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$*$-Semilattice

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Abstract

In this paper we define a $*$-Semi lattice A and prove that for each $a \in C(A)$, $\theta_a = \{(x, y) \mid a \wedge x = a \wedge y\}$ a congruence on A and also prove that θ is a factor congruence if and only if $\theta = \theta_a$ for some $a \in C(A)$. Also we prove that for each $a \in C(A), A_a = \{a \wedge x \mid x \in A\}$ is itself a $*$-Semi lattice and $A \cong A_a \times A_{a^*}$.

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1 Introduction

It is known that a semi lattice $(A, \wedge)$ with zero is non empty set A together with a binary operation $\wedge$ which is associative, idempotent, commutative and $0 \in A$ satisfy $0 \wedge x = x \wedge 0 = 0$. It is well known that if $(A, \wedge)$ is a semi lattice and define $x \leq y \Leftrightarrow x = x \wedge y$ is a partial order on A. We say two congruences θ, ϕ are pair of factor congruences if $\theta \cap \phi = \Delta$ and $\theta \circ \phi = \phi \circ \theta = A \times A$. Recall that if A is a Boolean algebra then $\theta_a = \{a \wedge x \mid x \in A\}$ is a factor congruence for every $a \in A$ [2, 4]. Also we have $A \cong A/\theta_a \times A/\theta_{a^*}$ and also $A \cong (a] \times [a)$. In this paper we defined a $*$-Semi lattice A and its central elements $C(A)$ and proved that for each $a \in C(A), \theta_a = \{(x, y) \mid a \wedge x = a \wedge y\}$ a congruence on A and also proved that θ is a factor congruence if and only if $\theta = \theta_a$ for some $a \in C(A)$. Also it is proved that $A/\theta_a \cong A_a$ where $A_a = \{a \wedge x \mid x \in A\}$, which is itself a $*$-Semi lattice and $A \cong A_a \times A_{a^*}$.

2 $*$-Semi lattice

In this section First we define $*$-Semi lattice and we shall prove various properties. First let us start with the definition of $*$-Semi lattice.

Definition 2.1. *Let $(A, \wedge, 0)$ be a semi lattice with '0'. If $*$ is a unary operation on A such that, for any x,y,a $\in A$,*
(1) $a \wedge a^* = 0$
(2) $0^* \wedge x = x$
(3) $a \wedge ((a \wedge x)^* \wedge (a^* \wedge y)^*)^* = a \wedge x, \qquad a^* \wedge ((a \wedge x)^* \wedge (a^* \wedge y)^*)^* = a^* \wedge y$
(4) $x = ((a \wedge x)^* \wedge (a^* \wedge x)^*)^*$.
Then A is a $*$-semi lattice.

Now we prove the following

Lemma 2.2. $y^{**} = y$, *for all* $y \in A$.

Proof. By Definition 2.1(3), let $a = 0$. Then
$0^* \wedge [(0 \wedge x)^* \wedge (0^* \wedge y)^*]^* = 0^* \wedge y \Rightarrow [0^* \wedge y^*]^* = y \Rightarrow y^{**} = y$. □

Now we define a relation $\theta_a = \{(p, q) \mid a \wedge p = a \wedge q\}$ on a $*$-Semi lattice A and the set of all central elements $C(A)$ of A.

Definition 2.3. *Let A be a $*$-semi lattice. An element $a \in A$ is called central element if a satisfies the following*
(i) $a \wedge x = a \wedge y \Rightarrow a \wedge x^* = a \wedge y^*$
(ii)$a^* \wedge x = a^* \wedge y \Rightarrow a^* \wedge x^* = a^* \wedge y^*$ *,for any x,y $\in A$.*

The set of all central elements is denoted by $C(A)$. Observe that if $a \in C(A)$ then $a^* \in C(A)$. In the following we prove θ_a is a congruence if $a \in C(A)$ and it is also proved that θ_a, θ_{a^*} are pair of factor congruences on A.

Lemma 2.4. *Let A be a $*$-Semi lattice and $a \in C(A)$. Then*
$\theta_a = \{(p, q) | a \wedge p = a \wedge q\}$ *is a congruence on A.*

Proof. Clearly θ_a is an equivalence relation on A.
Let $(x, y), (z, t) \in \theta_a$. Then $a \wedge x = a \wedge y$ and $a \wedge z = a \wedge t$. Now $a \wedge x \wedge z = a \wedge x \wedge a \wedge z = a \wedge y \wedge a \wedge t = a \wedge y \wedge t$. Therefore $(x \wedge z, y \wedge t) \in \theta_a$. Now, $(x, y) \in \theta_a$ then $a \wedge x = a \wedge y \Rightarrow a \wedge x^* = a \wedge y^*$. Thus $(x^*, y^*) \in \theta_a$. Therefore θ_a is a congruence relation on A.

Recall that in a Semi lattice $(A, \wedge)$, if we define "$\leq$" by $x \leq y$ if and only if $x \wedge y = x$ then "$\leq$" is a partial order on A. □

We prove the following.

Theorem 2.5. *Let A be a $*$-Semi lattice and $a \in C(A)$. Then θ_a is a factor congruence on A.*

Proof. Let $(x, y) \in \theta_a \cap \theta_{a^*}$. Then $a \wedge x = a \wedge y$ and $a^* \wedge x = a^* \wedge y$.

$$\begin{aligned} \text{Now } x &= ((a \wedge x)^* \wedge (a^* \wedge x)^*)^* \wedge x \\ &= ((a \wedge y)^* \wedge (a^* \wedge y)^*)^* \wedge x \\ &= y \wedge x \end{aligned}$$

Therefore $x \leq y$. Similarly we can prove $y \leq x$ and hence $x = y$. Therefore $\theta_a \cap \theta_{a^*} = \triangle$.
Let $x \neq y$ and $z = ((a \wedge x)^* \wedge (a^* \wedge y)^*)^*$. Now, $a \wedge z = a \wedge ((a \wedge x)^* \wedge (a^* \wedge y)^*)^* = a \wedge x$ and $a^* \wedge z = a^* \wedge ((a \wedge x)^* \wedge (a^* \wedge y)^*)^* = a^* \wedge y$. Therefore, $(x, z) \in \theta_a, (z, y) \in \theta_{a^*}$. Thus, $(x, y) \in \theta_{a^*} \circ \theta_a$. Therefore $\bigtriangledown \subseteq \theta_a \circ \theta_{a^*}$. Since $\theta_a \circ \theta_b \subseteq \bigtriangledown$ for any two congruences in particular, $\theta_a \circ \theta_{a^*} \subseteq \bigtriangledown, \theta_a \circ \theta_{a^*} = \bigtriangledown$. Therefore θ_a is a factor congruence on A.

□

Now we prove that θ is a factor congruence if and only if $\theta = \theta_a$ for some $a \in C(A)$.

Theorem 2.6. *Let A be a $*$-Semi lattice and θ is a congruence on A. Then θ is a factor congruence if and only if $\theta = \theta_a$ for some $a \in C(A)$.*

Proof. Let θ be a factor congruence on A. Then there exists a ϕ such that $\theta \circ \phi = \triangle$ and $\theta \cap \phi = \bigtriangledown$. Now, $(0, 0^*) \in \theta \circ \phi$ then there exists $x \in A$ such that $(0, x) \in \phi, (x, 0^*) \in \theta$. To prove $\theta = \theta_x$, let $(p, q) \in \theta_x$ that is $x \wedge p = x \wedge q$. We have, $(x, 0^*) \in \theta$ then $(x \wedge p, 0^* \wedge p), (x \wedge q, 0^* \wedge q) \in \theta$. Thus $(p, q) \in \theta$. Therefore $\theta_x \subseteq \theta$.
On the other hand, $(p, q) \in \theta$, that is $(x \wedge p, x \wedge q) \in \theta$. Since $(0, x) \in \phi, (0 \wedge p, x \wedge p)$ and $(0 \wedge q, x \wedge q)$ are in $\phi, (x \wedge q, x \wedge p) \in \phi$. Thus $(x \wedge p, x \wedge q) \in \theta \cap \phi = \triangle$. $(x \wedge p, x \wedge q) \in \triangle \Rightarrow x \wedge p = x \wedge q \Rightarrow (p, q) \in \theta_x$. Therefore $\theta \subseteq \theta_x$. Thus $\theta = \theta_x$. Now we show that $\phi = \theta_{x^*}$. Let $(p, q) \in \theta_{x^*}$.

$$\begin{aligned} \text{Now } (0, x) \in \phi &\Rightarrow (0^*, x^*) \in \phi \\ &\Rightarrow (0^* \wedge p, x^* \wedge p), (0^* \wedge q, x^* \wedge q) \in \phi \\ &\Rightarrow (p, x^* \wedge p), (q, x^* \wedge q) \in \phi \\ &\Rightarrow (p, q) \in \phi(\text{ since } x^* \wedge p = x^* \wedge q) \end{aligned}$$

Therefore $\theta_{x^*} \subseteq \phi$. On the other hand $(p, q) \in \phi$.

$$\begin{aligned} (x, 0^*) \in \theta &\Rightarrow (x^*, 0) \in \theta \\ &\Rightarrow (x^* \wedge p, 0 \wedge p), (x^* \wedge q, 0 \wedge q) \in \theta \\ &\Rightarrow (x^* \wedge p, 0), (x^* \wedge q, 0) \in \theta \\ &\Rightarrow (x^* \wedge p, x^* \wedge q) \in \theta \end{aligned}$$

Therefore $(x^* \wedge p, x^* \wedge q) \in \phi \cap \theta = \triangle$.
$x^* \wedge p = x^* \wedge q \Rightarrow (p, q) \in \theta_{x^*}$. Therefore $\phi \subseteq \theta_{x^*}$. Thus $\phi = \theta_{x^*}$. Now our claim is to show that $x \in C(A)$. Let $x \wedge t = x \wedge w$. Thus $(t, w) \in \theta_x = \theta \Rightarrow (t^*, w^*) \in \theta$ (since θ is a congruence) $\Rightarrow (t^*, w^*) \in \theta_x \Rightarrow x \wedge t^* = x \wedge w^*$.
Suppose $x^* \wedge t = x^* \wedge w \Rightarrow (t, w) \in \theta_{x^*} = \phi \Rightarrow (t^*, w^*) \in \phi = \theta_{x^*}$ (since ϕ is a congruence) $\Rightarrow x^* \wedge t^* = x^* \wedge w^*$. Therefore $x \in C(A)$. The converse is trivial by lemma 2.4.

□

Theorem 2.7. *Let* $(A, \wedge, 0)$ *be a* $*$*-semi lattice then for every* $a \in C(A)$ *define* $A_a = \{a \wedge x \mid x \in A\}$ *is itself a* $*$*-semi lattice where* $\wedge$ *is induced operation and* $'$ *is defined by* $(a \wedge x)' = a \wedge x^*$ *for all* $x \in A$.

Proof. Since $a \in C(A), a \wedge x = a \wedge y \Rightarrow a \wedge x^* = a \wedge y^* \Rightarrow (a \wedge x)' = (a \wedge y)'$. Hence the unary operation $'$ is well defined. Let $a \wedge x, a \wedge y \in A_a$. Then $(a \wedge x) \wedge (a \wedge y) = a \wedge x \wedge y$. Therefore $\wedge$ is closed on A_a. Also $0 \in A_a$ since $a \wedge 0 = 0$. Thus $(A_a, \wedge)$ is a semi lattice with 0. Now, $(a \wedge x) \wedge (a \wedge x)' = a \wedge x \wedge a \wedge x^* = a \wedge x \wedge x^* = a \wedge 0 = 0$. Further $0' = (a \wedge 0)' = a \wedge 0^* = a$. Thus $0' \wedge (a \wedge x) = a \wedge (a \wedge x) = a \wedge x$.
Let $a \wedge x, a \wedge y, a \wedge z \in A_a$.
Now $(a \wedge x) \wedge [[(a \wedge x) \wedge (a \wedge y)]' \wedge [(a \wedge x)' \wedge (a \wedge z)]']'$

$$\begin{aligned}
&= a \wedge x \wedge [[a \wedge (x \wedge y)^*] \wedge [a \wedge x^* \wedge a \wedge z]']' \\
&= a \wedge x \wedge [a \wedge (x \wedge y)^* \wedge [a \wedge (x^* \wedge z)^*]]' \\
&= a \wedge x \wedge a \wedge [(x \wedge y)^* \wedge (x^* \wedge z)^*]^* \\
&= a \wedge x \wedge [(x \wedge y)^* \wedge (x^* \wedge z)^*]^* \\
&= a \wedge x \wedge y \qquad \text{by definition 2.1(4)} \\
&= (a \wedge x) \wedge (a \wedge y)
\end{aligned}$$

Now $(a \wedge x)' \wedge [[(a \wedge x) \wedge (a \wedge y)]' \wedge [(a \wedge x)' \wedge (a \wedge z)]']'$

$$\begin{aligned}
&= a \wedge x^* \wedge [a \wedge (x \wedge y)^* \wedge [a \wedge x^* \wedge a \wedge z]']' \\
&= a \wedge x^* \wedge [a \wedge (x \wedge y)^* \wedge [a \wedge (x^* \wedge z)^*]]' \\
&= a \wedge x^* \wedge a \wedge [(x \wedge y)^* \wedge (x^* \wedge z)^*]^* \\
&= a \wedge x^* \wedge [(x \wedge y)^* \wedge [x^* \wedge z]^*]^* \\
&= a \wedge x^* \wedge z \qquad \text{by definition 2.1(3)} \\
&= a \wedge x^* \wedge a \wedge z \\
&= (a \wedge x)' \wedge (a \wedge z)
\end{aligned}$$

$$\begin{aligned}
[[(a \wedge x) \wedge (a \wedge y)]' \wedge [(a \wedge x)' \wedge (a \wedge y)]']' &= [a \wedge (x \wedge y)^* \wedge (a \wedge x^* \wedge y)']' \\
&= [a \wedge (x \wedge y)^* \wedge a \wedge (x^* \wedge y)^*]' \\
&= [a \wedge (x \wedge y)^* \wedge (x^* \wedge y)^*]' \\
&= a \wedge [(x \wedge y)^* \wedge (x^* \wedge y)^*]^* \\
&= a \wedge y \qquad \text{by definition 2.1(4)}
\end{aligned}$$

Therefore A_a is $*$-Semi lattice.

□

Theorem 2.8. *Let* A *be a* $*$*-Semi lattice. Then for any* $a \in C(A), f_a : A \to A_a$ *defined by* $f(x) = a \wedge x$ *is a homomorphism and* $A/\theta_a \cong A_a$.

Proof. Let $f_a(x \wedge y) = a \wedge x \wedge y = (a \wedge x) \wedge (a \wedge y) = f_a(x) \wedge f_a(y)$ and $f_a(x^*) = a \wedge x^* = (a \wedge x)' = [f_a(x)]'$. Therefore f_a is a homomorphism, clearly f is onto.
Now $kerf_a = \{(x, y) \mid f_a(x) = f_a(y)\} = \{(x, y) \mid a \wedge x = a \wedge y\} = \theta_a$. By fundamental theorem of homomorphism $A/Kerf_a \cong A_a$ which imply $A/\theta_a \cong A_a$. □

Theorem 2.9. *Let A be a $*$-Semi lattice. Then for any $a \in C(A), A \cong A_a \times A_{a^*}$.*

Proof. Since for each $a \in A$, θ_a is a factor congruence, we have $A \cong A/\theta_a \times A/\theta_{a^*}$. Therefore by above Theorem $A \cong A_a \times A_{a^*}$. □

Theorem 2.10. *Let A be $*$-Semilattice and $(x^* \wedge y^*)^* \wedge y = y$ for all $x, y \in A$. If we define $x \vee y = (x^* \wedge y^*)^*$. Then $< A, \wedge, \vee, 0, 0^* >$ is bounded ortholattice.*

Proof. Let A be *-Semilattice and $(x^* \wedge y^*)^* \wedge y = y$.
Define $x \vee y = (x^* \wedge y^*)^*$ then clearly $\vee$ is commutative and $x \vee x = (x^* \wedge x^*)^* = (x^*)^* = x$.
Now, $(x \vee y)^* = [(x^* \wedge y^*)^*]^* = x^* \wedge y^*$ and $(x \wedge y)^* = [(x^{**} \wedge y^{**})^*]^* = x^* \vee y^*$

$$\begin{aligned} x \vee (y \vee z) &= x \vee (y^* \wedge z^*)^* \\ &= (x^* \wedge (y^* \wedge z^*)^{**})^* \\ &= (x^* \wedge (y^* \wedge z^*))^* \\ &= ((x^* \wedge y^*) \wedge z^*))^* \\ &= ((x^* \wedge y^*)^{**} \wedge z^*)^* \\ &= (x^* \wedge y^*)^* \vee z \\ &= (x \vee y) \vee z \end{aligned}$$

Thus $\vee$ is associative.

$$Also, (x \wedge y) \vee y \;=\; [(x \wedge y)^* \wedge y^*]^* = [x^{**} \wedge y^{**})^* \wedge y^*]^* = y^{**} = y$$

Thus $< A, \wedge, \vee, >$ is lattice. Since $0 \wedge x = 0, 0^* \wedge x = x, x \wedge x^* = 0$ and $x \vee x^* = [x^* \wedge x^{**}]^* = [x \wedge x^*]^* = 0^*$, we have $0 \leq x \leq 0^*$. Therefore $*$ is ortho complementation. Hence $< A, \wedge, \vee, 0, 0^* >$ is an ortholattice.

□

3 Open Problems

1. Example of a $*$-Semi lattice which is not an ortholattice.

2. Is the set of all congruences on a $*$-Semi lattice a Boolean lattice?

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