *The fractional derivative in the Caputo sense is defined as*

$$\begin{aligned} D^{(\alpha)} f(t) &= \left(D^{(\alpha)} f\right)(t) = I^{n-\alpha}\left(\frac{d^n f}{dt^n}\right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\frac{d^n f(s)}{ds^n}\right) ds \end{aligned}$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$. In particular, if $0 < \alpha \leq 1$, then $D^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds$, where $f'(s) = \frac{df(s)}{ds}$.

The identity $\left(I^\alpha D^{(\alpha)} f\right)(t) = f(t) - f(0)$ and other properties of the fractional operators used in the general theory of fractional differential equations can be found in [4], [10], and [12].

Let $Y = C(J, \mathbf{R})$ be a Banach space of all real-valued continuous functions $x(t)$ on a compact interval $J = [0,T]$. The product space $Y^\otimes = Y \times Y \times ... \times Y$ ($2k+1$ times) is a Banach space endowed with the norm $\|z\| = \sum_{j=1}^{2k+1} \|y_j\|$ for any $z = (y_1, y_2, \cdots, y_{2k+1}) \in Y^\otimes$.

Consider the fractional nonlinear integrodifferential equation

$$\begin{cases} D^{(\alpha)}x(t) \\ = f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)), \\ x(0) = x_0 \in \mathbf{R}, \end{cases} \tag{1}$$

where $0 < \alpha - \alpha_j < \alpha + \alpha_j < 1; 1 \leq j \leq k$; $k \in \mathbb{N}$, $x \in Y$, and $f : J \times Y^{\otimes} \to Y$ is a function satisfies the following condition:

(Hf) $f : J \times Y^{\otimes} \to Y$ is jointly continuous and there exists a positive constant L such that

$$\|f(t, x_1, x_2, x_3, ..., x_{2k+1}) - f(t, y_1, y_2, y_3, ..., y_{2k+1})\| \leq L \sum_{j=1}^{2k+1} \|x_j - y_j\|$$

for any $t \in J$, $x_1, x_2, x_3, ..., x_{2k+1}$, $y_1, y_2, y_3, ..., y_{2k+1} \in Y$. Moreover, let $K = \sup_{t \in J} \left\| f(t, \overbrace{0, 0, ..., 0}^{2k+1}) \right\|$.

Eq.(1) is equivalent to the integral equation (see [6] for more details)

$$\begin{aligned} x(t) &= x_0 + I^{\alpha}(f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), \\ &\qquad I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t))) \qquad (2) \\ &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \\ &\qquad f(s, x(s), D^{(\alpha_1)}x(s), D^{(\alpha_2)}x(s), ..., D^{(\alpha_k)}x(s), I^{\alpha_1}x(s), I^{\alpha_2}x(s), \\ &\qquad ..., I^{\alpha_k}x(s))ds. \end{aligned}$$

3 Existence problems

We prove the existence of the fractional nonlinear integrodifferential equation (1) by using the well-known Banach fixed point theorem. The following condition is essential to get the contraction property.

(Hq) Let $0 < q < 1$, and r be a positive finite real number such that

$$\begin{cases} q \geq L\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \sum_{j=1}^{k}\left(\frac{T^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)} + \frac{T^{\alpha+\alpha j}}{\Gamma(\alpha+\alpha j+1)}\right)\right) \\ r \geq (1-q)^{-1}\left(\|x_0\|\left(1 + \sum_{j=1}^{k} \frac{T^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right) + \frac{KT^{\alpha}}{\Gamma(\alpha+1)}\right). \end{cases}$$

Moreover, let $B_r = \{y \in Y : \|y\| \leq r\}$.

Theorem 3.1 *If the hypotheses (Hf)-(Hq) are satisfied, then the fractional integrodifferential equation (1) has a unique solution on J.*
Proof. *We prove, by using the Banach fixed point, the operator $\Psi : Y \to Y$ given by*

$$\Psi x(t) = x_0 + I^{\alpha}(f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), \\ I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)))$$

has a fixed point on B_r. This fixed point is then a solution of Eq.(1). Firstly, we show that $\Psi B_r \subset B_r$, Let $x \in B_r$ then

$$\begin{aligned}
&\|\Psi x(t)\| \leq \|x_0\| \\
&+ I^{\alpha}\left(\left\|f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t))\right\|\right) \\
&= \|x_0\| \\
&+ I^{\alpha}(\| f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)) \\
&-f(t, \overbrace{0, 0, ..., 0}^{2k+1}) + f(t, \overbrace{0, 0, ..., 0}^{2k+1}) \|) \\
&\leq \|x_0\| + LI^{\alpha}\left(\|x(t)\| + \sum_{j=1}^{k}\left(\left\|D^{(\alpha_j)}x(t)\right\| + \|I^{\alpha_j}x(t)\|\right)\right) \\
&+ I^{\alpha}\left(\left\|f(t, \overbrace{0, 0, ..., 0}^{2k+1})\right\|\right)
\end{aligned}$$

by the semigroup property of the fractional operators (see [4]; Theorem 3.7), we have

$$\begin{aligned}
&\|\Psi x(t)\| \leq \|x_0\| \\
+ \quad & L\left(I^{\alpha}\|x(t)\| + \sum_{j=1}^{k}\left(I^{\alpha-\alpha_j}\left\|I^{\alpha_j}D^{(\alpha_j)}x(t)\right\| + I^{\alpha+\alpha_j}\|x(t)\|\right)\right) \\
&+I^{\alpha}\left(\left\|f(t, \overbrace{0, 0, ..., 0}^{2k+1})\right\|\right) \\
= \quad & \|x_0\| + L\left(I^{\alpha}\|x(t)\| + \sum_{j=1}^{k}\left(I^{\alpha-\alpha_j}\|x(t) - x_0\| + I^{\alpha+\alpha_j}\|x(t)\|\right)\right) \\
&+I^{\alpha}\left(\left\|f(t, \overbrace{0, 0, ..., 0}^{2k+1})\right\|\right),
\end{aligned}$$

hence

$$\begin{aligned}
&\|\Psi x(t)\| \\
&\leq \|x_0\|\left(1+\sum_{j=1}^{k}\frac{t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right)+\frac{Kt^{\alpha}}{\Gamma(\alpha+1)} \\
&+L\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\left(\frac{t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}+\frac{t^{\alpha+\alpha j}}{\Gamma(\alpha+\alpha j+1)}\right)\right)\|x\|
\end{aligned}$$

since $x \in B_r$, *we get*

$$\|\Psi x(t)\| \leq \|x_0\|\left(1+\sum_{j=1}^{k}\frac{t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right)+\frac{Kt^{\alpha}}{\Gamma(\alpha+1)}+qr \leq (1-q)\,r+qr=r.$$

Hence, the operator Ψ *maps* B_r *into itself. Next, we prove that* Ψ *is a contraction mapping on* B_r*. Let* $x, y \in B_r$*. Since* $x(0)=y(0)$*, then*

$$\begin{aligned}
&\quad \|\Psi x(t)-\Psi y(t)\| \\
&= \left\| I^{\alpha} f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)) \right. \\
&- \left. I^{\alpha} f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)) \right\| \\
&\leq LI^{\alpha}\left[\|x(t)-y(t)\|+\sum_{j=1}^{k}\left(\left\|D^{(\alpha_j)}(x(t)-y(t))\right\|+\|I^{\alpha_j}(x(t)-y(t))\|\right)\right] \\
&\leq L\left[I^{\alpha}\|x(t)-y(t)\|+\sum_{j=1}^{k}\left(I^{\alpha-\alpha_j}\|(x(t)-y(t))\|+I^{\alpha+\alpha_j}\|x(t)-y(t)\|\right)\right] \\
&\leq L\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\left(\frac{t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}+\frac{t^{\alpha+\alpha j}}{\Gamma(\alpha+\alpha j+1)}\right)\right]\|x-y\| \\
&\leq q\,\|x-y\|.
\end{aligned}$$

Hence, the operator Ψ *has a unique fixed point which is a solution to the integral Eq.(2), and hence a solution to Eq.(1).*

4 Nonlocal problem

We discuss in this section, the existence problem of the fractional nonlinear integrodifferential equation (1) with a nonlocal condition of the form

$$x(0)+g(x)=x_0, \tag{3}$$

where $g: Y \to Y$ is a given function that satisfies the following condition:

(Hg) g is a continuous function and there exists a positive constant G such that

$$\|g(x) - g(y)\| \leq G\|x - y\|, \text{ for } x, y \in Y.$$

In terms of the nonlocal condition (3), Eq.(1) is equivalent to the integral equation

$$x(t) = x_0 - g(x) + \\ I^\alpha(f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t))).$$

We need the following hypothesis:

(Hr) Let $0 < q < 1$ and r be a positive finite real number such that

$$\begin{cases} q \geq G + L\left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \sum\limits_{i=1}^{k}\left(\frac{(1+G)T^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)} + \frac{T^{\alpha+\alpha j}}{\Gamma(\alpha+\alpha j+1)}\right)\right] \\ r \geq (1-q)^{-1}\left(\|x_0\|\left(1 + \sum\limits_{i=1}^{k}\frac{T^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right) + \|g(0)\|\left(1 + \sum\limits_{i=1}^{k}\frac{T^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right)\right. \\ \left. +\frac{KT^\alpha}{\Gamma(\alpha+1)}\right). \end{cases}$$

Moreover, let $B_r = \{y \in Y : \|y\| \leq r\}$.

Theorem 4.1 *If the hypotheses (Hf),(Hr), and (Hg) are satisfied, then the fractional integrodifferential equation (1)-(3) has a unique solution on J.*
Proof. *We want to prove that the operator $\Phi : Y \to Y$ defined by*

$$\Phi x(t) = x_0 - g(x) + \\ I^\alpha(f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)))$$

has a fixed point on B_r. This fixed point is then a solution of Eqs.(1) and (3). Let $x \in B_r$, then

$$\|\Phi x(t)\| \leq \|x_0\| + \|g(0)\| + G\|x\| + \\ I^\alpha\left(\left\|f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t))\right\|\right) \\ \leq \|x_0\| + \|g(0)\| + G\|x\| \\ +LI^\alpha\left(\|x(t)\| + \sum_{i=1}^{k}\left(\left\|D^{(\alpha_j)}x(t)\right\| + \|I^{\alpha_j}x(t)\|\right)\right) + I^\alpha\left(\left\|f(t, \overbrace{0, 0, ..., 0}^{2k+1})\right\|\right)$$

by the semigroup property of the fractional operators, we have

$$\begin{aligned}
\|\Phi x(t)\| &\leq \|x_0\| + \|g(0)\| + G\,\|x\| \\
&\quad +L\left(I^{\alpha}\|x(t)\| + \sum_{i=1}^{k}\left(I^{\alpha-\alpha_j}\|x(t)-x_0+g(x)\| + I^{\alpha}I^{\alpha_j}\|x(t)\|\right)\right) \\
&\quad +I^{\alpha}\left(\left\| f(t,\overbrace{0,0,....,0}^{2k+1})\right\|\right) \\
&\leq \|x_0\|\left(1+\sum_{i=1}^{k}\frac{t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right) \\
&\quad +\|g(0)\|\left(1+\sum_{i=1}^{k}\frac{t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)}\right) + \frac{Kt^{\alpha}}{\Gamma(\alpha+1)} + G\,\|x\| \\
&+ L\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{k}\left(\frac{(1+G)t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)} + \frac{t^{\alpha+\alpha j}}{\Gamma(\alpha+\alpha j+1)}\right)\right)\|x\|
\end{aligned}$$

since $x \in B_r$, we get

$$\|\Phi x(t)\| \leq (1-q)\,r + qr = r.$$

Hence, the operator Φ maps B_r into itself. Next, we prove that Φ is a contraction mapping on B_r. Let $x, y \in B_r$, then

$$\begin{aligned}
& \|\Phi x(t) - \Phi y(t)\| \leq G\,\|x-y\| \\
+ & \left\| I^{\alpha} f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)) \right. \\
- & \left. I^{\alpha} f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)) \right\| \\
\leq & \ G\,\|x-y\| \\
+ & \ LI^{\alpha}\left[\|x(t)-y(t)\| + \sum_{i=1}^{k}\left(\left\|D^{(\alpha_j)}(x(t)-y(t))\right\| + \|I^{\alpha_j}(x(t)-y(t))\|\right)\right] \\
\leq & \ G\,\|x-y\| + LI^{\alpha}\,\|x(t)-y(t)\| \\
+ & \ L\left[\sum_{i=1}^{k}\left(I^{\alpha-\alpha_j}\|(x(t)-y(t)-g(y)+g(x)\| + I^{\alpha+\alpha_j}\|x(t)-y(t)\|\right)\right] \\
\leq & \ G\,\|x-y\| \\
+ & \ L\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{k}\left(\frac{(1+G)t^{\alpha-\alpha_j}}{\Gamma(\alpha-\alpha_j+1)} + \frac{t^{\alpha+\alpha j}}{\Gamma(\alpha+\alpha j+1)}\right)\right]\|x-y\| \\
\leq & \ q\,\|x-y\|.
\end{aligned}$$

Hence, the operator Φ has a unique fixed point which is a solution of the system (1)-(3).

The last result is based on the following well-known fixed point theorem.

Theorem 4.2 *(Krasnoselkii) Let S be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that*
(i) $Px + Qy \in S$ whenever $x; y \in S$;
(ii) P is a contraction mapping;
(iii) Q is compact and continuous.
Then there exists $z \in S$ such that $z = Pz + Qz$.

To apply the above theorem we need the following condition instead of the condition (Hf).

(HF) Let $f : J \times Y^{\otimes} \to Y$ be jointly continuous and there exists a positive constant C such that

$$\sup \|f(t, x_1(t), x_2(t), x_3(t), ..., x_{2k+1}(t))\| \leq C,$$

for all $(t, x_1, x_2, x_3, ..., x_{2k+1}) \in J \times Y^{\otimes}$

Theorem 4.3 *If the hypotheses (Hg) and (HF) are satisfied, and if $G < 1$, then the fractional integrodifferential equation (1) with nonlocal condition (3) has a solution on J.*
Proof. *Let $B_r = \{y \in Y : \|y\| \leq r\}$, where $r(1-G) \geq \left(\|x_0\| + \|g(0)\| + \frac{CT^\alpha}{\Gamma(\alpha+1)}\right)$. Define the operators P, and Q on B_r by*

$$\begin{cases} Px(t) = x_0 - g(x) \\ Qx(t) = I^\alpha(f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), \\ \qquad\qquad I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t))). \end{cases}$$

We observe that if $x, y \in B_r$, then $Px + Qy \in B_r$. Indeed it is easy to check the inequality

$$\|Px + Qy\| \leq \|x_0\| + Gr + \|g(0)\| + \frac{CT^\alpha}{\Gamma(\alpha + 1)} \leq r(1 - G) + Gr = r.$$

It is obvious that the operator P is a contraction. By the hypothesis (HF), the operator Q is continuous and uniformly bounded on B_r. The equicontinuity of $Qx(t)$ is already proved in [11], and so $Q(B_r)$ is relatively compact. By the Arzela Ascoli Theorem, Q is compact. Hence by the Krasnoselkii theorem there exists a solution to the system (1)-(3).

5 Open Problem

Let $Y = C^1(J, \mathbf{R})$ be a Banach space of all real-valued functions having at most continuous first derivative on a compact interval $J = [0, T]$.
Consider the fractional nonlinear integrodifferential equation

$$\begin{cases} D^{(\alpha)}x(t) \\ = f(t, x(t), D^{(\alpha_1)}x(t), D^{(\alpha_2)}x(t), ..., D^{(\alpha_k)}x(t), I^{\alpha_1}x(t), I^{\alpha_2}x(t), ..., I^{\alpha_k}x(t)), \\ x(T) = x_T \in \mathbf{R}, \end{cases} \tag{4}$$

where $1 < max_{1\leq j\leq k}(\alpha_j) < \alpha < 2$; $k \in \mathbb{N}$, $x \in Y$, and $f : J \times Y^{\otimes} \to Y$ satisfies appropriate condition. The problem may be solved if one can establish a mild integral solution of the above equation and then a fixed point theorem can be applied to prove the existence problem.

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