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Normal Families of Meromorphic Functions which Omit a Function Set

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Abstract

In this paper, a particular family of meromorphic functions, which omits a function set is considered. By using the famous Zalcman-Pang lemma, we derive a sufficient condition for the normality of this particular meromorphic functions family.

Keywords: *meromorphic functions family, normality, Zalcman-Pang lemma.*

1 Introduction and Main Results

In complex analysis, the analytic or meromorphic functions family with particular analytic or meromorphic structure is interesting and significant (see [1], [2], [3] and [4] for examples).

In this paper we deal with the normality of the meromorphic functions family omitting some functions. In [5], Yang proved that for a family of meromorphic functions $\mathcal{F}$ on a domain D in $\mathbb{C}$, and let h be a function holomorphic on D and $h(z) 0$. Suppose that for each $f \in \mathcal{F}$, $f(z) \neq 0$ and $f^{(k)}(z) \neq h(z)$ for $z \in D$, then $\mathcal{F}$ is a normal family on D.

More recently, Pang and Zalcman [6] and [7] observed the following results:

Theorem 1.1 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose zeros have multiplicity at least 4, and h be a function holomorphic on D such that $h(z) 0$. Suppose that for each $f \in \mathcal{F}$, $f(z) \neq 0$ and $f'(z) \neq h(z)$ for $z \in D$, then $\mathcal{F}$ is a normal family on D.*

Theorem 1.2 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose zeros have multiplicity at least $k+3$, and h be a function holomorphic on D such that $h(z) 0$. Suppose that for each $f \in \mathcal{F}$, $f(z) \neq 0$ and $f^{(k)}(z) \neq h(z)$ for $z \in D$, then $\mathcal{F}$ is a normal family on D.*

In this paper, we deal with a generalization of the meromorphic functions family omitting a function, and study a particular family of meromorphic functions which omit a function set. We prove that this particular family of meromorphic functions has a well normality by using the famous Zalcman-Pang lemma. Firstly we present the definition of meromorphic functions omitting a function set.

Definition 1.3 (Meromorphic Functions Omitting a Function Set) *Let D be a domain in $\mathbb{C}$, f be a function meromorphic on D and $\mathcal{S}$ is a set including finite meromorphic functions on D:*

$$S = \{h_i(z)\,|z \in D, i = 1, \cdots l\}.$$

If for arbitrary $1 \leq i \leq l$ and $z \in D$ we have $f(z) \neq h_i(z)$, then the meromorphic function f is said to omit the function set $\mathcal{S}$.

Remark 1. It obvious that the meromorphic functions family which omits a function set is a natural generalization of the one omitting a function.

In this paper we will generalize the results in [5], [6] and [7] for the meromorphic functions family which omits a function to the one omitting a function set, and the main result of our paper is as follows:

Theorem 1.4 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose zeros have multiplicity at least k, and $\mathcal{S} = \{h_i(z)\,|z \in D, i = 1, \cdots l\}$ be a holomorphic functions set on D such that $h_i(z)\,0$ for arbitrary $1 \leq i \leq l$, the zeros of $h_i(z)$ have multiplicity m_i which satisfies $k \nmid m_i$ for arbitrary $1 \leq i \leq l$. Suppose that for each $f \in \mathcal{F}$, $\left|f^{(k)}(z)\right| < \min\limits_{1 \leq i \leq l} |h_i(z)|$ whenever $f(z) = 0$ and $f^{(k)}(z)$ omits the function set $\mathcal{S}$, then $\mathcal{F}$ is a normal family on D.*

The paper is organized as follows. In section 2, we present some preliminary lemmas. In section 3, we prove Theorem 1.4 by using Zalcman-Pang's approach. In section 4, we give two interesting open problems.

2 Preliminary Results

In order to prove our main theorem, we need the following preliminary results.

Lemma 2.1 (Zalcman-Pang) *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $M \geq 1$, such that $\left|f^{(k)}(z)\right| \leq M$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, for each $-1 \leq \alpha \leq k$, there exist*

a) points $z_n \in \Delta, z_n \to z_0$;

b) functions $f_n \in \mathcal{F}$;
c) positive numbers $\rho_n \to 0^+$, such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi) \tag{1}$$

uniformly with respect to the spherical metric

$$\|f(z) - g(z)\| = \frac{|f(z) - g(z)|}{\sqrt{1 + |f(z)|^2}\sqrt{1 + |g(z)|^2}} \tag{2}$$

on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\xi) \le g^{\#}(0) = kM + 1$. In particular, g has order at most 2 and $g^{\#}$ denotes

$$g^{\#}(z_0) = \lim_{z \to z_0} \frac{\|g(z) - g(z_0)\|}{|z - z_0|} = \frac{|g'(z_0)|}{1 + |g(z_0)|^2}$$

Lemma 2.2 (Hurwitz) *Let $\{f_n(z)\}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$ and converge to $f(z)$ uniformly on compact subsets of D. If $f(z) = a$ has a solution on D, then when n is large enough, $f_n(z) = a$ also has solutions on D.*

3 Proof of the Main Theorem

With the help of above lemmas, we then prove our main theorem.

Proof of Theorem 1.4. First we show that $\mathcal{F}$ is normal on the subset D' of D, where $h_i(z) \ne 0$ for arbitrary $1 \le i \le l$. Suppose then that $\mathcal{F}$ is not normal at $z_0 \in D'$, we may assume that $D = \Delta$ and let $M = \min\limits_{1 \le i \le l} |h_i(z)| + 1 \ge 1$. By Lemma 2.1, there exist $f_n \in \mathcal{F}$, $z_n \in \Delta, z_n \to z_0$ and $\rho_n \to 0^+$ such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{k}} = g_n(\xi) \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have order at least k and satisfies

$$g^{\#}(\xi) \le g^{\#}(0) = kM + 1 = k\left(\min_{1 \le i \le l} |h_i(z)| + 1\right) + 1. \tag{3}$$

We then claim that:

Claim 3.1 *If $g(\xi) = 0$, then $\left|g^{(k)}(\xi)\right| \le \min\limits_{1 \le i \le l} |h_i(z_0)|$*

Proof of Claim 3.1. Indeed, suppose that if $g(\xi_0) = 0$ and $g(\xi)\, 0$, by Hurwitz's Theorem there exists $\xi_n \to \xi_0$ such that (for n sufficiently large)

$$\frac{f_n(z_n + \rho_n \xi_n)}{\rho_n^k} = g_n(\xi_n) = 0,$$

thus $f_n(z_n + \rho_n \xi_n) = 0$. It follows from the hypotheses on $\mathcal{F}$ that $\left|f_n^{(k)}(z_n + \rho_n \xi_n)\right| < \min\limits_{1 \le i \le l} |h_i(z_n + \rho_n \xi_n)|$, hence

$$\left|g_n^{(k)}(\xi_n)\right| = \left|f_n^{(k)}(z_n + \rho_n \xi_n)\right| < \min_{1 \le i \le l} |h_i(z_n + \rho_n \xi_n)|$$

Let $n \to \infty$, then we complete the proof of Claim 3.1.

Since

$$g_n^{(k)}(\xi) - h_i(z_n + \rho_n \xi) = f_n^{(k)}(z_n + \rho_n \xi) - h_i(z_n + \rho_n \xi) \neq 0$$

for arbitrary $1 \le i \le l$, by Hurwitz's Theorem we have either
(i) there exists $1 \le i_0 \le l$ such that $g^{(k)}(\xi) \equiv h_{i_0}(z_0)$, or
(ii) for each $1 \le i \le l$ we always have $g^{(k)}(\xi) \neq h_i(z_0)$.

If (i) satisfies, since the zeros of g have order at least k, we have $g(\xi) = \frac{h_{i_0}(z_0)}{k!}(\xi - \xi_0)^k$,
by Claim 3.1 it follows that

$$|h_{i_0}(z_0)| = \left|g^{(k)}(\xi_0)\right| \le \min_{1 \le i \le l} |h_i(z_0)|. \tag{4}$$

Moreover from the expression of g, one gets

$$g^{\#}(0) \le \begin{cases} \frac{k}{2}, & |\xi_0| \ge 1 \\ |h_{i_0}(z_0)|, & |\xi_0| < 1, \end{cases} \tag{5}$$

then (4) and (5) lead a contradiction to (3).

If (ii) satisfies, it follows that $g^{(k)}(\xi) = h_{i_0}(z_0) + e^{a\xi + b}$ for some $1 \le i_0 \le l$. We divided this case into two parts:

(a) If $a = 0$, then $g^{(k)}(\xi) = h_{i_0}(z_0) + c$, since the zeros of g have order at least k, we have $g(\xi) = \frac{h_{i_0}(z_0) + c}{k!}(\xi - \xi_1)^k$, then it follows from Claim 3.1 that

$$|h_{i_0}(z_0) + c| = \left|g^{(k)}(\xi_1)\right| \le \min_{1 \le i \le l} |h_i(z_0)|. \tag{6}$$

Moreover

$$g^{\#}(0) \le \begin{cases} \frac{k}{2}, & |\xi_1| \ge 1 \\ |h_{i_0}(z_0) + c|, & |\xi_1| < 1, \end{cases} \tag{7}$$

thus (6) and (7) also lead a contradiction to (3).

(b) If $a \neq 0$, we have $g(\xi) = \frac{h_{i_0}(z_0)}{k!}\xi^k + a_1\xi^{k-1} + \cdots + a_k + \frac{e^{a\xi+b}}{a^k}$. It follows that there exist infinite $\xi_n \to \infty$ such that $g(\xi_n) = 0$, that is to say

$$a^k\left(\frac{h_{i_0}(z_0)}{k!}\xi_n^k + a_1\xi_n^{k-1} + \cdots + a_k - \frac{h_{i_0}(z_0)}{a^k}\right) = -h_{i_0}(z_0) - e^{a\xi_n+b}.$$

By Claim 3.1 we have

$$\left|a^k\right|\left|\frac{h_{i_0}(z_0)}{k!}\xi_n^k + a_1\xi_n^{k-1} + \cdots + a_k - \frac{h_{i_0}(z_0)}{a^k}\right| = \left|h_{i_0}(z_0) + e^{a\xi_n+b}\right| = \left|g^{(k)}(\xi_n)\right| \leq \min_{1\leq i\leq l}\left|h_i(z_0)\right|,$$

which has a contradiction to $\xi_n \to \infty$.

By all of above, we prove that $\mathcal{F}$ is normal on the subset D' of D, where $h_i(z) \neq 0$ for arbitrary $1 \leq i \leq l$.

We now turn to prove $\mathcal{F}$ is normal at points for which exists $1 \leq i_0 \leq l$ such that $h_{i_0}(z) = 0$. Making standard normalizations, we may assume that $h_{i_0}(z) = z^m b(z)$
for $z \in D$, $m \geq 1$, $b(0) = 1$ and $h_{i_0}(z) \neq 0$ for $0 < |z| < 1$. Let

$$\mathcal{G} = \left\{\mathcal{G}(z) = \frac{f(z)}{z^m}\,|f \in \mathcal{F}\right\},$$

since $\left|f^{(k)}(0)\right| \neq \min_{1\leq i\leq l}|h_i(0)| = 0$, $f(0) \neq 0$, we have $F(0) = \infty$. We then prove $\mathcal{G}$ is normal at 0. Suppose not, then by Lemma 2.1, there exist $G_n \in \mathcal{G}$, $z_n \to 0$ and $\rho_n \to 0^+$ such that

$$g_n(\xi) = \frac{G_n(z_n + \rho_n\xi)}{\rho_n^k} = \frac{f_n(z_n + \rho_n\xi)}{\rho_n^k(z_n + \rho_n\xi)^m} \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have order at least k and satisfies

$$g^{\#}(\xi) \leq g^{\#}(0) = kM + 1. \tag{8}$$

We then consider the following two cases:

(i) Suppose $z_n/\rho_n \to \infty$, we have

$$f_n^{(k)}(z) = z^m G_n^{(k)}(z) + \sum_{j=1}^{k} c_j z^{m-j} G_n^{(k-j)}(z),$$

where

$$c_j = \begin{cases} m(m-1)(m-j+1), & j \le m \\ 0, & j > m. \end{cases}$$

Since $\rho_n^j g_n^{(k-j)}(\xi) = G_n^{(k-j)}(z_n + \rho_n \xi)$ for arbitrary $0 \le j \le m$, we obtain

$$\frac{f_n^{(k)}(z_n + \rho_n \xi)}{h_{i_0}(z_n + \rho_n \xi)} = \left(g_n^{(k)}(\xi) + \sum_{j=1}^{k} c_j \frac{g_n^{(k-j)}(z_n + \rho_n \xi)}{(z_n/\rho_n + \xi)^j} \right) \frac{1}{b(z_n + \rho_n \xi)}. \tag{9}$$

Now

$$\lim_{n\to\infty} \frac{c_j}{(z_n/\rho_n + \xi)^j} = 0 \tag{10}$$

for arbitrary $1 \le j \le m$ and

$$\lim_{n\to\infty} \frac{1}{b(z_n + \rho_n \xi)} = 1. \tag{11}$$

It follows from (9), (10) and (11) that

$$\frac{f_n^{(k)}(z_n + \rho_n \xi)}{h_{i_0}(z_n + \rho_n \xi)} \to g^{(k)}(\xi) \tag{12}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g.

By $\rho_n^j g_n^{(k-j)}(\xi) = G_n^{(k-j)}(z_n + \rho_n \xi)$ and (9) we obtain that the M in (8) is equal to 1, thus by using (12) we can prove the following claim as the proof of Claim 3.1.

Claim 3.2 *If $g(\xi) = 0$, then $\left|g^{(k)}(\xi)\right| \le 1$.*

Moreover since it follows from (12) that

$$g_n^{(k)}(\xi) + o(1) = \frac{f_n^{(k)}(z_n + \rho_n \xi)}{h_{i_0}(z_n + \rho_n \xi)} \ne 1,$$

Hurwitz's Theorem implies that either
(i) $g^{(k)}(\xi) \equiv 1$, or
(ii) $g^{(k)}(\xi) \ne 1$ for arbitrary ξ.

If (i) satisfies, since the zeros of g have order at least k, one get $g(\xi) = \frac{1}{k!}(\xi - \xi_0)^k$, thus

$$g^{\#}(0) \le \begin{cases} \frac{k}{2}, & |\xi_0| \ge 1 \\ 1, & |\xi_0| < 1, \end{cases}$$

which contradicts $g^{\#}(0) = k + 1$.

If (ii) satisfies, it follows that $g^{(k)}(\xi) = 1 + e^{a\xi+b}$. We divided this case into two parts:

(a) If $a = 0$, then $g^{(k)}(\xi) = 1 + c$, since the zeros of g have order at least k, it follows that $g(\xi) = \frac{1+c}{k!}(\xi - \xi_1)^k$. By Claim 3.2 we have

$$|1 + c| = \left|g^{(k)}(\xi_1)\right| \leq 1. \tag{13}$$

Moreover from the expression of g, one gets

$$g^{\#}(0) \leq \begin{cases} \frac{k}{2}, & |\xi_1| \geq 1 \\ |1 + c|, & |\xi_1| < 1, \end{cases} \tag{14}$$

thus (13) and (14) also lead a contradiction to $g^{\#}(0) = k + 1$.

(b) If $a \neq 0$, we have $g(\xi) = \frac{1}{k!}\xi^k + a_1\xi^{k-1} + \cdots + a_k + \frac{e^{a\xi+b}}{a^k}$. It follows that there exist infinite $\xi_n \to \infty$ such that $g(\xi_n) = 0$, that is to say

$$a^k\left(\frac{1}{k!}\xi_n^k + a_1\xi_n^{k-1} + \cdots + a_k - \frac{1}{a^k}\right) = -1 - e^{a\xi_n+b}.$$

By Claim 3.2, we have

$$\left|a^k\right|\left|\frac{1}{k!}\xi_n^k + a_1\xi_n^{k-1} + \cdots + a_k - \frac{1}{a^k}\right| = \left|1 + e^{a\xi_n+b}\right| = \left|g^{(k)}(\xi_n)\right| \leq 1,$$

which has a contradiction to $\xi_n \to \infty$.

(ii) So that we may assume that $z_n/\rho_n \to \alpha$, which is a finite complex number. Then we have

$$\frac{G_n(\rho_n\xi)}{\rho_n^k} = \frac{G_n(z_n + \rho_n(\xi - z_n/\rho_n))}{\rho_n^k} \to g(\xi - \alpha) = \tilde{g}(\xi),$$

the convergence being spherically uniform on compact sets of $\mathbb{C}$, hence uniform on compact disjoint from the poles of $\tilde{g}$. Clearly, all zeros of $\tilde{g}$ have order at least k, and the pole of $\tilde{g}$ at $\xi = 0$ has order at least m. Now

$$K_n(\xi) = \frac{f_n(\rho_n\xi)}{\rho_n^{k+m}} = \frac{G_n(\rho_n\xi)}{\rho_n^k}\frac{(\rho_n\xi)^m}{\rho_n^m} \to \xi^m\tilde{g}(\xi) = K(\xi) \tag{15}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of $\tilde{g}$, and $\lim\limits_{n\to\infty}\frac{h_{i_0}(\rho_n\xi)}{\rho_n^m} = \xi^m$ uniformly on compact subsets of $\mathbb{C}$. Note that since $\tilde{g}$ has a pole of order at least k at $\xi = 0$, $K(0) \neq 0$ and all zeros of K have order at least k. Furthermore since

$$K_n^{(k)}(\xi) - \frac{h_{i_0}(\rho_n\xi)}{\rho_n^m} = \frac{f_n^{(k)}(\rho_n\xi) - h_{i_0}(\rho_n\xi)}{\rho_n^m} \neq 0,$$

it follows from $\lim_{n\to\infty} \frac{h_{i_0}(\rho_n\xi)}{\rho_n^m} = \xi^m$ and Hurwitz's Theorem that either
(i) $K^{(k)}(\xi) \equiv \xi^m$, or
(ii) $K^{(k)}(\xi) \neq \xi^m$ for arbitrary ξ.

If (i) satisfies, it follows that K is a polynomial of multiplicity $m+k$. If all zeros of K has order k, we have $m+k = pk$, which has a contradiction to the zeros of $h_i(z)$ having multiplicity m_i such that $k \nmid m_i$ for arbitrary $1 \le i \le l$. If there exists a zero of K with order at least $k+1$, then $K^{(k)}(\xi)$ must vanish at any points where $K(\xi)$ vanishe. On the other hand, $K^{(k)}(\xi) \neq 0$ for $\xi \neq 0$, thus we have $K(0) = 0$, a contradiction.

If (ii) satisfies, we have $g^{(k)}(\xi) = 1 + e^{a\xi+b}$, firstly we claim that:

Claim 3.3 *If $K(\xi) = 0$, then* $\left|K^{(k)}(\xi)\right| \le |\xi|^m$.

Proof of Claim 3.3. Indeed, suppose that if $K(\xi_0) = 0$ and $K(\xi)0$, by Hurwitz's Theorem there exists $\xi_n \to \xi_0$ such that (for n sufficiently large)

$$\frac{f_n(\rho_n\xi_n)}{\rho_n^{k+m}} = K_n(\xi_n) = 0,$$

thus $f_n(\rho_n\xi_n) = 0$. It follows from the hypotheses on $\mathcal{F}$ that $\left|f_n^{(k)}(\rho_n\xi_n)\right| < \min_{1\le i\le l} |h_i(\rho_n\xi_n)|$,
hence

$$\left|K_n^{(k)}(\xi_n)\right| = \left|\frac{f_n^{(k)}(\rho_n\xi_n)}{\rho_n^m}\right| < \left|\frac{h_{i_0}(\rho_n\xi_n)}{\rho_n^m}\right| = \left|\frac{(\rho_n\xi_n)^m b(\rho_n\xi_n)}{\rho_n^m}\right| = |\xi_n|^m |b(\rho_n\xi_n)|.$$

Let $n \to \infty$, we complete the proof of Claim 3.3.

We then divided this case into two parts:

(a) If $a = 0$, as the proof of case (i) we obtain a contradiction.

(b) If $a \neq 0$, we have

$$K(\xi) = \frac{1}{(k+m)\cdots(m+1)}\xi^{k+m} + a_1\xi^{k-1} + \cdots + a_k + \frac{e^{a\xi+b}}{a^k}.$$

It follows that there exist infinite $\xi_n \to \infty$ such that $K(\xi_n) = 0$, that is to say

$$a^k\left(\frac{1}{(k+m)\cdots(m+1)}\xi_n^{k+m} + a_1\xi_n^{k-1} + \cdots + a_k - \frac{\xi_n^m}{a^k}\right) = -\xi_n^m - e^{a\xi_n+b}.$$

By using Claim 3.3, we have

$$\left|a^k\right|\left|\frac{1}{(k+m)\cdots(m+1)}\xi_n^{k+m}+a_1\xi_n^{k-1}+\cdots+a_k-\frac{\xi_n^m}{a^k}\right|=\left|\xi_n^m+e^{a\xi_n+b}\right|=\left|K^{(k)}\left(\xi_n\right)\right|\le\left|\xi_n\right|^m,$$

which also has a contradiction to $\xi_n\to\infty$.

The contradiction establishes $\mathcal{G}$ is normal at 0. It remains to prove that $\mathcal{F}$ is normal at 0. Since $\mathcal{G}$ is normal at 0 and $G(0)=\infty$ for each $G(z)\in\mathcal{G}$, there exists $\delta>0$ such that if $G(z)\in\mathcal{G}$, then $|G(z)|\ge 1$ for all $z\in\Delta_\delta=\{z\in||z|<\delta\}$. Thus $f(z)\ne 0$ for $z\in\Delta_\delta$ and for all $f\in\mathcal{F}$, which is equivalent to $1/f$ is analytic in Δ_δ for all $f\in\mathcal{F}$. Therefore, for all $f\in\mathcal{F}$, we have

$$\left|\frac{1}{f(z)}\right|=\left|\frac{1}{G(z)}\frac{1}{|z|^k}\right|\le\frac{2^k}{\delta^k},|z|=\frac{\delta}{2}. \tag{16}$$

By the Maximum Principle and Montel's Theorem, $\mathcal{F}$ is normal at $\xi=0$. This completes the proof of Theorem 1.4.

4 Open Problem

There are two interesting open problems related to our paper, one open problem is related to the meromorphic differential polynomial which omit a function set:

Problem 4.1 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose zeros have multiplicity at least k, and $\mathcal{S}=\{h_i(z)\,|z\in D,i=1,\cdots l\}$ be a holomorphic functions set on D such that $h_i(z)\,0$ for arbitrary $1\le i\le l$, the zeros of $h_i(z)$ have multiplicity m_i which satisfies $k\nmid m_i$ for arbitrary $1\le i\le l$. Suppose that for each $f\in\mathcal{F}$, $\max\limits_{1\le j\le k}\left|f^{(j)}(z)\right|<\min\limits_{1\le i\le l}|h_i(z)|$ whenever $f(z)=0$ and the differential polynomial $\sum\limits_{j=1}^{k}p_j(z)\,f^{(j)}(z)$ omits the function set $\mathcal{S}$, is $\mathcal{F}$ is a normal family on D?*

Another open problem is related to the meromorphic functions family which shares a function set, firstly we present the definition of meromorphic functions sharing a function set as follows:

Definition 4.2 (Meromorphic Functions Sharing a Function Set) *Let D be a domain in $\mathbb{C}$, f and g meromorphic on D and $\mathcal{S}$ is a set including finite meromorphic functions on D:*

$$\mathcal{S}=\{h_i(z)\,|z\in D,i=1,\cdots l\}.$$

If f and g satisfies

$$\begin{aligned} E_f(S) &= \{z \in D \,|f(z) = h_i(z), \exists 1 \le i \le l\} \\ &= \{z \in D \,|g(z) = h_j(z), \exists 1 \le j \le l\} \\ &= E_g(S), \end{aligned}$$

then the two meromorphic functions f and g on D are said to share the function set $\mathcal{S}$.

The open problem is as follows:

Problem 4.3 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose zeros have multiplicity at least k. If there exists a holomorphic function set on D*

$$S = \{h_i(z) \,|z \in D, i = 1, \cdots, l\},$$

where $h_i(z) \neq 0$ for arbitrary $z \in D$ and $1 \le i \le l$, such that for each $f \in \mathcal{F}$, $E_f(\mathcal{S}) = E_{f^{(k)}}(\mathcal{S})$ and $0 < \left|f^{(k)}(z)\right| \le \sup\limits_{1 \le i \le l} |h_i(z)|$ whenever $z \in E_f(0)$, is $\mathcal{F}$ is a normal family on D?

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References

[1] K. Hamai, T. Hayami, K. Kuroki and S. Owa, *Extremal Function and Coefficient Inequalities for Certain Analytic Functions*, Int. J. Open Problems Complex Analysis, Vol. 2, No. 3, (2010), pp. 174-180.

[2] Alina Alb Lupas, *A Note on a Subclass of Analytic Functions Defined by Multiplier Transformations*, Int. J. Open Problems Complex Analysis, Vol. 2, No. 2, (2010), pp. 154-159.

[3] N. Magesh, G. Murugusundaramoorthy, T. Rosy and K. Muthunagai, *Subordination and Superordination Results for Analytic Functions Associated with Convolution Structure*, Int. J. Open Problems Complex Analysis, Vol. 2, No. 2, (2010), pp. 67-81.

[4] Alina Alb Lupas, *A Note on a Subclass of Analytic Functions Defined by Ruscheweyh Derivativeand Multiplier Transformations*, Int. J. Open Problems Complex Analysis, Vol. 2, No. 2, (2010), pp. 60-66.

[5] L. Yang, *Normality for Families of Meromorphic Functions*, Sci. Sinica Ser. A, 29, (1986), pp. 1263-1274.

[6] X. C. Pang, L. Zalcman, *Normal Families of Meromorphic Functions with Multiple Zeros and Poles*, Israel J., 136, (2003), pp.1-9.

[7] X. C. Pang, D. G. Yang and L. Zalcman, *Normal Families of Meromorphic Functions whose Derivatives Omit a Function*, Comput. Methods Funct., 2, (2002), pp.257-265.