Normal Families of Meromorphic Functions which Omit a Function Set

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Abstract
In this paper, a particular family of meromorphic functions, which omits a function set is considered. By using the famous Zalcman-Pang lemma, we derive a sufficient condition for the normality of this particular meromorphic functions family.

Keywords: meromorphic functions family, normality, Zalcman-Pang lemma.

1 Introduction and Main Results

In complex analysis, the analytic or meromorphic functions family with particular analytic or meromorphic structure is interesting and significant (see [1], [2], [3] and [4] for examples).

In this paper we deal with the normality of the meromorphic functions family omitting some functions. In [5], Yang proved that for a family of meromorphic functions \( F \) on a domain \( D \) in \( \mathbb{C} \), and let \( h \) be a function holomorphic on \( D \) and \( h(z) \neq 0 \). Suppose that for each \( f \in F \), \( f(z) \neq 0 \) and \( f^{(k)}(z) \neq h(z) \) for \( z \in D \), then \( F \) is a normal family on \( D \).

More recently, Pang and Zalcman [6] and [7] observed the following results:

**Theorem 1.1** Let \( F \) be a family of functions meromorphic on a domain \( D \) in \( \mathbb{C} \), all of whose zeros have multiplicity at least 4, and \( h \) be a function holomorphic on \( D \) such that \( h(z) \neq 0 \). Suppose that for each \( f \in F \), \( f(z) \neq 0 \) and \( f^{(k)}(z) \neq h(z) \) for \( z \in D \), then \( F \) is a normal family on \( D \).

**Theorem 1.2** Let \( F \) be a family of functions meromorphic on a domain \( D \) in \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k+3 \), and \( h \) be a function holomorphic on \( D \) such that \( h(z) \neq 0 \). Suppose that for each \( f \in F \), \( f(z) \neq 0 \) and \( f^{(k)}(z) \neq h(z) \) for \( z \in D \), then \( F \) is a normal family on \( D \).
In this paper, we deal with a generalization of the meromorphic functions family omitting a function, and study a particular family of meromorphic functions which omit a function set. We prove that this particular family of meromorphic functions has a well normality by using the famous Zalcman-Pang lemma. Firstly we present the definition of meromorphic functions omitting a function set.

**Definition 1.3 (Meromorphic Functions Omitting a Function Set)**

Let $D$ be a domain in $\mathbb{C}$, $f$ be a function meromorphic on $D$ and $S$ is a set including finite meromorphic functions on $D$:

$$S = \{ h_i(z) | z \in D, i = 1, \cdots l \}.$$

If for arbitrary $1 \leq i \leq l$ and $z \in D$ we have $f(z) \neq h_i(z)$, then the meromorphic function $f$ is said to omit the function set $S$.

**Remark 1.** It obvious that the meromorphic functions family which omits a function set is a natural generalization of the one omitting a function.

In this paper we will generalize the results in [5], [6] and [7] for the meromorphic functions family which omits a function to the one omitting a function set, and the main result of our paper is as follows:

**Theorem 1.4** Let $F$ be a family of functions meromorphic on a domain $D$ in $\mathbb{C}$, all of whose zeros have multiplicity at least $k$, and $S = \{ h_i(z) | z \in D, i = 1, \cdots l \}$ be a holomorphic functions set on $D$ such that $h_i(z) \neq 0$ for arbitrary $1 \leq i \leq l$, the zeros of $h_i(z)$ have multiplicity $m_i$ which satisfies $k \nmid m_i$ for arbitrary $1 \leq i \leq l$. Suppose that for each $f \in F$, $|f^{(k)}(z)| < \min_{1 \leq i \leq l} |h_i(z)|$ whenever $f(z) = 0$ and $f^{(k)}(z)$ omits the function set $S$, then $F$ is a normal family on $D$.

The paper is organized as follows. In section 2, we present some preliminary lemmas. In section 3, we prove Theorem 1.4 by using Zalcman-Pang’s approach. In section 4, we give two interesting open problems.

## 2 Preliminary Results

In order to prove our main theorem, we need the following preliminary results.

**Lemma 2.1 (Zalcman-Pang)** Let $F$ be a family of functions meromorphic on the unit disc $\Delta$, all of whose zeros have multiplicity at least $k$, and suppose that there exists $M \geq 1$, such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$. Then if $F$ is not normal at $z_0$, for each $-1 \leq \alpha \leq k$, there exist

a) points $z_n \in \Delta$, $z_n \to z_0$;
b) functions \( f_n \in F \);

c) positive numbers \( \rho_n \to 0^+ \), such that

\[
\frac{f_n(z_n + \rho_n \xi)}{\rho_n^k} = g_n(\xi) \to g(\xi)
\]  

(1)

uniformly with respect to the spherical metric

\[
\|f(z) - g(z)\| = \frac{|f(z) - g(z)|}{\sqrt{1 + |f(z)|^2 \sqrt{1 + |g(z)|^2}}}
\]  

(2)

on compact subsets of \( \mathbb{C} \), where \( g \) is a nonconstant meromorphic function on \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k \), such that \( g^\#(\xi) \leq g^\#(0) = kM + 1 \). In particular, \( g \) has order at most 2 and \( g^\# \) denotes

\[
g^\#(z_0) = \lim_{z \to z_0} \frac{|g(z) - g(z_0)|}{|z - z_0|} = \frac{|g'(z_0)|}{1 + |g(z_0)|^2}.
\]

Lemma 2.2 (Hurwitz) Let \( \{f_n(z)\} \) be a family of functions meromorphic on a domain \( D \) in \( \mathbb{C} \) and converge to \( f(z) \) uniformly on compact subsets of \( D \). If \( f(z) = a \) has a solution on \( D \), then when \( n \) is large enough, \( f_n(z) = a \) also has solutions on \( D \).

3 Proof of the Main Theorem

With the help of above lemmas, we then prove our main theorem.

Proof of Theorem 1.4. First we show that \( F \) is normal on the subset \( D' \) of \( D \), where \( h_i(z) \neq 0 \) for arbitrary \( 1 \leq i \leq l \). Suppose then that \( F \) is not normal at \( z_0 \in D' \), we may assume that \( D = \Delta \) and let \( M = \min_{1 \leq i \leq l} |h_i(z)| + 1 \geq 1 \). By Lemma 2.1, there exist \( f_n \in F \), \( z_n \in \Delta \), \( z_n \to z_0 \) and \( \rho_n \to 0^+ \) such that

\[
\frac{f_n(z_n + \rho_n \xi)}{\rho_n^k} = g_n(\xi) \to g(\xi)
\]

spherically uniformly on compact subsets of \( \mathbb{C} \), where \( g \) is a nonconstant meromorphic function on \( \mathbb{C} \), all of whose zeros have order at least \( k \) and satisfies

\[
g^\#(\xi) \leq g^\#(0) = kM + 1 = k \left( \min_{1 \leq i \leq l} |h_i(z)| + 1 \right) + 1.
\]

(3)

We then claim that:

Claim 3.1 If \( g(\xi) = 0 \), then \( |g^{(k)}(\xi)| \leq \min_{1 \leq i \leq l} |h_i(z_0)| \)
Proof of Claim 3.1. Indeed, suppose that if \( g(\xi_0) = 0 \) and \( g(\xi) \neq 0 \), by Hurwitz’s Theorem there exists \( \xi_n \rightarrow \xi_0 \) such that (for \( n \) sufficiently large)

\[
\frac{f_n(z_n + \rho_n \xi_n)}{\rho_n^k} = g_n(\xi_n) = 0,
\]

thus \( f_n(z_n + \rho_n \xi_n) = 0 \). It follows from the hypotheses on \( F \) that \( |f_n^{(k)}(z_n + \rho_n \xi_n)| < \min_{1 \leq i \leq l} |h_i(z_n + \rho_n \xi_n)| \), hence

\[
|g_n^{(k)}(\xi_n)| = |f_n^{(k)}(z_n + \rho_n \xi_n)| < \min_{1 \leq i \leq l} |h_i(z_n + \rho_n \xi_n)|
\]

Let \( n \rightarrow \infty \), then we complete the proof of Claim 3.1.

Since

\[
g_n^{(k)}(\xi) - h_i(z_n + \rho_n \xi) = f_n^{(k)}(z_n + \rho_n \xi) - h_i(z_n + \rho_n \xi) \neq 0
\]

for arbitrary \( 1 \leq i \leq l \), by Hurwitz’s Theorem we have either

(i) there exists \( 1 \leq i_0 \leq l \) such that \( g^{(k)}(\xi) \equiv h_{i_0}(z_0) \), or

(ii) for each \( 1 \leq i \leq l \) we always have \( g^{(k)}(\xi) \neq h_i(z_0) \).

If (i) satisfies, since the zeros of \( g \) have order at least \( k \), we have \( g(\xi) = \frac{h_{i_0}(z_0)}{k!} (\xi - \xi_0)^k \), by Claim 3.1 it follows that

\[
|h_{i_0}(z_0)| = |g^{(k)}(\xi_0)| \leq \min_{1 \leq i \leq l} |h_i(z_0)|.
\]

Moreover from the expression of \( g \), one gets

\[
g^\#(0) \leq \begin{cases} \frac{k}{2}, & |\xi_0| \geq 1 \\ |h_{i_0}(z_0)|, & |\xi_0| < 1, \end{cases}
\]

then (4) and (5) lead a contradiction to (3).

If (ii) satisfies, it follows that \( g^{(k)}(\xi) = h_{i_0}(z_0) + a\xi + b \) for some \( 1 \leq i_0 \leq l \). We divided this case into two parts:

(a) If \( a = 0 \), then \( g^{(k)}(\xi) = h_{i_0}(z_0) + c \), since the zeros of \( g \) have order at least \( k \), we have \( g(\xi) = \frac{h_{i_0}(z_0)+c}{k!} (\xi - \xi_1)^k \), then it follows from Claim 3.1 that

\[
|h_{i_0}(z_0) + c| = |g^{(k)}(\xi_1)| \leq \min_{1 \leq i \leq l} |h_i(z_0)|.
\]

Moreover

\[
g^\#(0) \leq \begin{cases} \frac{k}{2}, & |\xi| \geq 1 \\ |h_{i_0}(z_0) + c|, & |\xi| < 1, \end{cases}
\]
thus (6) and (7) also lead a contradiction to (3).

(b) If $a \neq 0$, we have $g(\xi) = \frac{h_{i_0}(z_0)}{k!} \xi^k + a_1 \xi^{k-1} + \cdots + a_k + \frac{e^{a\xi+b}}{a^k}$. It follows that there exist infinite $\xi_n \to \infty$ such that $g(\xi_n) = 0$, that is to say

$$a^k \left( \frac{h_{i_0}(z_0)}{k!} \xi_n^k + a_1 \xi_n^{k-1} + \cdots + a_k - \frac{h_{i_0}(z_0)}{a^k} \right) = -h_{i_0}(z_0) - e^{a\xi_n+b}.$$ 

By Claim 3.1 we have

$$|a^k| \left| \frac{h_{i_0}(z_0)}{k!} \xi_n^k + a_1 \xi_n^{k-1} + \cdots + a_k - \frac{h_{i_0}(z_0)}{a^k} \right| = |h_{i_0}(z_0) + e^{a\xi_n+b}| = |g^{(k)}(\xi_n)| \leq \min_{1 \leq i \leq l} |h_i(z_0)|,$$

which has a contradiction to $\xi_n \to \infty$.

By all of above, we prove that $F$ is normal on the subset $D'$ of $D$, where $h_i(z) \neq 0$ for arbitrary $1 \leq i \leq l$.

We now turn to prove $F$ is normal at points for which exists $1 \leq i_0 \leq l$ such that $h_{i_0}(z) = 0$. Making standard normalizations, we may assume that $h_{i_0}(z) = z^m b(z)$ for $z \in D$, $m \geq 1$, $b(0) = 1$ and $h_{i_0}(z) \neq 0$ for $0 < |z| < 1$. Let

$$G = \left\{ G(z) = \frac{f(z)}{z^m} | f \in F \right\},$$

since $|f^{(k)}(0)| \neq \min_{1 \leq i \leq l} |h_i(0)| = 0$, $f(0) \neq 0$, we have $F(0) = \infty$. We then prove $G$ is normal at 0. Suppose not, then by Lemma 2.1, there exist $G_n \in G$, $z_n \to 0$ and $\rho_n \to 0^+$ such that

$$g_n(\xi) = \frac{G_n(z_n + \rho_n \xi)}{\rho_n^k} = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^k} \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have order at least $k$ and satisfies

$$g^#(\xi) \leq g^#(0) = kM + 1. \quad (8)$$

We then consider the following two cases:

(i) Suppose $z_n/\rho_n \to \infty$, we have

$$f_n^{(k)}(z) = z^m G_n^{(k)}(z) + \sum_{j=1}^k c_j z^{m-j} G_n^{(k-j)}(z),$$
where
\[ c_j = \begin{cases} m (m - 1) (m - j + 1), & j \leq m \\ 0, & j > m. \end{cases} \]

Since \( \rho_j g_n^{(k-j)} (\xi) = G_n^{(k-j)} (z_n + \rho_n \xi) \) for arbitrary \( 0 \leq j \leq m \), we obtain
\[
\frac{f_n^{(k)} (z_n + \rho_n \xi)}{h_{io} (z_n + \rho_n \xi)} = \left( g_n^{(k)} (\xi) + \sum_{j=1}^{k} c_j \frac{g_n^{(k-j)} (z_n + \rho_n \xi)}{(z_n/\rho_n + \xi)^j} \right) \frac{1}{b(z_n + \rho_n \xi)}. \tag{9}
\]

Now
\[
\lim_{n \to \infty} \frac{c_j}{(z_n/\rho_n + \xi)^j} = 0 \tag{10}
\]
for arbitrary \( 1 \leq j \leq m \) and
\[
\lim_{n \to \infty} \frac{1}{b(z_n + \rho_n \xi)} = 1. \tag{11}
\]

It follows from (9), (10) and (11) that
\[
\frac{f_n^{(k)} (z_n + \rho_n \xi)}{h_{io} (z_n + \rho_n \xi)} \to g^{(k)} (\xi) \tag{12}
\]
uniformly on compact subsets of \( \mathbb{C} \) disjoint from the poles of \( g \).

By \( \rho_j g_n^{(k-j)} (\xi) = G_n^{(k-j)} (z_n + \rho_n \xi) \) and (9) we obtain that the \( M \) in (8) is equal to 1, thus by using (12) we can prove the following claim as the proof of Claim 3.1.

**Claim 3.2** If \( g(\xi) = 0 \), then \( |g^{(k)} (\xi)| \leq 1. \)

Moreover since it follows from (12) that
\[
g_n^{(k)} (\xi) + o(1) = \frac{f_n^{(k)} (z_n + \rho_n \xi)}{h_{io} (z_n + \rho_n \xi)} \neq 1,
\]
Hurwitz’s Theorem implies that either
(i) \( g^{(k)} (\xi) \equiv 1 \), or
(ii) \( g^{(k)} (\xi) \neq 1 \) for arbitrary \( \xi \).

If (i) satisfies, since the zeros of \( g \) have order at least \( k \), one get \( g(\xi) = \frac{1}{k!} (\xi - \xi_0)^k \), thus
\[
g^# (0) \leq \begin{cases} \frac{k}{2}, & |\xi_0| \geq 1 \\ 1, & |\xi_0| < 1, \end{cases}
\]
which contradicts \( g^# (0) = k + 1. \)
If (ii) satisfies, it follows that \( g^{(k)}(\xi) = 1 + e^{a\xi + b} \). We divided this case into two parts:

(a) If \( a = 0 \), then \( g^{(k)}(\xi) = 1 + c \xi \), since the zeros of \( g \) have order at least \( k \), it follows that \( g(\xi) = 1 + c(\xi - \xi_1)^k \). By Claim 3.2 we have
\[
|1 + c| = |g^{(k)}(\xi_1)| \leq 1. \tag{13}
\]
Moreover from the expression of \( g \), one gets
\[
g^\#(0) \leq \begin{cases} \frac{k}{2}, & |\xi_1| \geq 1 \\ |1 + c|, & |\xi_1| < 1 \end{cases}, \tag{14}
\]
thus (13) and (14) also lead a contradiction to \( g^\#(0) = k + 1 \).

(b) If \( a \neq 0 \), we have \( g(\xi) = \frac{1}{k!} \xi^k + a_1 \xi^k + \cdots + a_k + e^{a\xi + b} \). It follows that there exist infinite \( \xi_n \to \infty \) such that \( g(\xi_n) = 0 \), that is to say
\[
a^k \left( \frac{1}{k!} \xi_n^k + a_1 \xi_n^{k-1} + \cdots + a_k - \frac{1}{a^k} \right) = -1 - e^{a\xi_n + b}.
\]
By Claim 3.2, we have
\[
|a^k| \left| \frac{1}{k!} \xi_n^k + a_1 \xi_n^{k-1} + \cdots + a_k - \frac{1}{a^k} \right| = |1 + e^{a\xi_n + b}| = |g^{(k)}(\xi_n)| \leq 1,
\]
which has a contradiction to \( \xi_n \to \infty \).

(ii) So that we may assume that \( z_n/\rho_n \to \alpha \), which is a finite complex number. Then we have
\[
\frac{G_n(\rho_n \xi)}{\rho_n^k} = \frac{G_n(z_n + \rho_n (\xi - z_n/\rho_n))}{\rho_n^k} \to g(\xi - \alpha) = \tilde{g}(\xi),
\]
the convergence being spherically uniform on compact sets of \( \mathbb{C} \), hence uniform on compact disjoint from the poles of \( \tilde{g} \). Clearly, all zeros of \( \tilde{g} \) have order at least \( k \), and the pole of \( \tilde{g} \) at \( \xi = 0 \) has order at least \( m \). Now
\[
K_n(\xi) = \frac{f_n(\rho_n \xi)}{\rho_n^{k+m}} = \frac{G_n(\rho_n \xi) (\rho_n \xi)^m}{\rho_n^m} \to \xi^m \tilde{g}(\xi) = K(\xi) \tag{15}
\]
uniformly on compact subsets of \( \mathbb{C} \) disjoint from the poles of \( \tilde{g} \), and \( \lim_{n \to \infty} h_{k_n}(\rho_n \xi) = \xi^m \) uniformly on compact subsets of \( \mathbb{C} \). Note that since \( \tilde{g} \) has a pole of order at least \( k \) at \( \xi = 0 \), \( K(0) \neq 0 \) and all zeros of \( K \) have order at least \( k \). Furthermore since
\[
K^{(k)}(\xi) - \frac{h_{k_n}(\rho_n \xi)}{\rho_n^m} = \frac{f_n(\rho_n \xi) - h_{k_n}(\rho_n \xi)}{\rho_n^m} \neq 0,
\]
it follows from \( \lim_{n \to \infty} \frac{h_n(\rho_n \xi)}{\rho_n^m} = \xi^m \) and Hurwitz’s Theorem that either
(i) \( K^{(k)}(\xi) \equiv \xi^m \), or
(ii) \( K^{(k)}(\xi) \neq \xi^m \) for arbitrary \( \xi \).

If (i) satisfies, it follows that \( K \) is a polynomial of multiplicity \( m+k \). If all zeros of \( K \) has order \( k \), we have \( m+k = pk \), which has a contradiction to the zeros of \( h_i(z) \) having multiplicity \( m_i \) such that \( k \mid m_i \) for arbitrary \( 1 \leq i \leq l \). If there exists a zero of \( K \) with order at least \( k+1 \), then \( K^{(k)}(\xi) \) must vanish at any points where \( K(\xi) \) vanishes. On the other hand, \( K^{(k)}(\xi) \neq 0 \) for \( \xi \neq 0 \), thus we have \( K(0) = 0 \), a contradiction.

If (ii) satisfies, we have \( g^{(k)}(\xi) = 1 + e^{a\xi+b} \), firstly we claim that:

**Claim 3.3** If \( K(\xi) = 0 \), then \( |K^{(k)}(\xi)| \leq |\xi|^m \).

**Proof of Claim 3.3.** Indeed, suppose that if \( K(\xi_0) = 0 \) and \( K(\xi) \), by Hurwitz’s Theorem there exists \( \xi_n \to \xi_0 \) such that (for \( n \) sufficiently large)
\[
\frac{f_n(\rho_n \xi_n)}{\rho_n^{k+m}} = K_n(\xi_n) = 0,
\]
thus \( f_n(\rho_n \xi_n) = 0 \). It follows from the hypotheses on \( \mathcal{F} \) that \( |f_n^{(k)}(\rho_n \xi_n)| < \frac{1}{\min_{1 \leq i \leq l} |h_i(\rho_n \xi_n)|} \), hence
\[
|K_n^{(k)}(\xi_n)| = \left| \frac{f_n^{(k)}(\rho_n \xi_n)}{\rho_n^{m}} \right| \leq \left| \frac{h_n(\rho_n \xi_n)}{\rho_n^{m}} \right| = \left| \frac{(\rho_n \xi_n)^m b(\rho_n \xi_n)}{\rho_n^{m}} \right| = |\xi_n|^m |b(\rho_n \xi_n)|.
\]
Let \( n \to \infty \), we complete the proof of Claim 3.3.

We then divided this case into two parts:

(a) If \( a = 0 \), as the proof of case (i) we obtain a contradiction.

(b) If \( a \neq 0 \), we have
\[
K(\xi) = \frac{1}{(k+m)\cdots(m+1)}\xi^{k+m} + a_1\xi^{k-1} + \cdots + a_k + \frac{e^{a\xi+b}}{a^k}.
\]
It follows that there exist infinite \( \xi_n \to \infty \) such that \( K(\xi_n) = 0 \), that is to say
\[
da^k \left( \frac{1}{(k+m)\cdots(m+1)}\xi_n^{k+m} + a_1\xi_n^{k-1} + \cdots + a_k - \frac{\xi_n^m}{a^k} \right) = -\xi_n^m - e^{a\xi_n+b}.
\]
By using Claim 3.3, we have
\[
\left| a_k \right| \left| \frac{1}{(k + m) \cdots (m + 1)} \xi_n^{k+m} + a_1 \xi_n^{k-1} + \cdots + a_k - \frac{\xi_n^m}{a_k} \right| = \left| \xi_n^m + e^{a_n+b} \right| = \left| K^{(k)}(\xi_n) \right| \leq |\xi_n|^m,
\]
which also has a contradiction to \( \xi_n \to \infty \).

The contradiction establishes \( G \) is normal at 0. It remains to prove that \( F \) is normal at 0. Since \( G \) is normal at 0 and \( G(0) = \infty \) for each \( G(z) \in G \), there exists \( \delta > 0 \) such that if \( G(z) \in G \), then \( |G(z)| \geq 1 \) for all \( z \in \Delta_\delta = \{ z : |z| < \delta \} \). Thus \( f(z) \neq 0 \) for \( z \in \Delta_\delta \) and for all \( f \in F \), which is equivalent to \( 1/f \) is analytic in \( \Delta_\delta \) for all \( f \in F \). Therefore, for all \( f \in F \), we have
\[
\left| \frac{1}{f(z)} \right| = \left| \frac{1}{G(z)} \frac{1}{|z|^k} \right| \leq \frac{2^k}{\delta^k} |z| = \frac{\delta}{2}, \quad (16)
\]
By the Maximum Principle and Montel’s Theorem, \( F \) is normal at \( \xi = 0 \). This completes the proof of Theorem 1.4.

4 Open Problem

There are two interesting open problems related to our paper, one open problem is related to the meromorphic differential polynomial which omit a function set:

**Problem 4.1** Let \( F \) be a family of functions meromorphic on a domain \( D \) in \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k \), and \( S = \{ h_i(z) \mid z \in D, i = 1, \cdots l \} \) be a holomorphic functions set on \( D \) such that \( h_i(z) \neq 0 \) for arbitrary \( 1 \leq i \leq l \), the zeros of \( h_i(z) \) have multiplicity \( m_i \) which satisfies \( k \nmid m_i \) for arbitrary \( 1 \leq i \leq l \). Suppose that for each \( f \in F \),
\[
\max_{1 \leq j \leq k} \left| f^{(j)}(z) \right| < \min_{1 \leq i \leq l} \left| h_i(z) \right|
\]
ever \( f(z) = 0 \) and the differential polynomial \( \sum_{j=1}^{k} p_j(z)f^{(j)}(z) \) omits the function set \( S \), is \( F \) is a normal family on \( D \)?

Another open problem is related to the meromorphic functions family which shares a function set, firstly we present the definition of meromorphic functions sharing a function set as follows:

**Definition 4.2** (Meromorphic Functions Sharing a Function Set) Let \( D \) be a domain in \( \mathbb{C} \), \( f \) and \( g \) meromorphic on \( D \) and \( S \) is a set including finite meromorphic functions on \( D \):
\[
\mathcal{S} = \{ h_i(z) \mid z \in D, i = 1, \cdots l \}.
\]
If \( f \) and \( g \) satisfies
\[
E_f(S) = \{ z \in D | f(z) = h_i(z), \exists 1 \leq i \leq l \}
= \{ z \in D | g(z) = h_j(z), \exists 1 \leq j \leq l \}
= E_g(S),
\]
then the two meromorphic functions \( f \) and \( g \) on \( D \) are said to share the function set \( S \).

The open problem is as follows:

**Problem 4.3** Let \( \mathcal{F} \) be a family of functions meromorphic on a domain \( D \) in \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k \). If there exists a holomorphic function set on \( D \)
\[ S = \{ h_i(z) \mid z \in D, i = 1, \ldots, l \}, \]
where \( h_i(z) \neq 0 \) for arbitrary \( z \in D \) and \( 1 \leq i \leq l \), such that for each \( f \in \mathcal{F}, E_f(S) = E_{f^{(k)}}(S) \) and \( 0 < |f^{(k)}(z)| \leq \sup_{1 \leq i \leq l} |h_i(z)| \) whenever \( z \in E_f(0) \), is \( \mathcal{F} \) is a normal family on \( D \)?

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**References**


