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About an Amalgamated Duplication of a Ring Along an Ideal

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Abstract

This paper investigates G-GCD, finite conductor, and v-coherence properties for various amalgamated duplication of a ring along an ideal contexts. Our results generate new families of examples of this classes of rings with zero-divisors.

Keywords: *Amalgamated duplication of a ring along an ideal, v-coherence, finite conductor ring, G-GCD ring.*

1 Introduction

All rings considered in this paper are commutative with identity element and all modules are unitary. A ring is called a GCD ring, if all principal ideals of R are projectives and the intersection of any two principal ideals of R is a principal ideal of R. And a ring is called a generalized GCD ring (G-GCD ring for short), if all principal ideals of R are projectives and the intersection of any two finitely generated flat ideals of R is a finitely generated flat ideal of R. For more details, see [4, 5].

In [4], Glaz extended the definition of a finite conductor domains to rings with zero-divisors. Hence, a ring R is a finite conductor, if $Ra \cap Rb$ and $(0, c)$ are finitely generated ideals of R for each $a, b, c \in R$. It is shown that R is a finite conductor if and only if any ideal of R with $\mu(I) \leq 2$ is finitely presented, where $\mu(I)$ denotes the cardinality of minimal set of generators of I; see [4, Proposition 2.1]. For instance, see [4].
Thus the class of G-GCD ring and the class of coherent rings are included in

the class of finite conductor rings.

Let I and J be two nonzero fractional ideals of R, and set $(I:J) = \{x \in Q(R)/xJ \subset I\}$ (where $Q(R)$ is the total ring of quotients) the fractional ideal of I and J. We denote by I^{-1} and I_v respectively the ideals $(R:I)$ and $(R:(R:I))$. A nonzero fractional ideal I is said to be v-finite, if $I_v = J_v$ for some finitely generated fractional ideal J of R.
A ring R is called v-coherent, if $(0:a)$ and $\cap_{1\leq i\leq n} Ra_i$ are a v-finite ideals of R for any element $a \in R$ and any finite set of elements $a_1....a_n \in R$. For instance, see [6].

For two rings $A \subset B$, we say that A is a module retract (or a subring retract) of B if there exists an A-module homomorphism $\varphi : B \rightarrow A$ such that $\varphi\mid_A = id\mid_A$. φ is called a module retraction map. If such a map φ exists, B contains A as an A- module direct summand.

The amalgamated duplication of a ring R along an R-module E submodule of the total ring of quotients $T(R)$; $R \bowtie E$, introduced by D'Anna and Fontana [2], is the following subring of $R \times T(R)$ (endowed with the usual componentwise operations):

$$R \bowtie E := \{\,(r, r+e) \mid r \in R \ \text{ and } e \in E\}.$$

It is obvious that, if in the R-module $R \oplus E$ we introduce a multiplicative structure by setting $(r,e)(s,f) := (rs, rf+se+ef)$, where $r, s \in R$ and $e, f \in E$ then, we get the ring isomorphism $R \bowtie E \cong R \oplus E$. When $E^2 = 0$, this new construction coincides with the Nagata's idealization. One main difference between this constructions, with respect to the idealization (or with respect to any commutative extension, in the sense of Fossum) is that the ring $R \bowtie E$ can be a reduced ring and it is always reduced if R is a domain. If $E = I$ is an ideal in R then, the ring $R \bowtie I$ is a subring of $R \times R$. This extension has been studied, in the general case, and from the different point of view of pullbacks, by D'Anna and Fontana [2]. As it happens for the idealization, one interesting application of this construction is the fact that it allows to produce rings satisfying (or not satisfying) preassigned conditions. Recently, D'Anna proved that, if R is a local Cohen-Macaulay ring with canonical module ω_R then, $R \bowtie I$ is a Gorenstein ring if and only if $I \cong \omega_R$, cf. [1]. For instance, see [1, 2]

In this paper, we study the possible transfer of G-GCD, finite conductor and v-coherence notions for various amalgamated duplication of a ring along an ideal contexts. Thereby, new examples are provided which, particulary, enrich the current literature with new classes of G-GCD rings, finite conductor rings, and v-coherence rings with zero-divisors.

2 Problem Formulations

We study the possible transfer of G-GCD, finite conductor and v-coherence notions for various amalgamated duplication of a ring along an ideal contexts.

3 Main Results

We begin by studying the transfer of G-GCD property between R and his amalgamated duplication along some ideal I of R.

Theorem 3.1 *Let R be a ring and let $I(\neq 0)$ be a proper principal ideal of R. Then $R \bowtie I$ is not a G-GCD ring in the following cases:*

1. *R is an integral domain.*
2. *R is a local ring.*

Proof.

1. Let $a \in I$ such that $I = Ra$. Then it is easy to see that $O_1 = \{(0,i), i \in I\}$ and $O_2 = \{(i,0), i \in I\}$ are a principal ideals of $R \bowtie I$. Consider the short exact sequence of $R \bowtie I$-modules:

   $$(1) \qquad 0 \to \ker(u) \to R \bowtie I \xrightarrow{u} O_1 \to 0$$

   where $u(r, r+i) = (r, r+i)(0,a) = (0, (r+i)a)$. Then, $\ker(u) = \{(r,0) \in R \bowtie I / r \in I\} = O_2$. Consider also the short exact sequence of $R \bowtie I$-modules:

   $$(2) \qquad 0 \to \ker(v) \to R \bowtie I \xrightarrow{v} O_2 \to 0$$

   where $v(r, r+i) = (r, r+i)(a,0) = (ra, 0)$. Then, $\ker(v) = \{(0,i) \in R \bowtie I / i \in I\} = O_1$.
   We claim that O_1 is not projective. Deny. The ideal O_1 is projective, and so the short exact sequence (1) splits. Then O_2 is generated by an idempotent element $(x,0)$ with $x(\neq 0) \in I$. Hence, $(x,0)^2 = (x,0)(x,0) = (x^2,0) = (x,0)$. Thus, $x^2 = x$ and so $x = 1$ or $x = 0$, a contradiction since a I is a nonzero proper ideal of R. Consequently, O_1 is not projective and so $R \bowtie I$ cannot be a G-GCD.

2. By hypothesis I is a principal ideal of R, then so is O_1. We claim that O_1 is not projective. Deny O_1 is projective. Since R is local, then so is $R \bowtie I$ (by [2, Theorem 3.5]). Then O_1 is free, a contradiction since $O_1 O_2 = 0$. Therefore, O_1 is not projective and so $R \bowtie I$ cannot be a G-GCD and this completes the proof of the Theorem.

Next, we give new examples of Noetherian rings that are not G-GCD. Note that, if R is a ring and M is an R-module, as usual we use $pd_R(M)$ and $fd_R(M)$ to denote the usual projective and flat dimensions of M, respectively. The classical global and weak dimension of R are respectively denoted by $gldim(R)$ and $wdim(R)$.

Example 3.2 *Let R be a Noetherian domain and let I be a principal ideal. Then:*

1. *$R \bowtie I$ is a Noetherian ring.*
2. *$R \bowtie I$ is not a G-GCD ring.*

Proof.

1. By hypothesis, R is a Noetherian domain. Then $R \bowtie I$ is a Noetherian ring by [2, Corollary 3.3].
2. Follows immediately from Theorem 4.1.

Example 3.3 *Let $\mathbb{Z}$ be the ring of integers and let n be an integer. Then $\mathbb{Z} \bowtie n\mathbb{Z}$ is a Noetherian ring that is not G-GCD by Theorem 4.1.*

Now, we study the transfer of v-coherence property between a ring R and his amalgamated duplication along some ideal I of R.

Theorem 3.4 *Let R be a ring and let I be an ideal of R. Then:*

1. *Assume that R is a total ring of quotients and $I \subseteq Nil(R)$. Then $R \bowtie I$ is a total ring of quotients; in particular $R \bowtie I$ is a v-coherent ring.*
2. *Assume that (R, M) is a local total ring of quotients. Then $R \bowtie I$ is a total ring of quotients; in particular $R \bowtie I$ is a v-coherent ring.*

Proof. In (1) and (2), we have to prove that each element $(r, r+i)$ of $R \bowtie I$ is invertible or zero-divisor.

(1) Since R is a total ring of quotients, r is invertible or zero-divisor.
If r is a zero-divisor element, by [**?**, Proposition 2.2], $(r, r+i)$ is a zero-divisor element of $R \bowtie I$.
If r is invertible in R, then (r, r) is invertible in $R \bowtie I$, and hence $(r, r+i) = (r, r) + (0, i)$ is invertible in $R \bowtie I$ as a sum of an invertible element and a nilpotent one (since $I \subseteq Nil(R)$).
(2) If $r \in M$, then r is non invertible in R, that's a zero-divisor in R (since R is a total ring of quotients). By [**?**, Proposition 2.2], $(r, r+i)$ is a zero-divisor element of $R \bowtie I$.
If $r \notin M$, then $(r, r+i) \notin M \bowtie I$, and $M \bowtie I$ is the maximal ideal of the local ring $R \bowtie I$. Then $(r, r+i)$ is invertible in $R \bowtie I$.
Therefore, each element of $R \bowtie I$ is invertible or zero-divisor, as desired.

Example 3.5 *Let $R = k[x, y]/(x, y)^2$ where k is a field, x, y are indeterminates over k. R coincides with its total ring of quotients. Then $k[x, y]/(x, y)^2 \bowtie (\overline{x, y})$ is a v-coherent ring which is not G-GCD ring (by Theorem 4.1 and Theorem 4.2).*

Finally we study the the transfer of finite conductor (resp., G-GCD) property, when I is a pure ideal of R.

Theorem 3.6 *Let R be a semi local ring and let I be a pure ideal of R. Then:*

1. *R is a finite conductor ring if and only if $R \bowtie I$ is a finite conductor ring.*

2. *R is a G-GCD ring if and only if $R \bowtie I$ is a G-GCD ring.*

To prove this Theorem, we need the following Lemmas.

Lemma 3.7 *[4, page 2835]*

1. *Faithfully flat ring extensions descend the finite conductor properties.*

2. *Let $(R_i)_{i=1,...,m}$ be a family of rings. If R_i is a finite conductor ring for each $i = 1, ..., m$, then so is $\prod_{i=1}^{m} R_i$.*

3. *Every localization of a finite conductor ring at maximal ideal is a finite conductor.*

Lemma 3.8 *Let R be a semi local ring with maximal ideals $m_1, ...m_n$ such that R_{m_i} are finite conductor rings for $1 \leq i \leq n$. Then R is a finite conductor ring.*

Proof. Since R_{m_i} are a finite conductor rings for all $1 \leq i \leq n$, then $S = \prod_{i=1}^{n} R_{m_i}$ is a finite conductor ring by Lemma 3.7 (2). Moreover, S is a faithfully flat extension of R. Consequently, R is a finite conductor ring by Lemma 3.7 (1).

Lemma 3.9 *Let R be a commutative ring and let I be a proper ideal of R. If R is a semi local ring, then so is $R \bowtie I$.*

Proof. Let M be a maximal ideal of $R \bowtie I$. If $m = Q \cap R$, then (by [2, Theorem 3.5]) $M \in \{m_1, m_2\}$ and m_i is a maximal ideal of R, where $m_1 = \{(p, p + i)/p \in m, i \in I\}$ and $m_2 = \{(p + i, p)/p \in m, i \in I\}$. Since R is semi local, it clear that $R \bowtie I$ is semi local.

Lemma 3.10 *Let R be a ring and let I be a flat ideal of R. If $R \bowtie I$ is a finite conductor (resp., G-GCD) ring, then so is R.*

Proof. Since I is a flat ideal of R, $R \bowtie I$ is a faithfully flat R-module. On the other hand, by hypothesis, $R \bowtie I$ is a finite conductor (resp., G-GCD) ring. Then, by Lemma 3.7 (1) (resp., [4, page 2837]), R is a finite conductor (resp., G-GCD) ring.

Corollary 3.11 *Let R be a local total ring of quotients and let I be a finitely generated flat ideal of R. If R is a not finite conductor, then $R \bowtie I$ is a v-coherent ring that is not finite conductor .*

Proof. Follows immediately from Theorem 4.2 and Lemma 3.10.
Proof. [of Theorem 4.3] The sufficient conditions in (1) and (2) follows from Lemma 3.10. Hence, we have to prove the necessary conditions.

(1) Suppose that R is a finite conductor ring. To prove that $R \bowtie I$ is a finite conductor ring, we have to show that $(R \bowtie I)_M$ is a finite conductor ring whenever M is a maximal ideal of $R \bowtie I$ (by Lemma 3.8 and Lemma 3.9 since $R \bowtie I$ is semi local). For such ideal, set $m := M \cap R$. Then necessarily $M \in \{M, M'\}$ where $M = \{(r, r+i)/r \in m, i \in I\}$ and $M' = \{(r+i, r)/r \in m, i \in I\}$ (by [2, Theorem 3.5(b)]). Since I is a pure ideal, $I_m \in \{0, R_m\}$ for any maximal ideal m of R (by [3, Theorem 1.2.15]). Then, testing all cases of [1, Proposition 7] , we have two cases:

(a) $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.

(b) $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ or $I \subseteq m$.

Since R is a finite conductor ring, then R_m is also finite conductor ring. Therefore, so is $R_m \times R_m$ (by Lemma 3.7(2)). Hence, $(R \bowtie I)_M$ is a finite conductor ring. Accordingly, by Lemma 3.8, $R \bowtie I$ is a finite conductor ring since $R \bowtie I$ is semi local.

(2) Assume that R is a G-GCD ring. By [5, Theorem 3.2], to show that $R \bowtie I$ is a G-GCD ring, we have to prove that $R \bowtie I$ is a finite conductor ring and that $(R \bowtie I)_M$ is a GCD domain whenever M is a maximal ideal of $R \bowtie I$. Using [5, Theorem 3.2], it is clear by (1) that $R \bowtie I$ is a finite conductor ring. On the other hand, since R is a G-GCD ring, R_M is a GCD ring. Moreover, since I is a pure ideal, inspecting the cases above, we obtain that $(R \bowtie I)_M$ is a GCD domain whenever M is a maximal ideal of $R \bowtie I$. Consequently, $R \bowtie I$ is a G-GCD ring and this completes the proof of the Theorem.

A simple example of Theorem 4.3 is given by introducing the notion of the trace of modules. Recall that if M is an R-module, the trace of M, $Tr(M)$, is the sum of all images of morphisms $M \to R_R$, see [8]. Clearly $Tr(M)$ is an ideal of R.

Example 3.12 *Let A be a finite conductor ring, set $S = A \setminus \cup P_i$ where P_i are finitely many prime ideal, and set $R = S^{-1}A$. Let M be a projective R-module. Then:*

1. *$Tr(M)$ is a pure ideal since M is projective R-module (by [9, pp. 269-270]).*
2. *R is a semi local finite conductor ring (by [4, page 2835]).*
3. *$R \bowtie Tr(M)$ is a finite conductor ring (by Theorem 4.3).*

4 Conclusion

These are the main results of the paper.

Theorem 4.1 *Let R be a ring and let $I(\neq 0)$ be a proper principal ideal of R. Then $R \bowtie I$ is not a G-GCD ring in the following cases:*

1. *R is an integral domain.*

2. *R is a local ring.*

Theorem 4.2 *Let R be a ring and let I be an ideal of R. Then:*

1. *Assume that R is a total ring of quotients and $I \subseteq Nil(R)$. Then $R \bowtie I$ is a total ring of quotients; in particular $R \bowtie I$ is a v-coherent ring.*

2. *Assume that (R, M) is a local total ring of quotients. Then $R \bowtie I$ is a total ring of quotients; in particular $R \bowtie I$ is a v-coherent ring.*

Theorem 4.3 *Let R be a semi local ring and let I be a pure ideal of R. Then:*

1. *R is a finite conductor ring if and only if $R \bowtie I$ is a finite conductor ring.*

2. *R is a G-GCD ring if and only if $R \bowtie I$ is a G-GCD ring.*

5 Open Problem

Question. Let R be a commutative ring and let I be a proper ideal of R. Is R finite conductor (resp., v-coherent) if and only if so is $R \bowtie I$, in general ?

References

[1] M. D'Anna; *A construction of Gorenstein rings*, J. Algebra 306 (2006), no. 2, 507-519.

[2] M. D'Anna and M. Fontana; *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. 6 (2007), no. 3, 443-459.

[3] S. Glaz; *Commutative Coherent Rings*, Springer-Verlag, Lecture Notes in Mathematics, 1371 (1989).

[4] S. Glaz; *Finite conductor rings*,Proc. A.M.S. 129(10) (2000), 2833-2843.

[5] S. Glaz; *Finite conductor properties of $R(X)$ and $R < X >$*, Marcel Dekker Lecture Notes in Pure and Applied Mathematics, 220 (2001) 231-250.

[6] S. Kabbaj and N. Mahdou; *Trivial Extensions Defined by coherent-like condition*, Comm. Algebra, 32 (10) (2004), 3937-3953.

[7] M. Kabbour and N. Mahdou; *On weakly Von Neumann regular rings*, Int. Journal. Open Problems Compt. Math, Vol. 4, No. 1, March 2011.

[8] WM. McGovern, G. Puninski, P. Rothmaler; *When every projective module is a direct sum of finitely generated modules*, J. Algebra 315 (2007), 454-481.

[9] W. V. Vasconcelas; *Finiteness in projective ideals*, J. Algebra, 25 (1973), 269-278.