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On the Computations of Eigenvalues of the Fourth-order Sturm Liouville Problems

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Abstract

Locations of eigenvalues of Fourth-order Sturm-Liouville problems are investigated numerically by a combination of both Tau and Lanczos methods. Numerical and theoretical results indicate that the present method is efficient and accurate. Some complexity results are also studied.

Keywords: *Fourth-Order Sturm-Liouville Problem, Tau Method, Lanczos method.*

1 Introduction

In the present paper we investigate the solution of the fourth-order Sturm-Liouville problems given by

$$(p(x)y''(x))'' = (s(x)y'(x))' + (\lambda w(x) - q(x))\, y(x), \quad -1 \leq x \leq 1. \tag{1}$$

subject to

$$\begin{array}{rcl} \alpha_1 y(-1) & + & \beta_1 y(1) = 0 \\ \alpha_2 y'(-1) & + & \beta_2 y'(1) = 0 \\ \alpha_3 y''(-1) & + & \beta_3 y''(1) = 0 \\ \alpha_4 y'''(-1) & + & \beta_4 y'''(1) = 0 \end{array} \tag{2}$$

where $p,\ s,\ w$ and q are piecewise continuous functions with $p(x),\ w(x) \geq 0$. Here α_i and β_i (for $i = 1, 2, 3, 4$) are constants.

The fourth-order Sturm-Liouville problems play very important roles in both theory and applications. This is due to their use to describe a large number of physical and engineering phenomenons such as the deformation of an elastic beam under a variety of boundary conditions, for more details see [1],[2],[6],[7], [9], and [11].

In this paper, we present a method for locating the eigenvalues of problem (1)-(2). This method depends on applying the Tau method to discretize equation (1) to obtain a system of the form

$$M(\lambda)\mathbf{Y} = \mathbf{0},$$

where $M(\lambda)$ is a matrix function of λ. Then we apply the Lenczos method to scan a given interval to determine a value for λ.

This paper is organized as follows: The numerical methods of solution and problem statement are presented in section (2). Numerical results are presented in Section 3 followed by some conclusion remarks.

2 Tau-Lanczos Method

In this section, we present the numerical method used to solve problem (1)-(2). Essentially, this approach depends on applying the Tau method on equation (1) to obtain a system of the form

$$M(\lambda)\mathbf{Y} = \mathbf{0},$$

where $M(\lambda)$ is a matrix function of λ. According to the fact that problem (1) has nonzero eigenvectors then $M(\lambda)$ should be a singular matrix when λ is an eigenvalue of equation (1). Moreover, to find a value for such λ in a specific interval, we apply the Lenczos method. The full discussion of the methods of solution for problem (1) is presented in the following three subsections:

2.1 Tau Method

For sake of simplicity we discuss the more general form of equation (1) given by

$$\mathcal{P}(x)y''''(x)+\mathcal{A}(x)y'''(x)+\mathcal{B}(x)y''(x)+\mathcal{C}(x)y'(x)+\mathcal{Q}(x)y(x)-\lambda\mathcal{W}(x)y(x) = 0. \tag{3}$$

Firstly, we assume that the exact solution of (1) is approximated in terms of the Chebyshev polynomials as

$$y(x) \approx Y_N(x) = \sum_{k=0}^{N+4} y_k \, T_k(x). \tag{4}$$

where y_k $(k = 0 : N + 4)$ are unknown constants need to be determined. Similarly, the functions $\mathcal{P}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{Q}$ and $\mathcal{W}$ can also be approximated as

$$\begin{aligned}
\mathcal{P}(x) \approx \mathcal{P}_N(x) &= \sum_{k=0}^{N+4} p_k \, T_k(x), \\
\mathcal{A}(x) \approx \mathcal{A}_N(x) &= \sum_{k=0}^{N+4} a_k \, T_k(x), \\
\mathcal{B}(x) \approx \mathcal{B}_N(x) &= \sum_{k=0}^{N+4} b_k \, T_k(x), \\
\mathcal{C}(x) \approx \mathcal{C}_N(x) &= \sum_{k=0}^{N+4} c_k \, T_k(x), \\
\mathcal{Q}(x) \approx \mathcal{Q}_N(x) &= \sum_{k=0}^{N+4} q_k \, T_k(x), \\
\mathcal{W}(x) \approx \mathcal{W}_N(x) &= \sum_{k=0}^{N+4} w_k \, T_k(x),
\end{aligned} \tag{5}$$

where p_k, a_k, b_k, c_k, q_k and w_k are known constants. Direct substitutions of (4) and (5) into (3) yield

$$\mathcal{R}_N(x) = \mathcal{P}_N {Y_N}'''' + \mathcal{A}_N {Y_N}''' + \mathcal{B}_N \, {Y_N}'' + \mathcal{C}_N {Y_N}' + \mathcal{Q}_N Y_N - \lambda \mathcal{W}_N Y_N, \tag{6}$$

where $\mathcal{R}_N$ represents the residual function. In particular, the functions ${Y_N}'$, ${Y_N}''$, ${Y_N}'''$ and ${Y_N}''''$ can be written in the following forms

$$\begin{aligned}
Y_N'(x) &= \sum_{k=0}^{N+3} y_k^{(1)} \, T_k(x), \\
Y_N''(x) &= \sum_{k=0}^{N+2} y_k^{(2)} \, T_k(x), \\
Y_N'''(x) &= \sum_{k=0}^{N+1} y_k^{(3)} \, T_k(x), \\
Y_N''''(x) &= \sum_{k=0}^{N} y_k^{(4)} \, T_k(x),
\end{aligned} \tag{7}$$

where $y_k^{(1)}$, $y_k^{(2)}$, $y_k^{(3)}$ and $y_k^{(4)}$ are linear combinations of the unknown coefficients y_i $(i = 1, \cdots, N+4)$ through the following recurrence relations

$$c_{n-1} y_{n-1}^{(q)} - y_{n+1}^{(q)} = 2n y_n^{(q-1)}, \qquad n \geq 1, q = 1, 2, 3, 4. \tag{8}$$

Here $c_0 = 2$, $c_n = 1$ for $n \geq 1$ and $y_n^{(0)} = y_n$. One can see that the approximation of the function y given in equations (4) converge uniformly if $y \in C^4[-1,1]$ and $y^{(5)}(x)$ is piecewise continuous function on $[-1,1]$. Define ζ_k, η_k, ν_k, ω_k, τ_k and σ_k so that

$$\begin{aligned}
\mathcal{P}_N(x) {Y_N}''''(x) &\approx \sum_{k=0}^{N} \zeta_k \, T_k(x), \\
\mathcal{A}_N(x) {Y_N}'''(x) &\approx \sum_{k=0}^{N} \eta_k \, T_k(x), \\
\mathcal{B}_N(x) {Y_N}''(x) &\approx \sum_{k=0}^{N} \nu_k \, T_k(x), \\
\mathcal{C}_N(x) {Y_N}'(x) &\approx \sum_{k=0}^{N} \omega_k \, T_k(x), \\
\mathcal{Q}_N(x) Y_N(x) &\approx \sum_{k=0}^{N} \tau_k \, T_k(x), \\
\mathcal{W}_N(x) Y_N(x) &\approx \sum_{k=0}^{N} \sigma_k \, T_k(x).
\end{aligned} \tag{9}$$

Consequently, equation (6) can be written in the form

$$\mathcal{R}_N(x) \approx \sum_{k=0}^{N} (\zeta_k + \eta_k + \nu_k + \omega_k + \tau_k - \lambda\sigma_k)\, T_k(x). \tag{10}$$

To obtain the coefficients of $T_k(x)$ in the right hand side of equation (10), we use the fact that $T_k(x)$ and $R_N(x)$ are orthogonal for $k = 0 : N$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, i.e.

$$\langle R_N(x), T_k(x) \rangle = \zeta_k + \eta_k + \nu_k + \omega_k + \tau_k - \lambda\sigma_k = 0, \qquad \text{for } k = 0 : N, \tag{11}$$

where

$$\langle R_N(x), T_k(x) \rangle = \int_{-1}^{1} \frac{R_N(x)\, T_k(x)}{\sqrt{1-x^2}} dx.$$

It is more convenient to rewrite equations (11) in the following vector form

$$\zeta + \eta + \nu + \omega + \tau - \lambda\sigma = \mathbf{0}, \tag{12}$$

where

$$\begin{array}{llll} \zeta & = (\zeta_0, \zeta_1, ..., \zeta_N)^T, \quad \eta & = (\eta_0, \eta_1, ..., \eta_N)^T, \nu & = (\nu_0, \nu_1, ..., \nu_N)^T, \\ \omega & = (\omega_0, \omega_1, ..., \omega_N)^T, \tau & = (\tau_0, \tau_1, ..., \tau_N)^T, \sigma & = (\sigma_0, \sigma_1, ..., \sigma_N)^T. \end{array}$$

If we assume that

$$\begin{array}{llll} \mathbf{Y} & = (y_0, y_1, ..., \zeta_{N+4})^T, & \mathbf{Y}^{(1)} & = \left(y_0^{(1)}, y_1^{(1)}, ..., y_{N+3}^{(1)}\right)^T, \\ \mathbf{Y}^{(2)} & = \left(y_0^{(2)}, y_1^{(2)}, ..., y_{N+2}^{(2)}\right)^T, & \mathbf{Y}^{(3)} & = \left(y_0^{(3)}, y_1^{(3)}, ..., y_{N+1}^{(3)}\right)^T, \\ \mathbf{Y}^{(4)} & = \left(y_0^{(4)}, y_1^{(4)}, ..., y_N^{(4)}\right)^T, & & \end{array}$$

then equation (9) ensures the existence of six matrices $\mathbf{G}_{01}$, $\mathbf{G}_{02}$, $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$ and $\mathbf{G}_4$ with dimensions $(N+1)\times(N+5)$, $(N+1)\times(N+5)$, $(N+1)\times(N+4)$, $(N+1)\times(N+3)$, $(N+1)\times(N+2)$ and $(N+1)\times(N+1)$ respectively; so that

$$\begin{aligned} \sigma &= \mathbf{G}_{01}\,\mathbf{Y}, \quad \tau = \mathbf{G}_{02}\,\mathbf{Y}, \quad \omega = \mathbf{G}_1\,\mathbf{Y}^{(1)} \\ \nu &= \mathbf{G}_2\,\mathbf{Y}^{(2)}, \quad \eta = \mathbf{G}_3\,\mathbf{Y}^{(3)}, \quad \zeta = \mathbf{G}_4\,\mathbf{Y}^{(4)}. \end{aligned} \tag{13}$$

However, equation (8) ensures the existence of another four matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ with dimensions $(N+4)\times(N+5)$, $(N+3)\times(N+5)$, $(N+2)\times(N+5)$ and $(N+1)\times(N+5)$ respectively; so that

$$\mathbf{Y}^{(1)} = \mathbf{A}_1\ \mathbf{Y}, \ \mathbf{Y}^{(2)} = \mathbf{A}_2\ \mathbf{Y}, \ \mathbf{Y}^{(3)} = \mathbf{A}_3\,\mathbf{Y}, \ \text{ and } \ \mathbf{Y}^{(4)} = \mathbf{A}_4\,\mathbf{Y}. \tag{14}$$

Consequently, inserting the results from (13) and (14) into (12) leads to the following eigenvalue problem

$$\mathbf{\Lambda Y} = \lambda \mathbf{G}_{01}\mathbf{Y}, \tag{15}$$

where $\mathbf{\Lambda} = \mathbf{G}_4\mathbf{A}_4 + \mathbf{G}_3\mathbf{A}_3 + \mathbf{G}_2\mathbf{A}_2 + \mathbf{G}_1\mathbf{A}_1 + \mathbf{G}_{02}$ is $(N+1)\times(N+5)$ matrix. On the other hand, the boundary conditions (2) can be expressed as

$$\alpha_0 y(-1) + \beta_0 y(1) = 0 \Rightarrow \sum_{k=0}^{N+4} \epsilon_{0_k}\ y_k = 0 \tag{16}$$

$$\alpha_1 y'(-1) + \beta_1 y'(1) = 0 \Rightarrow \sum_{k=0}^{N+4} \epsilon_{1_k}\ y_k = 0 \tag{17}$$

$$\alpha_2 y''(-1) + \beta_2 y''(1) = 0 \Rightarrow \sum_{k=0}^{N+4} \epsilon_{2_k}\ y_k = 0 \tag{18}$$

$$\alpha_3 y'''(-1) + \beta_3 y'''(1) = 0 \Rightarrow \sum_{k=0}^{N+4} \epsilon_{3_k}\ y_k = 0, \tag{19}$$

where

$$\begin{aligned}
\epsilon_{0_k} &= (-1)^k \alpha_1 + \beta_1, \quad \epsilon_{1_k} = (-1)^{k+1} k^2 \alpha_2 + \beta_2 k^2, \\
\epsilon_{2_k} &= (-1)^n n^2 \left(\tfrac{n^2-1}{3}\right) \alpha_2 + n^2 \left(\tfrac{n^2-1}{3}\right)\ \beta_2, \quad \text{and} \\
\epsilon_{3_k} &= (-1)^{n+1} n^2 \left(\tfrac{n^2-1}{3}\right) \left(\tfrac{n^2-4}{5}\right) \alpha_3 + n^2 \left(\tfrac{n^2-1}{3}\right) \left(\tfrac{n^2-4}{5}\right) \beta_3.
\end{aligned}$$

Note that the above compact forms for ϵ_{i_k}, for $i = 1, 2, 3, 4$ and $k = 0 : N+4$ are proven using the following form of the pth derivative of T_n (for $n = 0, \cdots$) at ± 1:

$$T_n^{(p)}(\pm 1) = (\pm 1)^{n+p} \prod_{k=0}^{p-1} \frac{n^2 - k^2}{2k+1}.$$

Engaging the results of (15)-(19) gives the following algebraic eigenvalue problem

$$M(\lambda)\mathbf{Y} = \mathbf{0}, \tag{20}$$

where

$$M(\lambda) = \begin{pmatrix} \mathbf{\Lambda} \\ \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} - \lambda \begin{pmatrix} \mathbf{G}_{01} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and $\epsilon_i = \left(\epsilon_{i_0}, \epsilon_{i_1}, \cdots, \epsilon_{i_{N+4}}\right)$ (for $i = 0, 1, 2, 3$). Using the fact that the fourth-order Sturm-Liouville problems (1) has nonzero eigenvectors, system (20) must have nonzero solutions; therefore,

$$\det(M(\lambda)) = \mathbf{0} \tag{21}$$

when λ is an eigenvalue of Problem (1). It is found that $M(\lambda)$ is a large sparse matrix, hence we apply the Lanczos method to locate those eigenvalues, $\lambda' s$, and to approximate them as we explain in the next subsection.

2.2 Lanczos Method

Particulary, problem (21) is unstable problem; hence we replace it by a stable problem using the following theorem (see [5]).

Theorem 1 *The following statements are equivalent:*

(C1) $\det(M(\lambda)) = 0$.

(C2) $\min\left\{\|M(\lambda)u\|^2 : u \in \Re^s \text{ and } \|u\| = 1\right\} = 0.$

(C3) $\min\left\{\frac{\|M(\lambda)u\|^2}{\|u\|^2} : u \in \Re^s \text{ and } u \neq 0\right\} = 0.$

(C4) The smallest eigenvalue of $M(\lambda)^* M(\lambda)$ is zero,

where $*$ means the transpose of the matrix and $\|.\|$ denotes the Euclidean norm. Therefore, in our study we replace problem (21) with problem $(C2)$. For convenience, in what follow we assume that $\mathbb{M} = M(\lambda)^* M(\lambda)$. It is noted that $\mathbb{M}$ is a large, square, and symmetric matrix; therefore the most suitable method to use, in this case, is the Lanczos method which is described below.

Consider the Rayleigh quotient

$$R(u) = \frac{u^* \mathbb{M} u}{u^* u}, \quad u \neq 0, \tag{22}$$

then, the minimum of $R(u)$ is the smallest eigenvalue of $\mathbb{M}$. Moreover, if we suppose that $e_l \subset \Re^s$ is the Lanczos orthonormal vectors and $E_l = [\varrho_1 \ \ \varrho_2 \ \ \dots \ \ \varrho_l]$, then

$$E_l^* \mathbb{M} E_l = \Pi_l = \begin{bmatrix} \varkappa_1 & \varrho_1 & \cdots & 0 \\ \varrho_1 & \varkappa_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \varrho_{l-1} & \varkappa_l \end{bmatrix}. \tag{23}$$

One can verify that

$$m_l = \min_{u \neq 0} R(E_l u) \geq \lambda_{\min}.$$

Hence, $m_1 \geq m_2 \geq \cdots \geq m_s = \Lambda_{\min}$, for more details see [5]. In general, we obtain a good approximation for $\Lambda_{\min}$ from m_l for some $l << s$. However, we should note that Π_l is tridiagonal matrix, so we can write it in the computer as $l \times 3$ matrix. The following algorithm shows how to calculate $\varkappa_1, \varkappa_2, \dots, \varkappa_l, \varrho_1, \dots, \varrho_{l-1}$.

Algorithm 2

- ***Input:*** *Tol*
- ***Output:*** *calculate $\varkappa_1, \varkappa_2, \dots, \varkappa_l, \varrho_1, \dots, \varrho_{l-1}$*
- ***Step 1:*** *Set $i = 0; \Upsilon = 0;$ and $\varrho_0 = 1$.*
- ***Step 2:*** *while $\varrho_j \geq tol$ and $i < l$, do steps 3-7*
- ***Step 3:*** *If $i \neq 0$, do steps 4-5*
- ***Step 4:*** *For $k = 1 : s$, do step 5*

- ***Step 5:*** *Set* $\pi = \psi_k, \psi_k = \frac{\Upsilon_k}{\varrho_i}, \Upsilon_k = -\varrho_i \pi$.
- ***Step 6:*** *Set* $\Upsilon = \Upsilon + \mathbb{M}\psi$.
- ***Step 7:*** *Set* $i = i+1, \varkappa_i = \psi^* \Upsilon; \Upsilon = \Upsilon - \varkappa_i \psi;$ *and* $\varrho_i = \|\Upsilon\|_2$.

The following remarks on Algorithm 2 should be noted:

1. In each step we do only one evaluation for $A\psi$. This means that if we want to compute Π_l, we need only l evaluations of $A\psi$.
2. We compute $\mathbb{M}\psi$ as follows:
   (a) Compute $u = M(\lambda)\psi$.
   (b) Compute $M(\lambda)^* u$.
3. If $M(\lambda)$ has an average of about h nonzero per row, then approximately $(3h+8)s$ flops are involved in a single Lanczos step.
4. The Lanczos vectors are generated in the s-vector w.
5. Unfortunately, during the calculations in Algorithm 2, we lose the orthogonality of the Lanczos vectors which is due to the cancelation.

To avoid the problem of loss of the orthogonality, one can use either the complete reorthogonalization or the selective orthogonalization. The first method is very expensive and complicated to use. Therefor, we use the selective orthogonalization in this paper.

Suppose that the symmetric QR method is applied to Π_l, see [5]. Assume that $\theta_1, \theta_2, ..., \theta_l$ are the computed Ritz values and Ω_l is nearly orthogonal matrix of eigenvectors. Let

$$F_l = [f_1 \ f_2 \ ... \ f_l] = E_l \Omega_l$$

Then it can be shown that

$$|e_{l+1} f_i| \approx \frac{eps \ \|\mathbb{M}\|_2}{|\varrho_l| \, |\Omega_{li}|}$$

and

$$\|\mathbb{M} f_i - \theta_i f_i\| \approx |\varrho_l| \, |\Omega_{li}| = \varrho_{li}$$

where eps is the machine precision. The computed Ritz pair (θ, f) is called "good" if

$$\|\mathbb{M} f - \theta f\| \approx \sqrt{eps} \ \|\mathbb{M}\|_2 .$$

We can measure the loss of orthogonality of E_i by

$$\kappa_i = \|I_i - E_i^* E_i\| \text{ and } \kappa_1 = \|1 - e_1^* e_1\|.$$

It can be easily seen that κ_i is an increasing function of i. We can measure the value of κ_{i+1} using κ_i via the following theorem.

Theorem 3 *If $\kappa_i \leq \mu$, then $\kappa_{i+1} \leq \frac{1}{2}(\mu + eps + \sqrt{(\mu - eps)^2 + 4\|E_i^* e_{i+1}\|^2})$.*

Fix κ, say for example $\kappa = 0.01$. If $\kappa_i \leq \kappa$, then q_{i+1} is orthogonal on all columns of E_i. In this case we will not do any reorthogonalization. If $\kappa_i > \kappa$, then we orthogonalize e_{i+1} against each "good" Ritz vector. In the selective orthogonalization we apply the symmetric QR method on Π_l which has small size comparing with the size of $\mathbb{M}$. Then, we apply the Rayleigh quotient iteration to compute the smallest eigenvalue of the matrix Π_j using the following algorithm.

Algorithm 4

- ***Input:*** *$X^{(0)}$ such that $\left\|X^{(0)}\right\| = 1$.*
- ***Output:*** *the smallest eigenvalue of the matrix Π_j*
- ***Step 1:*** *For $k = 0, 1, ...,$ do step 2-4.*
- ***Step 2:*** *Compute $\mu_k = \frac{X^{(k)*} T_l X^{(k)}}{X^{(k)*} X^{(k)}}$.*
- ***Step 3:*** *Solve $(\Pi_l - \mu_k I_l) Z^{(k+1)} = X^{(k)}$ for $Z^{(k+1)}$.*
- ***Step 4:*** *Set $X^{(k+1)} = \frac{Z^{(k+1)}}{\left\|Z^{(k+1)}\right\|}$.*

For more details about the selective orthogonalization, see [5],[8] and [12]. Finally, we can summarize our technique in this section as follows.

1. Compute the matrix $M\ (\lambda)$; as we did in section 2.
2. Set $\mathbb{M} = M^*(\lambda) M(\lambda),\ l = 1$.
3. Use the Selective orthogonalization and the Lanczos method to compute the matrix T_l.
4. Use Algorithm 9 to approximate m_l.
5. If m_l is good approximation for $\Lambda_{\min}$, stop
   else $l = l + 1$; and repeat steps 3-5.
6. Stop.

3 Numerical Results

It is important to mention that the present method was applied on many different examples, but herein we only present the following two examples to illustrate the efficiency and accuracy of this method. Consider the fourth-order eigenvalue problem

$$y^{(4)}(z) - \mu y(z) = 0, z \in (0,1)$$

subject to

$$y(0) = y'(0) = y(1) = y''(1) = 0.$$

Using the transformation

$$x = 2z - 1,$$

one obtains the following eigenvalue problem

$$y^{(4)}(x) - \lambda y(x) = 0, x \in (-1,1)$$

subject to

$$y(-1) = y'(-1) = y(1) = y''(1) = 0$$

where $\lambda = \frac{\mu}{16}$. We scan for a solution of the $\mu-$parameter in the interval $[237, 148634.50]$ where the increment is 0.05. Table (1) shows the minimal eigenvalues Λ and the number of evaluations of $\mathbb{M}\psi$ which were necessary to obtain Λ, say ν.

Consider the fourth-order eigenvalue problem

$$y^{(4)}(z) = 0.02z^2y'' + 0.04zy' - (0.0001z^4 - 0.02)y + \mu y(z), z \in (0,5)$$

subject to

$$y(0) = y''(0) = y(5) = y''(5) = 0.$$

Using the transformation

$$x = \frac{2z-5}{5},$$

one obtains the following eigenvalue problem

$$y^{(4)}(x) = (\frac{625}{16}) \quad [0.02(x+1)^2y'' + 0.04(x+1)y' - (0.0001(\tfrac{5x+5}{2})^4 \\ -0.02)y] + \lambda y(x), \qquad x \in (-1,1)$$

subject to

$$y(-1) = y'(-1) = y(1) = y''(1) = 0$$

where $\lambda = \frac{625\mu}{16}$. The solution of the $\mu-$parameter is scanned in the interval $500, 173881.35]$ with increment 0.05. We make an entirely analogous analysis to do that of Example 1. Table (2) shows the minimal eigenvalues Λ and the number of evaluations of $\mathbb{M}\psi$ which were necessary to obtain Λ, say ν.

μ	Λ	ν	μ	Λ	ν
237	$8.3e^{-02}$	7	2496.40	$6.2e^{-06}$	5
237.05	$7.4e^{-02}$	7	2496.45	$2.2e^{-06}$	4
237.10	$6.7e^{-02}$	7	2496.50	$1.1e^{-07}$	4
237.15	$4.2e^{-02}$	7	$\vdots$		
237.20	$3.3e^{-02}$	7	10867.50	$7.9e^{-06}$	5
237.25	$2.2e^{-02}$	6	10867.55	$5.7e^{-06}$	4
237.30	$1.1e^{-02}$	6	10867.60	$2.3e^{-07}$	4
237.35	$7.5e^{-03}$	6	$\vdots$		
237.40	$2.5e^{-03}$	6	31780.05	$8.4e^{-06}$	6
237.45	$1.8e^{-03}$	6	31780.10	$1.4e^{-07}$	5
237.50	$1.7e^{-03}$	6	$\vdots$		
237.55	$2.7e^{-04}$	6	74000.80	$9.1e^{-07}$	5
237.60	$3.1e^{-05}$	6	74000.85	$2.2e^{-09}$	4
237.65	$4.2e^{-06}$	6	74000.90	$5.5e^{-06}$	4
237.70	$6.2e^{-7}$	5	$\vdots$		
237.75	$2.1e^{-06}$	6	148634.45	$4.8e^{-07}$	4
237.80	$9.2e^{-06}$	6	148634.50	$3.3e^{-06}$	4

Table (1)

μ	Λ	ν	μ	Λ	ν
500	$9.9e^{-02}$	6	3803.50	$4.4e^{-07}$	4
500.05	$9.4e^{-02}$	6	3803.55	$3.1e^{-08}$	4
500.10	$8.7e^{-02}$	6	3803.60	$7.5e^{-07}$	4
500.15	$5.2e^{-02}$	6	$\vdots$		
500.20	$3.1e^{-03}$	6	14617.60	$9.7e^{-08}$	5
500.25	$2.1e^{-04}$	6	14617.65	$4.5e^{-09}$	5
500.30	$9.6e^{-05}$	6	14617.70	$3.2e^{-08}$	5
500.35	$8.5e^{-05}$	6	$\vdots$		
500.40	$2.5e^{-06}$	6	39943.75	$1.1e^{07}$	4
500.45	$1.9e^{-06}$	6	39943.80	$5.4e^{-09}$	4
500.50	$1.8e^{-07}$	5	$\vdots$		
500.55	$1.1e^{-08}$	5	89135.35	$9.6e^{-08}$	4
500.60	$4.1e^{-07}$	5	89135.40	$8.9e^{-09}$	4
500.65	$8.2e^{-06}$	5	89135.45	$6.7e^{-07}$	4
500.70	$7.4e^{-06}$	5	$\vdots$		
500.75	$5.5e^{-05}$	5	173881.30	$5.1e^{-08}$	4
500.80	$6.7e^{-04}$	5	173881.35	$4.3e^{-07}$	4

Table (2)

4 Open problem

Future work will be on the extension of the present study to higher-order Sturm-Liouville problems such as sixth-order of the form

$$[p(x)y'''(x)]''' = [s(x)y''(x)]'' - [r(x)y'(x)]' - [\lambda w(x) - q(x)]\, y(x), \qquad (24)$$

subject to

$$\alpha_j y^{(j)}(-1) + \beta_j y^{(j)}(1) = 0, \quad (j = 0, \cdots, 5), \qquad (25)$$

where $p,\ s,\ r,\ w$ and q are piecewise continuous functions with $p(x) > 0$ and $w(x) \geq 0$ for all $x \in (-1, 1)$. Here α_j and β_j (for $j = 0, \cdots, 5$) are constants.

5 Conclusion

We discuss the determination of the location of eigenvalues for the fourth order Sturm–Liouville problems using a combination of Tau and Lanczos methods. The numerical results for the examples demonstrate the efficacy and accuracy of this method. Moreover, the number of evaluations ν shows that the present approach is not expensive.

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