Int. J. Open Problems Compt. Math., Vol. 4, No. 3, September 2011
ISSN 1998-6262; Copyright © ICSRS Publication, 2011
www.i-csrs.org

Estimates Normalized Eigenfunction to the Boundary Value Problem in Different Cases of Weight Functions

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Abstract

In this, paper we estimates the normalized eigenfunctions to the boundary value problem of the form (1)-(3), in different cases of weight functions and some properties of our problem are showed.

Keywords: Boundary value problem, eigenvalues, normalized eigenfunctions, weight functions.

2000 Mathematics Subject Classification: 47E05, 34B05, 34B07, 34B07

1. Introduction

Numerous problems of the theory of fluctuations of the spatially-distributed systems lead to necessity of research of characteristic constant and eigenfunctions of differential operators corresponding them, and also to the questions connected with research of various functions from eigenvalues and eigenfunctions. Interest to the problem of the spectral analysis has especially increased when it was found out, that the spectral analysis of the self-adjoint differential operators is the basic mathematical device at the decision of problems of quantum mechanics. As it is known, many problems of mathematical physics, mechanics, theory of elasticity, optimal control lead to the problem of research of a spectrum of differential operators and series expansion of any function on eigenfunctions of such operator. Classical results in this direction belong to [3,4, and 7].Aigounov and Tamila [1-2] , Jwamer ana Aigounov [6] and Jwamer [5] are obtained the estimation of the normalized eigenfunction to the T.Regge problem in different cases of weights functions ,but in our works we are also obtained the estimation of the normalized eigenfunctions but to the problem of the type described in below.

Let numbers a > 0, m and M (0< m ≤ M) be fixed. Let L^+ [0, a] denote the family of all integrable functions ρ(x) on the closed interval [0, a] that satisfy the condition m ≤ ρ(x) ≤ M, equipped with the usual L_1 metric. In what follows, we refer to these functions as weight functions.

Consider the spectral problem

$$-y'' + q(x)y = \lambda^2\rho(x)y \qquad (1)$$

$$y'(0) = y(a) + y'(a) = 0 \qquad (2)$$

$$\left(\int_0^a \rho(x)|y(x)|^2\,dx\right)^{1/2} = 1 \qquad (3)$$

Where λ spectral parameter.

Theorem 1:

If λ is an Eigen value corresponding to the eigenfunction y(x) of the problem (1)− (3) and $\rho(x) = \rho$ and $q(x) = q$ are constants if $\delta \neq 0$ then λ is real.

Proof:

Multiply equation (1) by $\bar{y}(x)$ and integrate the resulting equation from 0 up to a, as a result we shall get:

$$-\int_0^a y''(x)\,\bar{y}(x)dx + \int_0^a q(x)y(x)\,\bar{y}(x)dx = \int_0^a \lambda^2\,\rho(x)y(x)\bar{y}(x)dx$$

$$-y'(x)\bar{y}(x)]_0^a + \int_0^a |y'(x)|^2 + \int_0^a q(x)|y(x)|^2dx = \int_0^a \lambda^2\,\rho(x)|y(x)|^2dx$$

$$-y'(a)\bar{y}(a) + y'(0)\bar{y}(0) + \int_0^a |y'(x)|^2dx + \int_0^a q(x)|y(x)|^2dx$$

$$= \int_0^a \lambda^2\,\rho(x)|y(x)|^2dx$$

in view of boundary conditions(2) we have:

$$y(a)\bar{y}(a) + 0 + \int_0^a |y'(x)|^2dx + \int_0^a q(x)|y(x)|^2dx = \int_0^a \lambda^2\,\rho(x)|y(x)|^2dx$$

$$|y(a)|^2 + \int_0^a |y'(x)|^2dx + \int_0^a q(x)|y(x)|^2dx = \int_0^a \lambda^2\,\rho(x)|y(x)|^2dx \qquad (4)$$

If in equation (1) and in boundary condition (2) to pass to complex interfaced values we get:

$$-\bar{y}'' + q(x)\bar{y}(x) = \overline{\lambda^2}\rho(x)\bar{y}(x)$$

$$\bar{y}'(0) = \bar{y}(a) + \bar{y}'(a) = 0$$

Multiplying differential equation on y(x) and integrate from 0 up to a

$$|y(a)|^2 + \int_0^a |y'(x)|^2 dx + \int_0^a q(x)|y(x)|^2 dx = \int_0^a \overline{\lambda^2}\rho(x)\,|y(x)|^2 dx \qquad (5)$$

Subtracting equations (4) to (5) we obtain:

$$(\lambda^2 - \overline{\lambda^2}) \int_0^a \rho\,(x)|y(x)|^2\, dx = 0 \quad since \quad \int_0^a \rho\,(x)|y(x)|^2\, dx = 1$$

$$\rightarrow \left(\lambda^2 - \overline{\lambda^2}\right) = 0 \rightarrow \left(\lambda - \bar{\lambda}\right)\left(\lambda + \bar{\lambda}\right) = 0$$

$$\rightarrow \left(\lambda - \bar{\lambda}\right) = 0 \text{ or } \left(\lambda + \bar{\lambda}\right) = 0$$

But $\left(\lambda + \bar{\lambda}\right) \neq 0$ because $\delta \neq 0$, then $\left(\lambda - \bar{\lambda}\right) = 0$

Then $\lambda = \bar{\lambda}$. Hence λ is real.

Theorem 2:
Let $\rho(x) \in L^+[0,a]$ and q(x) $\in L[0,a]$ then the eigenfunction to the problem (1) - (3) satisfy the inequality $\underset{x\in[0,a]}{Max}|y(x)| < c|\lambda|^{\frac{1}{2}}$ where c not depend on ρ *and* q.

Proof:
Let's consider the identity

$$|y(x)|^2 = y(x)\bar{y}(x) = \int_0^x [\bar{y}(s)y'(s) + y(s)\bar{y}'(s)]ds$$

$$= \int_0^x \frac{\sqrt{\rho(s)}.\,[\bar{y}(s)y'(s) + y(s)\bar{y}'(s)]}{\sqrt{\rho(s)}}\, ds$$

Hence and from inequality $\rho(s) \geq m$ follows that

$$|y(x)|^2 \leq \int_0^x \frac{\sqrt{\rho(s)}\,\left|\bar{y}(s)y'(s) + y(s)\bar{y}'(s)\right|}{\sqrt{m}}\, ds$$

$$= \frac{1}{\sqrt{m}}\left[\int_0^x \sqrt{\rho(s)}\,\left|\bar{y}(s).\,y'(s)\right| ds + \int_0^x \sqrt{\rho(s)}\,\left|y(s).\,\bar{y}'(s)\right| ds\right]$$

$$\leq \frac{1}{\sqrt{m}}\left[\int_0^x \sqrt{\rho(s)}\;|\bar{y}(s)|\left|y'(s)\right|\, ds + \int_0^x \sqrt{\rho(s)}\;|y(s)|\left|\bar{y}'(s)\right| ds\right]$$

$$= \frac{2}{\sqrt{m}}\int_0^x \sqrt{\rho(s)}\,|\bar{y}(s)|\left|y'(s)\right| ds \leq \frac{2}{\sqrt{m}}\int_0^a \sqrt{\rho(s)}\,|\bar{y}(s)|\left|y'(s)\right| ds$$

Estimating last integral on Cauchy inequality we shall get

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}}\left[\int_0^a \rho(s).\,|y(s)|^2 ds\right]^{1/2}\left[\int_0^a \left|y'(s)\right|^2 ds\right]^{1/2}$$

Let's increase now equation (1) in our problem $\bar{y}(x)$ and integrate from o up to a We get the following:

$$|y(a)|^2 + \int_0^a |y'(x)|^2 dx + \int_0^a q(x)|y(x)|^2 dx = \int_0^a \lambda^2 \rho(x)\, |y(x)|^2 dx \tag{6}$$

And if in equation (1) and boundary condition (2) to pass to complex interfaced values and multiply differential equation by $y(x)$ and integrating from 0 up to a We shall get:

$$|y(a)|^2 + \int_0^a |y'(x)|^2 dx + \int_0^a q(x)|y(x)|^2 dx = \int_0^a \overline{\lambda^2} \rho(x)\, |y(x)|^2 dx \tag{7}$$

If to increase equation (6) $on\ \overline{\lambda}$ and equation (7) on λ we get:

$$(\overline{\lambda} + \lambda)\left[|y(a)|^2 + \int_0^a |y'(x)|^2 dx + \int_0^a q(x)|y(x)|^2 dx\right] = (\overline{\lambda} + \lambda)|\lambda|^2 \int_0^a \rho(x)\, |y(x)|^2 dx$$

$$|y(a)|^2 + \int_0^a |y'(x)|^2 dx + \int_0^a q(x)|y(x)|^2 dx = |\lambda|^2 \int_0^a \rho(x)\, |y(x)|^2 dx$$

$$(\,since\ \overline{\lambda} + \lambda \neq 0)$$

$$\int_0^a |y'(x)|^2 dx = |\lambda|^2 \int_0^a \rho(x)\, |y(x)|^2 dx - \int_0^a q(x)|y(x)|^2 dx - |y(a)|^2$$

And

$$|y(x)|^2 \le \frac{2}{\sqrt{m}}\left[\int_0^a \rho(s).\,|y(s)|^2 ds\right]^{1/2}\left[\int_0^a |y'(s)|^2 ds\right]^{1/2}$$

$$|y(x)|^2 = \frac{2}{\sqrt{m}}\left[\int_0^a \rho(s).\,|y(s)|^2 ds\right]^{1/2}\left[|\lambda|^2 \int_0^a \rho(x)\, |y(x)|^2 dx - \int_0^a q(x)|y(x)|^2 dx - |y(a)|^2\right]^{1/2}$$

Then

$$|y(x)|^2 \le \frac{2|\lambda|}{\sqrt{m}}\left[\int_0^a \rho(s).\,|y(s)|^2 ds\right]\left[1 - \left(\frac{\int_0^a q(x)|y(x)|^2 dx + |y(a)|^2}{|\lambda|^2 \int_0^a \rho(x)\, |y(x)|^2 dx}\right)\right]^{1/2}$$

$$\frac{|y(x)|^2}{\int_0^a \rho(s).\,|y(s)|^2 ds} \le \frac{2|\lambda|}{\sqrt{m}}\left[1 - \left(\frac{\int_0^a q(x)|y(x)|^2 dx + |y(a)|^2}{|\lambda|^2 \int_0^a \rho(x)\, |y(x)|^2 dx}\right)\right]^{1/2} \le \frac{2|\lambda|}{\sqrt{m}}$$

$$\frac{|y(x)|}{\left(\int_0^a \rho(s).\,|y(s)|^2 ds\right)^{1/2}} \le |\lambda|^{1/2} \sqrt[4]{\frac{4}{m}} \to \underset{x\in[0,a]}{Max}|y(x)| < C|\lambda|^{\frac{1}{2}}$$

Where $C = \sqrt[4]{\frac{4}{m}}$ not depends ρ *and* q

Theorem 3:

Let $\rho(x)$ and q(x) are constants of the problem (1) - (3) and if y(x) is an eigenfunction of the problem (1) - (3), then $0 \leq M_1 \leq \underset{x\in[0,a]}{Max}|y(x)| \leq M_2$. Where M_1 and M_2 are constants.

Proof:

From equ. (1), we have $y'' + \lambda^2\rho y - qy = 0$ this second order linear differential equation with constant coefficients,then general solution is

$$y(x) = C_1 e^{i\sqrt{\lambda^2\rho-q}\,x} + C_2 e^{-i\sqrt{\lambda^2\rho-q}\,x}$$

Clearly, that $y'(0) = 0$ we get $C_1 = C_2$, then we have

$$y(x) = C_1(e^{i\sqrt{\lambda^2\rho-q}\,x} + e^{-i\sqrt{\lambda^2\rho-q}\,x}).$$

Then

$$y'(x) = iC_1\sqrt{\lambda^2\rho - q}\,(e^{i\sqrt{\lambda^2\rho-q}\,x} - e^{-i\sqrt{\lambda^2\rho-q}\,x})$$

From the second boundary condition we obtain

$$iC_1\sqrt{\lambda^2\rho - q}\,\left(e^{i\sqrt{\lambda^2\rho-q}\,a} - e^{-i\sqrt{\lambda^2\rho-q}\,a}\right) + C_1(e^{i\sqrt{\lambda^2\rho-q}\,a} + e^{-i\sqrt{\lambda^2\rho-q}a}) = 0$$

Dividing both sides of this equation by C_1 we obtain

$$\frac{\sqrt{\lambda^2\rho-q}+i}{\sqrt{\lambda^2\rho-q}-i} = e^{2i\sqrt{\lambda^2\rho-q}\,a} \tag{8}$$

The resulting equation is an equation for determining eigen values of our problem. To determine the coefficientC_1, we use the normalization condition (3)

$$\int_0^a \rho\,|C_1|^2\left|e^{i\sqrt{\lambda^2\rho-q}\,x} + e^{-i\sqrt{\lambda^2\rho-q}\,x}\right|^2 dx = 1$$

or

$$\rho|C_1|^2.\int_0^a \left|e^{i\sqrt{\lambda^2\rho-q}\,x} + e^{-i\sqrt{\lambda^2\rho-q}\,x}\right|^2 dx = 1$$

We introduce the notation $\delta + i\sigma = i\sqrt{\lambda^2\rho - q}$ (where δ and σ are real numbers). Then,

$$\rho|C_1|^2.\int_0^a \left|e^{(\delta+i\sigma)x} + e^{-(\delta+i\sigma)\,x}\right|^2 dx = 1$$

Since $\left|e^{(\delta+i\sigma)x} + e^{-(\delta+i\sigma)\,x}\right|^2 = 2(cosh2\delta x + cos2\sigma x)$, then above equation can write as $2\rho|C_1|^2\left(\int_0^a cosh2\delta x\,dx + \int_0^a cos2\sigma x\,dx\right) = 1$

Or

$$\rho|C_1|^2\left[\frac{cosh2\delta x}{2\delta}]_0^a + \frac{cos2\sigma x}{2\sigma}]_0^a\right] = 1$$ (under the assumption, that $\delta.\sigma \neq 0$)

Or

$2\rho|C_1|^2.\left(\frac{sinh2\delta a}{2\delta} + \frac{sin2\sigma a}{2\sigma}\right) = 1$ (in the case $\delta = 0$, we obtain $\frac{sinh2\delta a}{\delta} = 2a$, and in the case σ=0, we obtain $\frac{sin2\sigma a}{\sigma} = 2a$).

Therefore, we obtain

$$|C_1| = \frac{1}{\sqrt{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}} \ .$$

We conclude, that

$$y(x) = C_0 \frac{e^{i\sqrt{\lambda^2\rho-q}\,x}+e^{-i\sqrt{\lambda^2\rho-q}\,x}}{\sqrt{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}} \qquad (9)$$

Where C_0 arbitrary complex number with module is one (i.e $|C_0| = 1$).

If λ satisfies the equation (8) (i.e λ eigenvalue), then equation (9) gives eigenfunction our problem (corresponding to the eigenvalue λ).

The case $\delta = \sigma = 0$ leads to $\lambda^2\rho - q = 0$, which is not considered .

Now we determine $Max_x|\,y(x)|$ and its behavior depends on $,\delta\ and\ \sigma$. In deriving equation (3) found that

$$\left|e^{(\delta+i\sigma)x} + e^{-(\delta+i\sigma)\,x}\right|^2 = 2(cosh2\delta x + cos2\sigma x)$$

Therefore

$$|\,y(x)| = \left|C_0 \frac{e^{i\sqrt{\lambda^2\rho-q}\,x}+e^{-i\sqrt{\lambda^2\rho-q}\,x}}{\sqrt{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}\right| = \left|\frac{e^{i\sqrt{\lambda^2\rho-q}\,x}+e^{-i\sqrt{\lambda^2\rho-q}\,x}}{\sqrt{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}\right| = \sqrt{\frac{2(cosh2\delta x+cos2\sigma x)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}$$

Then

$$\sqrt{\frac{2(cosh2\delta x-1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}} \le |\,y(x)| \le \sqrt{\frac{2(cosh2\delta x+1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}$$

Let $Max_x|\,y(x)|$ be achieved at the point of x_0, then

$$Max_x|\,y(x)| = |\,y(x_0)| \le \sqrt{\frac{2(cosh2\delta x_0+1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}} \le \sqrt{\frac{2(cosh2\delta a+1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}$$

Since cosh2δx is monotonic increasing on$[0, a]$, on the other hand

$$|y(x_0)| = Max_x|\,y(x)| \ge |\,y(a)| \ge \sqrt{\frac{2(cosh2\delta a-1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}$$

Therefore

$$\sqrt{\frac{2(cosh2\delta a-1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}} \le \max_{x\in[0,a]}|y(x)| \le \sqrt{\frac{2(cosh2\delta a+1)}{\rho\left(\frac{sinh2\delta a}{\delta}+\frac{sin2\sigma a}{\sigma}\right)}}$$

or

$$\sqrt{\frac{2\left(\frac{cosh2\delta a-1}{sinh2\delta a}\right)\delta}{\rho\left(1+\frac{\delta.sin2\sigma a}{\sigma.sinh2\delta a}\right)}} \le \max_{x\in[0,a]}|y(x)| \le \sqrt{\frac{2\left(\frac{cosh2\delta a+1}{sinh2\delta a}\right)\delta}{\rho\left(1+\frac{\delta.sin2\sigma a}{\sigma.sinh2\delta a}\right)}}$$

We note, that

$$\frac{\delta.sin2\sigma a}{\sigma.sinh2\delta a}=\frac{2a.\delta.sin2\sigma a}{2a.\sigma.sinh2\delta a}=\frac{sin2\sigma a}{2a.\sigma}.\frac{2a.\delta}{sinh2\delta a}$$

And since

$\left|\frac{sinx}{x}\right|<1\,,\frac{x}{sinhx}<1$ for x$\neq 0$, then$\frac{\delta.sin2\sigma a}{\sigma.sinh2\delta a}<1\,,if\ \delta^2+\sigma^2\neq 0$.

And the continuity of the functions $\frac{\sin x}{x}$ and $\frac{y}{sinhy}$ it follow that $\left|\frac{sinx}{x}.\frac{y}{sinhy}\right|$ continuous function on a pair of arguments(x,y), as well as the obvious $\lim_{\sqrt{x^2+y^2}\to 0}\left|\frac{sinx}{x}.\frac{y}{sinhy}\right|=0$, then for any sufficiently small w>0 there exist $\Delta>0$ such that$\left|\frac{sinx}{x}.\frac{y}{sinhy}\right|<1-w$, if $\sqrt{x^2+y^2}>\Delta$. It follow that outside a certain range of $\delta^2+\sigma^2>\Delta_1$ the inequalities

$$w<1+\frac{\delta.sin2\sigma a}{\sigma.sinh2\delta a}<2-w \quad \text{And therefore}$$

$$\sqrt{\frac{2\left(\frac{cosh2\delta a-1}{sinh2\delta a}\right)\delta}{\rho(2-w)}}\leq \max_{x\in[0,a]}|y(x)|\leq\sqrt{\frac{2\left(\frac{cosh2\delta a+1}{sinh2\delta a}\right)\delta}{\rho.w}}$$

Since $\cosh 2\delta a$ and $\frac{\delta}{sinh2\delta a}$ are even function, then the last inequality can be considered for $\delta\geq 0$. Obviously $\frac{cosh2\delta a-1}{sinh2\delta a}$ monotonic increases from 0 to 1, when δ increases from 0 to $+\infty$, and $\frac{cosh2\delta a+1}{sinh2\delta a}$ monotonic decreasing from $+\infty$ to 1, when δ increases from 0 to $+\infty$ and therefore, if $|\delta|\geq|\delta_0|>0$, then

$$\sqrt{\frac{2\left(\frac{cosh2|\delta_0|a-1}{sinh2|\delta_0|a}\right)|\delta|}{\rho(2-w)}}\leq \max_{x\in[0,a]}|y(x)|\leq\sqrt{\frac{2\left(\frac{cosh2|\delta_0|a+1}{sinh2|\delta_0|a}\right)|\delta|}{\rho.w}} \qquad (10)$$

If $\delta=0$, then we obtain the inequality

$$0\leq \underset{x\in[0,a]}{Max}|y(x)|\leq\sqrt{\frac{2}{\rho wa}} \qquad (11)$$

In the resulting estimates (10) used parameter δ , which is clearly not part of equation (1) and boundary conditions (2) and normalized condition (3). Therefore We express δ and σ through, ρ and q.

Suppose arg $\lambda=\varphi$, then $\lambda^2=|\lambda|^2(cos2\varphi+isin2\varphi)$

$\lambda^2\rho-q=\rho|\lambda|^2cos2\varphi-q+i\rho|\lambda|^2sin2\varphi$

On the other hand $-(\lambda^2\rho-q)=(\delta+i\sigma)^2=\delta^2-\sigma^2+2i\delta\sigma$,

Hence

$$\left\{\begin{array}{c}\delta^2-\sigma^2=-\rho|\lambda|^2cos2\varphi+q\\ 2\delta\sigma=-\rho|\lambda|^2sin2\varphi\end{array}\right\} \text{ or } \left\{\begin{array}{c}\delta^2-\sigma^2=-\rho|\lambda|^2cos2\varphi+q\\ 4\delta^2\sigma^2=\rho^2|\lambda|^4sin^22\varphi\end{array}\right\}$$

Solving these two last system of equations , we obtain

$$\delta^2=\frac{-\rho|\lambda|^2cos2\varphi+q+\sqrt{(\rho|\lambda|^2cos2\varphi-q)^2+\rho^2|\lambda|^4sin^22\varphi}}{2}$$

and

$$\sigma^2 = \frac{\rho^2|\lambda|^4 sin^2 2\varphi}{2\left[-\rho|\lambda|^2 cos2\varphi + q + \sqrt{(\rho|\lambda|^2 cos2\varphi - q)^2 + \rho^2|\lambda|^4 sin^2 2\varphi}\right]}$$

(Since $\delta^2 \geq 0$, then chosen non negative root). Separating out the factor $\rho|\lambda|^2$ from the last relations, we obtain

$$\delta^2 = \rho|\lambda|^2 \left(\frac{-cos2\varphi + \frac{q}{\rho|\lambda|^2} + \sqrt{1 - 2\frac{q}{\rho|\lambda|^2} cos2\varphi + \left(\frac{q}{\rho|\lambda|^2}\right)^2}}{2} \right)$$

$$\sigma^2 = \frac{\rho^2|\lambda|^4 sin^2 2\varphi}{2\,\rho|\lambda|^2 \left(-cos2\varphi + \frac{q}{\rho|\lambda|^2} + \sqrt{1 - 2\frac{q}{\rho|\lambda|^2} cos2\varphi + \left(\frac{q}{\rho|\lambda|^2}\right)^2} \right)}$$

And the condition $|\delta| \geq |\delta_0|$ from inequality (10) takes the form

$$\sqrt{\frac{\rho}{2}}.\sqrt{-cos2\varphi + \frac{q}{\rho|\lambda|^2} + \sqrt{1 - 2\frac{q}{\rho|\lambda|^2} cos2\varphi + \left(\frac{q}{\rho|\lambda|^2}\right)^2}}\;.|\lambda| \geq |\delta_0|$$

or

$$\frac{\rho}{2}.\left[-cos2\varphi + \frac{q}{\rho|\lambda|^2} + \sqrt{1 - 2\frac{q}{\rho|\lambda|^2} cos2\varphi + \left(\frac{q}{\rho|\lambda|^2}\right)^2}\right].|\lambda|^2 \geq |\delta_0|^2$$

or

$$\sqrt{1 - 2\frac{q}{\rho|\lambda|^2} cos2\varphi + \left(\frac{q}{\rho|\lambda|^2}\right)^2} \geq cos2\varphi + \frac{2{\delta_0}^2 - q}{\rho|\lambda|^2}$$

If $cos2\varphi + \frac{2{\delta_0}^2 - q}{\rho|\lambda|^2} \leq 0$, then this inequality obviously holds, if not, then it is equivalent to the following inequality

$$1 - 2\frac{q}{\rho|\lambda|^2} cos2\varphi + \left(\frac{q}{\rho|\lambda|^2}\right)^2 \geq \left(\frac{2{\delta_0}^2 - q}{\rho|\lambda|^2}\right)^2 + 2\frac{2{\delta_0}^2 - q}{\rho|\lambda|^2} cos2\varphi + cos^2 2\varphi$$

or

$$cos^2 2\varphi + \frac{4{\delta_0}^2}{\rho|\lambda|^2}\; cos2\varphi - \left[1 - \frac{4{\delta_0}^2(-q + {\delta_0}^2)}{\rho^2|\lambda|^4}\right] \leq 0,$$

Then

$$-\frac{2{\delta_0}^2}{\rho|\lambda|^2} - \sqrt{1 + \frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}} \leq cos2\varphi \leq -\frac{2{\delta_0}^2}{\rho|\lambda|^2} + \sqrt{1 + \frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}}$$

For sufficiently large $|\lambda|$ it is obvious

$$-\frac{2{\delta_0}^2}{\rho|\lambda|^2} + \sqrt{1 + \frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}} \leq \frac{q - 2{\delta_0}^2}{\rho|\lambda|^2}\,,$$

And since

$$-\frac{2{\delta_0}^2}{\rho|\lambda|^2}-\sqrt{1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}}\leq -1\ ,$$

Then we finally get

$$cos2\varphi \leq -\frac{2{\delta_0}^2}{\rho|\lambda|^2}+\sqrt{1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}}\,.$$

Inequality $-\frac{2{\delta_0}^2}{\rho|\lambda|^2}+\sqrt{1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}}\geq \frac{q-2{\delta_0}^2}{\rho|\lambda|^2}$ equivalent to

$$\sqrt{1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}}\geq \frac{q}{\rho|\lambda|^2}\quad .$$

Hence, if , $q<0$, then suffices $1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}\geq 0$ or $-\frac{4q{\delta_0}^2}{\rho^2}\leq |\lambda|^4$, if $q\geq 0$, then $1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}\geq \frac{q^2}{\rho^2|\lambda|^4}$ Or $|\lambda|^4\geq \frac{q^2-4q{\delta_0}^2}{\rho^2}$, i.e in any case, it suffices

$$|\lambda|^4\geq \frac{q^2+4q{\delta_0}^2}{\rho^2}.$$

Consequently, the condition $|\delta|\geq|\delta_0|$ satisfied, if

$$\left\{\begin{matrix} |\lambda|^4\geq \frac{q^2+4q|\delta_0|^2}{\rho^2} \\ \cos2\varphi \leq \sqrt{1+\frac{4q{\delta_0}^2}{\rho^2|\lambda|^4}}-\frac{2{\delta_0}^2}{\rho|\lambda|^2} \end{matrix}\right\}$$

Inequality (10) is rewritten as

$$\sqrt{\frac{2\left(\frac{ch2|\delta_0|a-1}{sh2|\delta_0|a}\right)}{\rho(2-w)}}\sqrt[4]{\frac{\rho}{2}\left[-cos2\varphi+\frac{q}{\rho|\lambda|^2}+\sqrt{1-2\frac{q}{\rho|\lambda|^2}cos2\varphi+\left(\frac{q}{\rho|\lambda|^2}\right)^2}\right]}\quad\sqrt{|\lambda|}$$

$$\leq \underset{x\in[0,a]}{Max}|y(x)| \quad \leq$$

$$\sqrt{\frac{2\left(\frac{ch2|\delta_0|a+1}{sh2|\delta_0|a}\right)}{\rho.w}}\sqrt[4]{\frac{\rho}{2}\left[-cos2\varphi+\frac{q}{\rho|\lambda|^2}+\sqrt{1-2\frac{q}{\rho|\lambda|^2}cos2\varphi+\left(\frac{q}{\rho|\lambda|^2}\right)^2}\right]}\sqrt{|\lambda|}$$

This completes the proof of theorem 3.

Conclusion

In this work, we concluded that if change the boundary condition which defined in [1, 4] from our boundary condition, the eigenvalues also changed from pure imaginary into real, but the estimation of the eigenfunctions remain as [1,4].

Open Problem

In the states theorems 2 and 3 , we proved them only in general we don't says if $\rho(a)=1$ or $\rho(a)\neq 1$,means irregular and regular as appear in paper [1], if we takes these case what happen in the estimation of normalized eiegnfuntions?

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