Int. J. Open Problems Compt. Math., Vol. 4, No. 3, September 2011
ISSN 1998-6262; Copyright © ICSRS Publication, 2011
www.i-csrs.org

Resolving Bertrand's Probability Paradox

Jinchang Wang and Rodger Jackson

Jinchang Wang, School of Business, Richard Stockton College of New Jersey, USA. e-mail: jinchang.wang@stockton.edu
Rodger Jackson, School of Arts and Humanities, Richard Stockton College of New Jersey, USA. e-mail: rodger.jackson@stockton.edu

Abstract

In this article we resolve Bertrand's probability paradox and show why it has perplexed researchers for 120 years. This paradox appears to have three equally plausible, yet incompatible, solutions. We begin by showing two facts, the random chords referred to in Bertrand's paradox are homogeneously distributed and there is no disagreement among people on this point. Based on these facts, we rigorously prove that two of the three alleged answers are not sound because of their false assumptions and that only one solution is correct. The paradox is therefore no longer paradoxical. We also reveal two significant stumbling blocks that have persistently caused puzzles with regard to this paradox: 1) misunderstanding of the nature of homogeneously distributed chords, and 2) a disparity between the problem as originally conceived in the mind and its subsequent representation. We conclude by presenting our reflections on the subtleties of the paradox and on some prevalent misconceptions that have historically plagued attempts to resolve it.

Keywords: Paradox, Probability theory, Probability distribution, Geometric probability

2010 Mathematics Subject Classification: 60A05, 60D05, 97E20

1 Introduction

In 1889, the French mathematician Joseph Louis Bertrand put forward a "paradox" in his book *Calcul des probabilités* [2]:

Drawing a chord at random in a circle, what is the probability that the chord is longer than a side of the inscribed equilateral triangle of the circle?

For this problem that looked to have just one solution, Bertrand provided three different but equally plausible solutions. For convenience, we will use term '*Bertrand's paradox*' to refer to this paradox, '*Bertrand-chords*' to represent the chords referred to in Bertrand's paradox, '*equil-tri-side*' to represent 'a side of the inscribed equilateral triangle of a circle', and '*Bertrand-probability*' to represent the 'probability a Bertrand-chord is longer than equil-tri-side.' The three solutions and their supporting arguments are as follows:

Solution-1. A Bertrand-chord is longer than an equil-tri-side if its midpoint lies within the concentric circle with half the original radius. Since the area of this inner circle is a quarter of that of the original circle, Bertrand-probability is 1/4.

Solution-2. A Bertrand-chord is longer than an equil-tri-side if the chord-angle is between 0°~30°, where *chord-angle* refers to the angle between a chord and the radius passing through one endpoint of the chord (to be strictly defined in 4.2). Since the range of a possible chord-angle is 0°~90°, Bertrand-probability is 1/3.

Solution-3. A Bertrand-chord is longer than equil-tri-side if its midpoint lies on the inner half of the radius bisecting the chord. Hence, Bertrand-probability is 1/2.

It has been 120 years since Bertrand put forward his paradox. It still remains in the list of unsolved paradoxes in classic books on paradoxes [18] [21] and in current philosophy dictionaries [15]. It is certainly not just an amusing puzzle. Scholars have argued that it demonstrates that there is a flaw in the principle of indifference [1] [19] [11] [7] [8] [2], and that it is a threat to the principle of maximum entropy [5] [6]. Since the three solutions were derived from no more than basic rules of logic, geometry, and probability, the paradox poses a blatant threat to the fundamentals of our system of knowledge. Tolerating this paradox and leaving it unsolved would amount to admitting that something is wrong in plane geometry, probability theory, or logic. It was this extraordinary challenge that initially motivated us to investigate the mysteries contained in it.

While there have been many attempts to solve Bertrand's paradox, the most recent and substantive attempts have been by Jaynes [9], Marinoff [11], Shackel [19], and Wang [24]. Jaynes went a long way toward proving 1/2 was the correct solution. He also suggested a pragmatic, albeit somewhat naïve, method of generating Bertrand-chords: - tossing broom straws from a standing position onto a 5-in diameter circle drawn on the floor. His proofs are strict and tenable. His method of generating Bertrand-chords is conceptually correct. Unfortunately, he did not show that the other two solutions were incorrect, and he failed to indicate what caused the absurdity. Moreover, his argument that the problem is well posed was incorrect. Marinoff on the other hand, asserted that the paradox was an ill-posed problem. He argued that the idea of a random chord was vague, and that a number of different solutions could be derived self-consistently from different ways of generating random chords. Shackel agreed with Marinoff that the problem was not well posed, calling it "a determinate probability problem which lacks a unique solution." However, he disagreed with both Jaynes' and Marinoff's respective approaches, concluding that "Bertrand's paradox continues to stand in refutation of the principle of indifference." Both Marinoff and Shackel assumed that that Bertrand's paradox was ill posed, but their interpretation of what it means to be ill-posed in this case was mistaken and thereby resulted in irrelevant arguments and multiple solutions. Shackel was also wrong (along with J. Bertrand and many others) in tying Bertrand's paradox to the principle of indifference. Wang indicated that the misunderstanding of the distribution of straight lines caused the paradox, but he did not elaborate this point, nor did he actually prove it. Although he located a central stumbling block to this puzzle, his idea remained only as an intriguing hypothesis without proofs. In his 2010 article, "On Bertrand's paradox", Bangu worked on a different but related Bertrand's paradox in one dimensional space [1]. The author argues that two intervals A and B, where B is a mapping of A with some transformation, are probabilistically different, which actually verifies the thesis in probability theory that the distribution of $\theta(X)$ is in general different from the distribution of X, where X is random variable and $\theta(X)$ is a function of X [17].

We can see that although scholars have provided valuable insights into aspects of the problem, there are critical flaws and lacunae in their arguments, and no one has compellingly explained what exactly went wrong with this paradox. Hence, the perplexity remains in people's minds. The absurdity in this paradox arises from the fact that we cannot rid ourselves of the feeling that there can be only one solution, even though the arguments of three solutions all seem plausible and straightforward. To resolve this paradox completely and compellingly, one must thoroughly release the perplexities in people's minds. One must show whether Bertrand's problem has one or many solutions, because almost all the debates regarding Bertrand's paradox stemmed from this issue. If there is one solution, just deriving the solution is not enough. One must also prove why the other purported "solutions" are wrong. After showing what is correct and why the others are wrong, one must further show what made people take the wrong pass as opposed to the correct one, i.e., what on earth went wrong with this paradox for 120 years. That is, one must accomplish the following ‘must-do’ tasks in order to fully resolve Bertrand’s paradox and to fully resolve the perplexities in people’s mind:

Task One: Show whether the problem has one or multiple solutions.

Task Two: If the problem has one solution, then derive the solution and show why the other purported solutions are incorrect; if the problem has multiple solutions, then show all the correct solutions.

Task Three: Point out the stumbling blocks that generated the various perplexities and absurdities of the paradox. That is, explain what went wrong.

To fail at any of these three must-do tasks would constitute an incomplete, and therefore inadequate, response to the paradox. No one so far has accomplished all three tasks. In the previous efforts, Task One was either dealt with incorrectly or not addressed at all. Marinoff [11] and Shackel [19] both contended, incorrectly, that the problem had multiple solutions since it was ill posed (the falsity will be shown in Section 2 and discussed in Section 8). Their subsequent work based on these faulty assumptions strayed away from the original Bertrand-problem. Jaynes [9] pragmatically showed, with his “broom-straw-tossing method” that the problem had one solution and calculated the correct solution, 1/2. He thus accomplished part of Task One and part of Task Three. But he failed to show why the other two purported solutions were not correct, nor did he explain the nature of the stumbling blocks that created the absurdity. Wang [24] claimed to have identified the stumbling block in his analysis of random chords. But he did not show *why* the problem must have one solution, nor did he demonstrate why two of the three purported solutions were not correct, leaving Tasks One and Two unaccomplished. As the result of these works that are either incomplete or erroneous, Bertrand’s paradox persists as an unsolved paradox in current literatures.

In this article we complete all three must-do tasks, and therefore claim to have successfully resolved the paradox. We show that there must be only one solution to the Bertrand problem in Section 2. We then derive the only solution and disprove the other two purported “solutions” in Sections 3 through 5. In Section 6 we explicate the key stumbling blocks that have turned what was merely a problem into a paradox, and in Section 8 we present our reflections on the subtleties of Bertrand’s paradox and comment, candidly and respectfully, on the prevalent misconceptions in literatures related to this paradox. In order to accomplish this task we may have to reiterate some key arguments as we respond to questions and critiques we received from colleagues.

We need to distinguish between Bertrand's *paradox* on one hand and Bertrand's *problem* on the other. *Bertrand's problem* refers to the problem "drawing a chord at random in a circle, what is the probability that the chord is longer than a side of the inscribed equilateral triangle of the circle?" *Bertrand's paradox* is Bertrand's problem plus the three purported solutions. We will show that Bertrand's problem entails a tacit and unambiguous graphical understanding in people's minds, and it has only one solution. When this problem was written down, however, it in effect transformed into another problem that has three purported solutions. The three purported solutions to the new written problem have been taken as the solutions to the original problem, which thereby sets up the contradiction represented in Bertrand's paradox.

In Bertrand's original essay the three alleged solutions were derived by using no more than geometry and simple probability concepts which can be grasped by anyone with a middle school education. We believe that confusions are best clarified at the place where they occur. Hence, we will avoid employing more advanced mathematics than geometry and basic probability concepts in our arguments.

2 Bertrand-chords Are Uniformly Distributed and so Bertrand's Problem Has One Solution

Bertrand-chords constitute a central issue in understanding Bertrand's problem. Almost all the disagreements regarding Bertrand's paradox stemmed from it. In this section, we investigate and clarify the fundamental issues: What are Bertrand-chords like? Do people agree with each other on what those chords are like? Does Bertrand's problem have one or many solutions?

We believe that a simple fact has been missed all along by everyone who has contended with this paradox, namely: *everyone agrees on what Bertrand-chords are like – they are homogeneously or uniformly distributed over the circle.*

To see this fact, let us start with asking the reader to examine the two sets of chords shown in Fig. 1 and Fig. 2, and give answer to the question, "In which figure are chords more likely Bertrand-chords?"

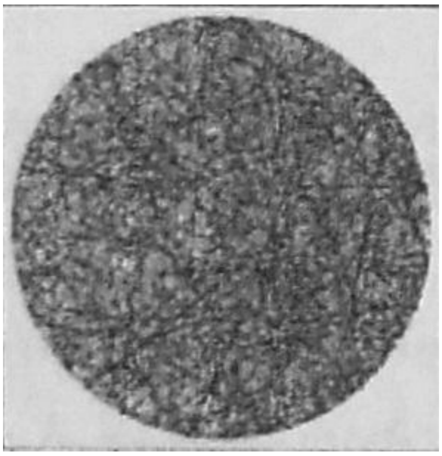

Fig. 1.

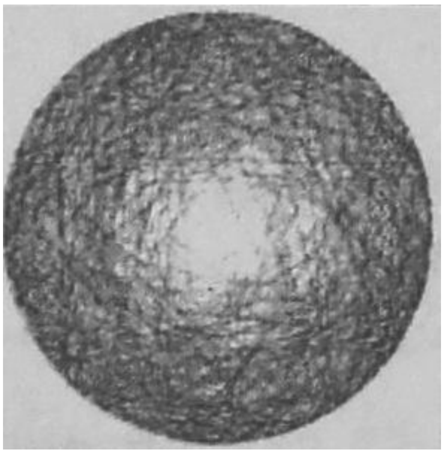

Fig. 2

We believe that readers would unanimously select Fig. 1 as the answer. Fig. 2 would not be selected since there are significantly fewer lines in the central area than in the area closer to the peripheral and hence the chords in this figure are not distributed homogeneously. One would think that if chords had been drawn purely at random, or the chords were truly random, then they would not have systematically missed the central area. Such a consensus illustrates that there is no disagreement among people on what Bertrand-chords are like in their minds: - They are homogeneously and uniformly spread over the circle. Suppose that we substituted 'Bertrand-chords' in the above question with 'random chords', people would still pick Fig. 1 as the answer. So, *we in fact all agree that both "Bertrand-chords" and "random chords" mean the chords that are homogenously or uniformly distributed over the circle.*

Another way of showing the existence of the fact that people are consentient on what Bertrand-chords are like is by considering what actually causes the tension in the paradox. The perplexity comes from the absurdity of three plausible solutions to a problem that ought to have just one solution. Yet, *why* do people think that this problem *ought to* have just one solution? The reason can only be: Bertrand-chords are homogeneously distributed in people's minds. To see this, consider this question, "what is the percentage of points on average which are greater than 0.5 among the points randomly drawn from [0, 1]?" We would all agree that this question has only one answer, which is 50%, since we assume the points randomly drawn from [0, 1] are truly random therefore homogeneously distributed over [0, 1]. Similarly, we believe that Bertrand's problem has only one answer because we assume, implicitly, Bertrand-chords are truly random therefore homogeneously distributed over the circle. If, on the other hand, we did not assume homogeneity of Bertrand-chords, we would not insist on a unique solution. This is because while homogeneity has only one distribution pattern, non-homogeneity may have countless patterns. With a non-homogeneous distribution, for example, chord densities may vary in many ways. The area with higher density may be located at the central area, at the peripheral area, or at any place in the circle; and the density in one area can be 2 time, 3 times, 8.5 times, or 1.67 times as much as that in some other areas. So, if Bertrand-chords were *not* homogeneously distributed, there would be many solutions to the problem, each of which is associated with a pattern. As a result, we would not think in the first place there ought to be only one solution to the problem, and would not take the problem for a paradox. Therefore, Bertrand-chords must be homogeneously distributed in every one's mind.

This fact can also be seen in the unstated assumption of the three solutions. J. Bertrand used the phrase "drawing chords at random" to describe his Bertrand-chords. Then, he derived the three solutions, 1/4, 1/3, and 1/2. By reviewing his arguments of the three solutions carefully, we can see that each solution was based on an unstated assumption of homogeneity. For instance, the chords' midpoints must be assumed to be homogeneously distributed in the circle in order to derive 1/4 as Solution-1; and the chord-angles must be assumed to be homogeneously distributed between 0° and 90° so as to derive 1/3 as Solution-2; and the chords' midpoints must be assumed to be homogeneously distributed on a radius of the circle in order to derive 1/2 as Solution-3. Behind those unstated assumptions of homogeneity in the arguments for the three purported solutions is another unstated but obvious assumption: - Bertrand-chords are homogeneously or uniformly distributed.

Why didn't Bertrand, and the scholars thereafter, explicitly state these assumptions? Probably because they did not think it was necessary since "chords drawn at random" would, needless to say, be homogeneously distributed just like "points drawn at random" are surely homogeneously distributed, and there was just one pattern of homogeneous distribution. If Bertrand had assumed any distribution *but* homogeneity for his chords, he would have specified the pattern of the distribution in order to avoid conceptual ambiguity. But he did not specify it, no one has since bothered to specify it, and no scholar has ever claimed that Bertrand-chords are *not* homogeneously distributed. That is because we in fact have never disagreed with each other on what Bertrand-chords are like.

Therefore, it is manifest that Bertrand-chords are homogeneously distributed, which is tacitly accepted by all of us. With this fact revealed and clarified, we can see that *Bertrand's problem must have one and only one solution* since homogeneously distributed Bertrand-chords have just one distribution pattern: - homogeneity. However, this *tacitly* taken fact has not been *explicitly* recognized. This lapse has led to innumerable difficulties. To make matters worse, "multiple ways of generating Bertrand-chords" was erroneously viewed as the cause of the puzzle even in recent major articles as [19].

We now need to determine *what* is the only solution of Bertrand's problem and to disprove the other purported "solutions". To do that, we need to formally define Bertrand-chords, as we are going to do in the next section.

3 Defining Bertrand-chords or Homogeneously Distributed Chords

Random lines have been defined mathematically in the literature in terms of Poisson process or Poisson distribution [12] [6] [20] [10]. The essence of Poisson process is "pure randomness". Pure randomness leads to homogeneity. The essence of random lines, therefore, is homogeneity [12]. But, to our surprise, those exact descriptions of random lines have not even been cited in the recent articles on Bertrand paradox such as [11] [19], much less been utilized to solve the paradox.

As we have showed in the last section, Bertrand-chords are homogeneously distributed random chords over a circle, as those in Fig. 1. Bertrand-chords are purely random lines contained in a circle. In this section, we define Bertrand-chords in plain, but exact, words, trying to avoid excessively technical terms such as 'Poisson process' with the purpose of making our arguments understandable by anyone who is intrigued by Bertrand paradox.

Suppose an X-Y coordinate system is placed in the space on which chords lie. The *chord-direction* of a chord *h* is the angle that *h* has to turn clockwise in order to be parallel to the X-axis of the coordinate system. So, the range of a chord-direction is [0°, 180°). Since it is a continuous range composed of infinitely many numbers, for a given number β in [0°, 180°), probability for a random chord to have chord-direction β is zero. Considering that, when we say a set of parallel chords with chord-direction β, we mean those chords with chord-directions within the range $[\beta°-\Delta\theta°, \beta°+\Delta\theta°]$, where $\Delta\theta$ is a very small amount.

Let $\phi(O, r)$ denote a circle with radius r and the center at O. Let C represent a set of chords that are drawn at random on $\phi(O, r)$. Let C_α represent a subset of C which contains the chords with chord-direction within the range $[\alpha°-\Delta\theta°, \alpha°+\Delta\theta°]$ where α is a number between 0° and 180°, and $\Delta\theta$ is an arbitrarily small but fixed amount. C_α represents a set of parallel chords with chord-direction α. Let $\mathbf{D}_{\alpha\text{-normal}}$ be the diameter of $\phi(O, r)$ normal to the parallel chords in C_α, i.e. the chord-direction of $\mathbf{D}_{\alpha\text{-normal}}$ is $\alpha+90°$ if $\alpha\leq 90°$ or $\alpha-90°$ if $\alpha>90°$.

Uniformly distributed *points* are well defined in probability theory. We now use the concept of points to define homogeneous distributed Bertrand-chords.

Definition-A.

Chords in C are Bertrand-chords or homogeneously distributed chords if and only if their chord-directions are uniformly distributed over range [0°, 180°), and for any α between 0 and 180, the intersecting points of the chords in C_α with diameter $\boldsymbol{D}_{\alpha\text{-normal}}$ are uniformly distributed along $\boldsymbol{D}_{\alpha\text{-normal}}$.

□

This definition is similar to those in [12] [6]. The difference is that we avoid using the terminology "Poisson process". Points associated with a Poisson process are purely random points, which are hence uniformly distributed. Definition-A says that the chords in C are Bertrand-chords if any subset of parallel chords uniformly intersects the diameter perpendicular to them. Since the chords defined in Definition-A are homogeneously spread over the circle, points in the circle must have same chance to be on a chord in C. So we have an alternative definition based on the homogeneity of the chords directly, as below.

Definition-B.

Let Δr denote an arbitrarily small but fixed amount. Chords in C are Bertrand-chords or homogeneously distributed if and only if their chord-directions are uniformly distributed over range [0°, 180°), and for any two points P and Q in circle $\phi(O, r)$, probability that circle $\phi(P, \Delta r)$ is on a chord in C is same as probability that circle $\phi(Q, \Delta r)$ is on a chord in C.

□

"$\phi(P, \Delta r)$ is on a chord" in Definition-B means "a chord passing through circle $\phi(P, \Delta r)$". For convenience, in the text hereafter, we use a shorter phrase "chance for a point P to be on a chord" to represent the longer phrase "probability for circle $\phi(P, \Delta r)$ to be on a chord where Δr is an arbitrarily small but fixed amount". The above definition is thus shortened as:

Definition-C.

A set of chords, C, are Bertrand-chords or homogeneously distributed if and only if their chord-directions are uniformly distributed over range [0°, 180°) and any point in the circle has same chance to be on a chord in C.

□

By Definition-C, if some points in the circle do not have same chance to be on a chord in C then the chords are not Bertrand-chords nor uniformly distributed chords.

The three definitions are equivalent. Each can be used independently to determine whether a set of chords are Bertrand-chords. In Fig. 2, for example, a point in the central area of the circle has less chance to be on a chord than a point in the area close to the peripheral. So, according to Definition-C, the chords in Fig. 2 are neither Bertrand-chords nor homogeneously distributed chords.

4 Proving the Correct Solution and Disproving the Incorrect Solutions

In Section 2 we showed that we all have the same image of what Bertrand-chords are like, - they are homogeneously distributed over the circle, and hence, Bertrand's problem has only one solution. In Section 3 we defined homogeneously distributed chords or Bertrand-chords. We now derive *what* the unique solution is and show why the other purported "solutions" are not correct.

The way to address our solution has been inspired by what we take to be a "Bertrandian" approach. Bertrand himself did not rely on complicated, high level mathematics to set up the problem. With the "Bertrandian approach", the absurdity in Bertrand's paradox can be seen by people who possess little beyond a basic grasp of middle school mathematics. We believe that Bertrand's paradox is best resolved with the "Bertrandian approach". We follow that way by managing to set up our arguments and proofs on just plane geometry and basic rules of logic and to make our arguments understandable by anyone who sees the puzzle.

We provide the basic logical structure of our proofs in this section, leaving some details of the proofs to the subsequent one. Our hope is that this will help understand our proofs by allowing readers to examine the general structure of our arguments without being interrupted by the technical details, even though those meticulous expositions are necessary parts of the proofs.

4.1 The 1/4 solution is unsound

Bertrand's Solution-1 is 1/4. The argument is as follows:

A chord is longer than an equil-tri-side if its midpoint lies within a concentric circle with half the original radius. Since the area of this inner circle is a quarter that of the original circle, the Bertrand-probability would be 1/4 (as in Fig. 3) (4.1-A)

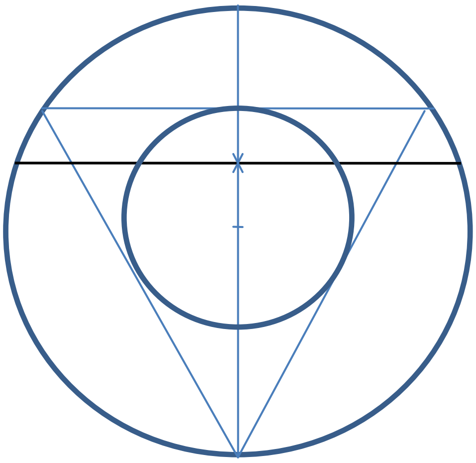

Fig. 3. For Solution-1

The argument (4.1-A) is acceptable only if the midpoints of those Bertrand-chords are uniformly distributed in the circle, so as to derive the probability 1/4 from the ratio between the area of the inner circle and the area of the original circle. The unstated assumption can be put as follows:

If a set of chords are Bertrand-chords, then their midpoints are homogeneously distributed in the circle. (4.1-B)

Since Bertrand-chords are meant to be homogeneously distributed chords, (4.1-B) can be equivalently put as:

If a set of chords are homogeneously distributed over the circle, then their midpoints are homogeneously distributed in the circle. (4.1-C)

(4.1-B) and (4.1-C) are equivalent. They are necessary for deriving the solution 1/4. If these assumptions are false, then the solution 1/4 is not sound.

The assumption addressed in (4.1-B) and (4.1-C) is *indeed false*. To show it, consider a method of drawing chords at random as follows:

Chord-Drawing-Method-1.

Select a point M at random in the circle and draw the chord whose middle point is M.

The middle points of the chords drawn in this way are homogeneously distributed in the circle, because those points are selected purely at random. But the chords drawn in this way are not homogeneously distributed. This fact is stated in the following statement:

If the middle points of a set of chords drawn at random are homogeneously distributed in the circle, then that set of chords are not homogeneously distributed over the circle. (4.1-D)

We have proved that statement (4.1-D) is true. The proof is put in Section 5.1.

Since (4.1-D) is true, it leads automatically to the truth of its contraposition as below:

If a set of chords drawn at random are homogeneously distributed over the circle, then their middle points are not homogeneously distributed in the circle.

(4.1-E)

Statements (4.1-E) and (4.1-C) have identical antecedents but opposite consequents. Since (4.1-E) has been proved to be true, (4.1-C) must be false, so must be its equivalence (4.1-B). Therefore, the assumption of solution 1/4 is false and the solution is unsound.

4.2 The 1/3 solution is unsound

Bertrand's Solution-2 is 1/3. The argument is as follows:

A chord with its end point at any place of the circumference is longer than an equil-tri-side if its chord-angle is between 0°~30° which is one third of the possible range 0°~90°. (as in Fig. 4) (4.2-A)

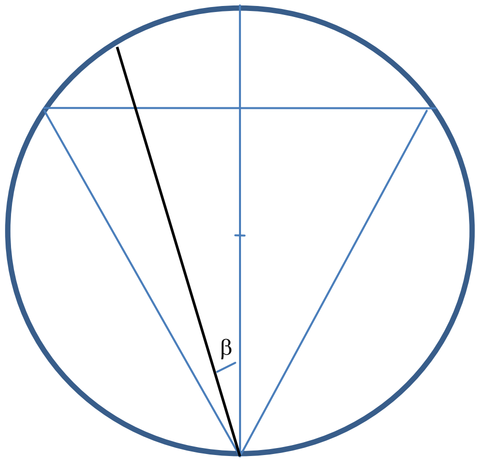

Fig. 4. For Solution-2

The term *chord-angle* of a chord, as used in (4.2-A), refers to the angle between a chord and the radius passing through one endpoint of the chord. Let us define it with mathematic rigor. Let A and B be the two endpoints of chord AB on the circumference of circle $\phi(O, r)$, as in Fig. 5. The *clockwise-angle* of chord AB at A, $\angle A_c$, is defined as the angle that chord AB needs to turn clockwise about point A to overlap with the radius OA. The clockwise-angle of chord AB at B, $\angle B_c$, is defined in the same way. Apparently, $\angle A_c + \angle B_c = 180°$. The *chord-angle of chord AB* is defined as the minimum of $\angle A_c$ and $\angle B_c$. Clearly, the chord-angle of a chord is between 0° and 90°. In Fig.5, angle β is the chord-angle of chord AB. Each chord has one and only one chord-angle. Furthermore, given an angle 0°≤β°≤90° and a point Y on circumference, there is one and only one chord with Y as one of its end point and with β° as its chord-angle.

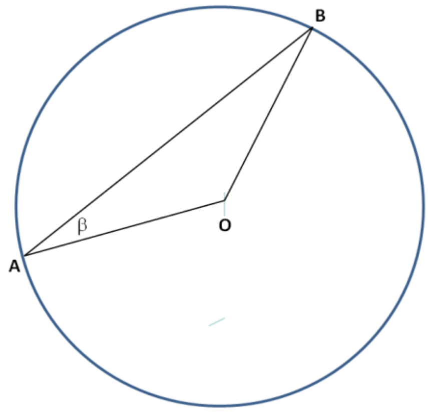

Fig. 5

Note that *chord-angle* here is different from *chord-direction* defined in Section 3. A chord-angle is relative to the radius passing through one endpoint of the chord. The radius changes its direction with the chord. A chord-direction is relative to the X-axis of the coordinate system which remains fixed for all chords.

The argument for the solution in (4.2-A) is acceptable only if the chord-angles of those Bertrand-chords are uniformly distributed between 0° and 90°, so as to derive the probability 1/3 from the ratio between 30° and 90°. The unstated assumption can be put as follows:

If a set of chords are Bertrand-chords, then their chord-angles are uniformly distributed between 0° and 90°. (4.2-B)

Since Bertrand-chords are meant to be homogeneously distributed chords, (4.2-B) can be equivalently put as:

If a set of chords drawn at random are homogeneously distributed over the circle, then their chord-angles are uniformly distributed between 0° and 90°.

(4.2-C)

(4.2-B) and (4.2-C) are equivalent. They are necessary for deriving the solution 1/3. If the assumptions are false, then the solution 1/3 would is not sound.

The assumptions addressed in (4.2-B) and (4.2-C) are *indeed false*. To show it, consider a method of drawing chords at random as follows:

Chord-Drawing-Method-2.

Randomly select a number β between 0 and 90. Randomly select a point A on the circumference. Draw the chord through A whose chord-angle is β°.

The chord-angles of the chords drawn in this way are uniformly distributed between 0° and 90°, because the chord angles are selected purely at random in that range. But the chords drawn in this way are not homogeneously distributed. This fact is stated in the following statement:

If the chord-angles of a set of chords drawn at random are uniformly distributed between 0° and 90°, then these chords are not homogeneously distributed over the circle. (4.2-D)

We have proved that statement (4.2-D) is true. The proof is put in Section 5.2.

Since (4.2-D) is true, it leads automatically to the truth of its contraposition as below:

If a set of chords drawn at random are homogeneously distributed over the circle, then their chord-angles are not uniformly distributed between 0° and 90°. (4.2-E)

Statements (4.2-E) and (4.2-C) have identical antecedents but opposite consequents. Since (4.2-E) has been proved to be true, (4.2-C) must be false, so must be its equivalence (4.2-B). Therefore, the assumption of solution 1/3 is false and the solution is unsound.

4.3. The 1/2 solution is correct.

Bertrand's Solution-3 is 1/2. The argument is as follows:

A chord is longer than equil-tri-side if its midpoint lies on the inner half of the radius bisecting the chord. (as in Fig. 6) (4.3-A)

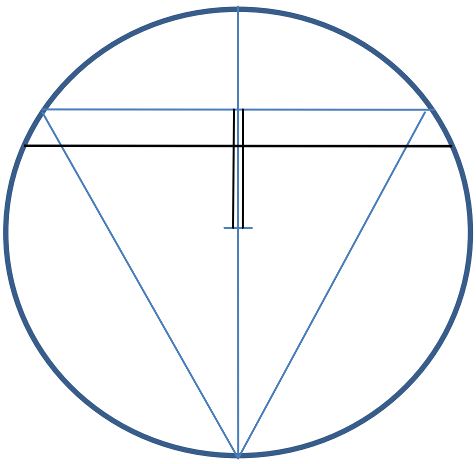

Fig. 6. For Solution-3

The argument (4.3-A) is true only if the midpoints of those Bertrand-chords perpendicular to a given radius R are uniformly distributed along R, so as to derive the probability 1/2 from the ratio between the half length of R and the full length of R. The unstated assumption can be put as follows:

If a set of chords, C, are Bertrand-chords, then for any given radius R the intersection points of R with those chords perpendicular to R are uniformly distributed on R. (4.3-B)

(4.3-B) is obviously true according to Definition-A of Bertrand-chords given in Section 3. Take a given radius R as a part of diameter $\mathbf{D}_{\alpha\text{-normal}}$ for some α, and tale the chords perpendicular to R as the parallel chords normal to $\mathbf{D}_{\alpha\text{-normal}}$. Since the chords in C are homogeneously distributed Bertrand-chords, the intersection points of those parallel chords on R, a part of $\mathbf{D}_{\alpha\text{-normal}}$, must be uniformly distributed along R according to Definition-A.

So, the assumption of (4.3-A) is true. Since the other arguments in (4.3-A) are obviously valid, Solution-3 is sound and Bertrand-probability is equal to 1/2.

5 Proofs of (4.1-D) and (4.2-D)

This section contains the details of proofs for statements (4.1-D) and (4.2-D) cited in Section 4.

5.1. Proving statement (4.1-D) to be true.

If the middle points of a set of chords drawn at random are homogeneously distributed in the circle, then that set of chords are not homogeneously distributed over the circle. (4.1-D)

Let C_1 denote a set of chords that are generated by Chord-Drawing-Method-1 in Section 4.1. Obviously, the middle points of the chords in C_1 are homogeneously distributed in the circle, which satisfies the antecedent of statement (4.1-D). We now prove that the chords in C_1 cannot be homogeneously distributed, therefore (4.1-D) is true. To do that, we need to establish a lemma.

Lemma 1.

Let X be an internal point of circle $\phi(O, r)$, H the middle point of line segment OX, and t the half of the length of line segment OX, i.e. $t=0.5|OX|$, as shown in Fig. 7. A chord runs through X if and only if its middle point is on circle $\phi(H, t)$.

Proof:

Firstly we prove that if the middle point of a chord is on circle $\phi(H, t)$, then the chord runs through X. Suppose a chord AB's middle point M is on circle $\phi(H, t)$, where A and B are on the circle, as shown in Fig. 7. Connect X with M, and connect M with O. $\angle XMO=90°$ since M is on circle $\phi(H, t)$ and XO is the diameter of $\phi(H, t)$. And $\angle AMO = \angle BMO = 90°$ since M is the middle point of chord AB. Since $\angle XMO = \angle AMO$, X must be on chord AB or its extended line. But X cannot be on the extended line of chord AB since X is an internal point of $\phi(H, t)$. So, chord AB runs through X.

We next prove that if a chord runs through X, then the chord's middle point must be on circle $\phi(H, t)$. Suppose chord AB runs through X whose chord middle point is M. ΔXMO is a right triangle with $\angle XMO=90°$ and H the middle point of the hypotenuse, because M is the middle point of chord AB which runs through X. So, $|MH|=|XH|$, which follows a theorem in geometry saying "the length of the central line of the hypotenuse of a right triangle is equal to half of the hypotenuse." But $|XH|=t$ that is the radius of circle $\phi(H, t)$, which means the distance between M and H is t. Therefore M is on circle $\phi(H, t)$ with center at H.

□

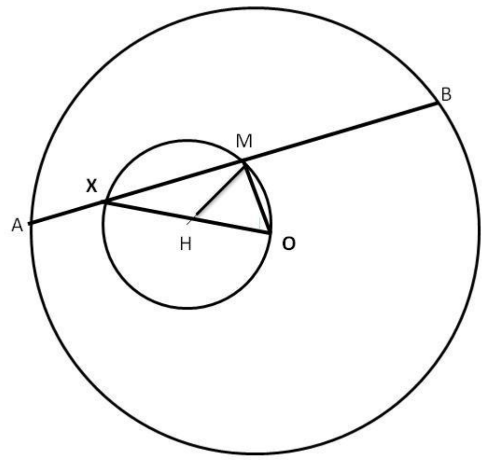

Fig. 7.

Lemma 1 shows that the middle points of those chords passing through a given internal point X form a circle that takes line segment OX as the diameter, and that a chord runs through X if and only if the chord middle point is on that circle.

Lemma 2 below shows that points in the circle do not have equal chance to be on a chord in C_1. Instead, a point close to the center of the circle has a smaller chance than a point close to the circumference. In other words, if a set of chords are generated by Chord-Drawing-Method-1, so that their middle points are homogeneously distributed, then the density of chords in the area close to the circumference is higher than the density in the central area, as those chords in Fig. 2. So, those chords must be not homogeneously distributed, and (4.1-D) is true.

Lemma 2.

The chords in C_1, which are generated by Chord-Drawing-Method-1, cannot be homogenously distributed, therefore statement (4.1-D) is true.

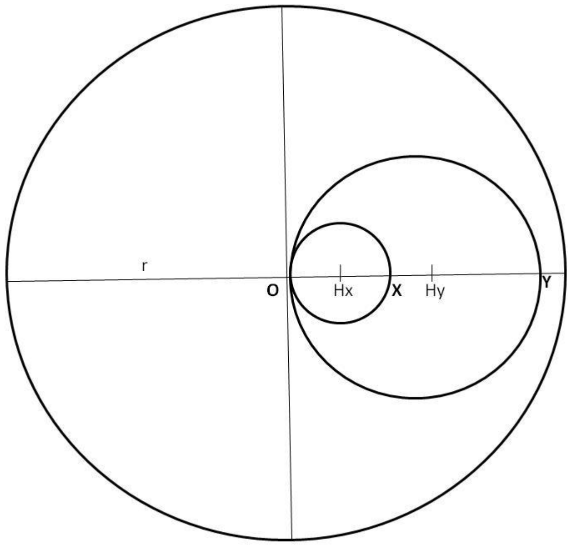

Fig. 8

Proof:

Consider two internal points X and Y in ϕ(O, r), and assume OX<OY, as in Fig.8. Let H_x and H_y denote middle points of segments OX and OY respectively. Let t_x=0.5|OX| and t_y=0.5|OY|. According to Lemma 1, X is on a chord if and only if a point on circle $\phi(H_x, t_x)$ happens to be selected as the chord middle point. Similarly, Y is on a chord if and only if a point on circle $\phi(H_y, t_y)$ happens to be selected as the chord middle point. Since the chord middle point is randomly selected in Chord-Generation-Method-1, a point on circle $\phi(H_x, t_x)$ has a smaller chance to be selected than a point on $\phi(H_y, t_y)$ because the perimeter of $\phi(H_x, t_x)$, $|OX|\pi$, is shorter than the perimeter of $\phi(H_y, t_y)$, $|OY|\pi$. Therefore, point X has a smaller chance to be on a chord in C_1 than point Y does (recall that 'chance a point P to be on a chord' means 'probability for $\phi(P, \Delta r)$ to be on a chord where Δr is an arbitrarily small but fixed amount' as specified in Section 3) . According to Definition-C, the chords in C_1 cannot be homogeneously distributed.

Statement (4.1-D) is thereby true since chords in C_1 satisfy the antecedent of (4.1-D), whereas they cannot be homogeneously distributed as we have just proved.

□

5.2. Proving statement (4.2-D) to be true

If the chord-angles of a set of chords drawn at random are uniformly distributed between 0° and 90°, then these chords are not homogeneously distributed over the circle. (4.2-D)

Let C_2 denote a set of chords that are generated by Chord-Generation-Method-2 in Section 4.2. Obviously, the chord-angles of the chords in C_2 are uniformly distributed between 0° and 90°, which satisfies the antecedent of statement (4.2-D). We now prove that the chords in C_2 cannot be homogeneously distributed, therefore (4.2-D) is true. To do that, we need to establish a few lemmas.

Lemma 3.

Let X denote an internal point in circle ϕ(O, r). Given a point P on the circumference of ϕ(O, r), there is one and only one chord PX running through X. Let $\theta_{x,P}$ denote the chord-angle of chord PX. Let $\theta_x=\theta_{x,A}$ denote the particular chord-angle of X such that A is on the circumference and AX⊥OX (Fig. 9). Then,

$$\theta_x = \underset{among\,all\,P's\,on\,circumference}{MAX} \theta_{x,P} .$$

Proof:

As shown in Fig. 9, the chord-angle of X at P, $\theta_{x,P} = \angle XPO$. In triangle OXP, $\frac{OX}{\sin\angle XPO} = \frac{OP}{\sin\angle OXP}$. By cross-multiplying, and noting that OP=r, we have $\sin\angle XPO = \frac{OX}{r}\sin\angle OXP$. Since OX and r are constants, the maximum value of sin∠XPO occurs when sin∠OXP=1, i.e. PX⊥OX. Therefore, ∠XPO takes its maximum value θ_x when P is at the point A such that AX⊥OX.

□

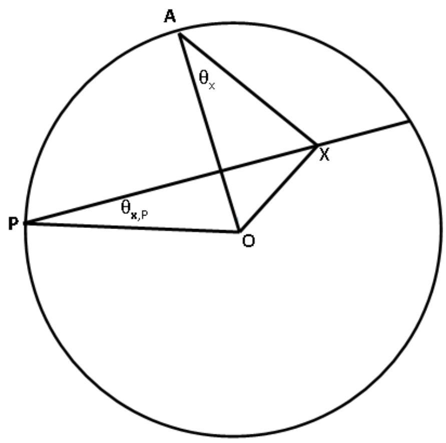

Fig. 9

Lemma 3 characterizes a special chord passing through a point X. The special chord is perpendicular to the radius OX and its chord-angle is the largest compared to the chord-angles of the other chords passing through X. That is, for each point X inside the circle, there is a number θ_x associated with X so that any chord passing through X must have its chord-angle less than or equal to θ_x. In other words, given a internal point X, it is impossible to have a chord passing through X with the chord-angle greater than θ_x. Only those chords with the chord-angles less than or equal to θ_x can pass through the given internal point X. θ_x is called the maximum-chord-angle of chords passing through X, or simply maximum-chord-angle of X.

Lemma 4.

If X and Y are two internal points of circle $\phi(O, r)$ and OX<OY, then $\theta_x < \theta_y$, where θ_x is the maximum-chord-angle of X and θ_y is the maximum-chord-angle of Y .

Proof:

Let A_x and A_y be the two points on the circumference so that the maximum-chord-angle of X, θ_x, occurs at A_x and maximum-chord-angle of Y, θ_y, occurs at A_y. By Lemma 3, $\sin\theta_x = OX/r$ and $\sin\theta_y = OY/r$. Since OX<OY, we have $\sin\theta_x<\sin\theta_y$ and $\theta_x<\theta_y$.

□

Lemma 4 states that the maximum-chord-angle of a point close to the circle center is smaller than the maximum-chord-angle of a point close to the circumference. Lemma 5 below shows that two points have an equal chance to be on the chord of chord-angle β if β is less than the maximum-chord-angle of anyone of the two points.

Lemma 5.

Take two internal points X and Y of $\phi(O, r)$ such that OX<OY. Take a number β such that $\beta< \theta_x$ and $\beta< \theta_y$ where θ_x and θ_y are maximum-chord-angle of X and Y respectively. Let A be a randomly selected point on the circumference of $\phi(O, r)$. The chance that X is on a chord through A with chord-angle $\beta°$ is the same as the chance that Y is on a chord through A with chord-angle $\beta°$.

Proof:

According to Lemma 3, it is possible to have a chord with chord-angle β passing through X since $\beta< \theta_x$, and it is also possible to have a chord with chord-angle β passing through Y since $\beta< \theta_y$.

For the given chord-angle β and internal point X, there are two points A_{x1} and A_{x2} on the circumference of $\phi(O, r)$ such that chords XA_{x1} and XA_{x2} pass through X with the chord-angle β (Fig. 10). So, given chord-angle β, the chance that X is on a chord is equal to the chance that the circumference point A_{x1} or A_{x2} is selected as an end point of the chord. By the same token, for the given β and internal point Y, there are two points A_{y1} and A_{y2} on the circumference of $\phi(O, r)$

such that chords YA_{y1} and YA_{y2} pass through Y with chord-angle β. So, given chord-angle β, the chance that Y is on a chord is equal to the chance that the circumference point A_{y1} or A_{y2} is selected as an end point of the chord.

Since A is a point randomly selected on the circumference, the four circumference points A_{x1}, A_{x2}, A_{y1}, and A_{y2} have same chance to be selected as A. Therefore, the chance that X is on a chord running through A with chord-angle β is the same as the chance that Y is on a chord running through A with chord-angle β. □

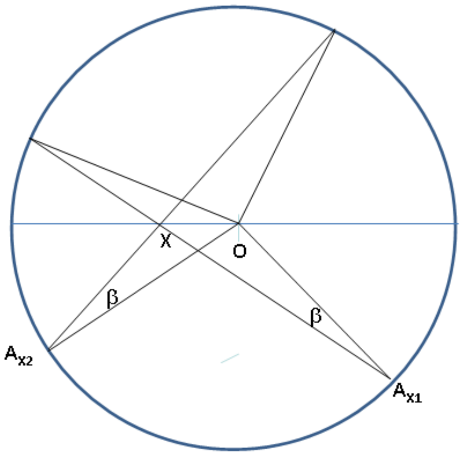

Fig. 10

Lemma 6 below shows that points in the circle do not have an equal chance to be on a chord that is generated by Chord-Drawn-Method-2. That is, the chords in C_2 cannot be homogeneously distributed. The chord density in the area close to the circumference is higher than that in the area close to the center, as those chords in Fig. 2. Therefore, chords in C_2 must not be homogeneously distributed, and statement (4.2-D) is true.

Lemma 6.

Chords in C_2, which are generated by Chord-Drawn-Method-2, cannot be homogeneously distributed, therefore statement (4.2-D) is true.

Proof:

When generating a chord in C_2 by using Chord-Drawn-Method 2, we pick a number β between 0° and 90° as the chord-angle of the chord, and then pick a point A on circumference as an end point of the chord. Consider two internal points X and Y such that OX<OY. By Lemma 4, $\theta_x < \theta_y$.

If $\beta > \theta_y$, then $\beta > \theta_x$. So, neither Y nor X can be on the newly generated chord according to Lemma 3, no matter whatever point A on the circumference is selected as the end point of the chord.

If $\theta_y \geq \beta > \theta_x$, then there exist two points, A_{y1} and A_{y2}, according to the proof of Lemma 5, on the circumference so that if anyone of the two is selected as an end point A then the chord will run through Y. That is, there is some chance for Y to be on the newly generated chord. But, by Lemma 3, there is no chance at all for X to be on the newly generated chord since $\beta > \theta_x$.

If $\beta \leq \theta_x$, then $\beta \leq \theta_y$ since $\theta_x < \theta_y$. According to Lemma 5, both X and Y can be on the newly generated chord with an equal chance.

In summary, whenever X has a chance to be on a chord, Y has the same chance; while at the time X does not have a chance to be on a chord, Y still has; and at the time Y does not have a chance to be on a chord, X does not have either. Therefore, Y has larger chance to be on a chord generated by this method, and the chords in C_2 must be non-homogeneously distributed over the circle according Definition-C.

Statement (4.2-D) is thereby true since chords in C_2 satisfy the antecedent of (4.2-D), whereas they cannot be homogeneously distributed as we have just proved. □

6 Stumbling Blocks to Resolving Bertrand's Paradox

We have proved that two of the purported solutions, 1/3 and 1/4, are unsound and that only 1/2 is the solution to Bertrand's problem. We will now step out from the technical details of the proofs and give our thoughts on the stumbling blocks that have caused the accompanying perplexities and prevented people from fully resolving the paradox for 120 years.

The first stumbling block is about what Bertrand-chords are like. Almost all the disagreements regarding Bertrand's paradox stemmed from it. Some scholars asserted that the chords referred to in Bertrand's problem were, in some way, vague; and the other scholars disagreed but failed to show that Bertrand-chord is an unambiguous conception. The debate on what Bertrand-chords are like led to the debate on how many solutions Bertrand's problem had. In their articles [11] [19] Marinoff and Shackel contended that Bertrand-chords were an ill-posed conception and, hence, Bertrand's problem did not have a unique solution. But people were not convinced by the arguments of Marinoff and Shackel, because their instincts told them that Bertrand's problem *ought* to have only one solution. We have cleared away this stumbling block by showing in Section 2 that people's instincts are correct, and Marinoff and Shackel are wrong. Bertrand-chords have a clear and consistent picture in people's minds, rather than being vague or understood differently by people. They are homogeneously distributed. Therefore, Bertrand's problem has only one solution. Having cleared away this stumbling block, we can further build up our arguments to compellingly resolve the paradox. Without clearing away this stumbling block, on the other hand, any “solution” would not be tenable because it would not be convincing as to what Bertrand-chords are like in the first place, as those “solutions” reported previously by the scholars such as Marinoff and Shackel.

The second stumbling block is the lack of knowledge about homogeneously distributed chords. Two different concepts, homogeneously distributed Bertrand-chords and randomly drawn chords, have been mistakenly conflated by scholars who worked on this paradox. We have cleared away this stumbling block by showing, in Section 4 and Section 5, that the chords drawn at random by using Chord-Drawn-Method-1 or Chord-Drawn-Method-2 are not Bertrand-chords, and purported solutions 1/3 and 1/4 derived from them are therefore not the solution to Bertrand’s problem. Hence, *randomly drawn chords are not necessarily Bertrand-chords*.

To see why randomly drawn chords are not necessarily homogeneously distributed, let us compare the concepts of randomly drawn *chords* with randomly drawn *points*. In probability theory, points drawn at random from the range [0,1] are uniformly or homogeneously distributed over [0,1]. But this feature of random points cannot be simply extrapolated to random chords. The distribution of a set of randomly drawn chords would be dependent on the method of “drawing”. By probability theory, the distribution of a function of random variable X is in general not the same as the distribution of X. Particularly, if X is uniformly distributed, X^2 is not. Generally, if we do some transformations or conversions on X, the resulting distribution would typically deviate from the original distribution of X. This fact was re-visited in Bangu’s recent article [1]. When drawing *chords* from randomly picked *points*, “transformations” are always needed in order to form the chords from the “randomly selected” points. Even though the points are purely random, the distribution of the chords obtained from some “transformation” on top of the randomness would not necessarily be purely random. Therefore, chords drawn at random may not be homogeneously distributed,

This important fact, chords drawn at random are not necessarily homogeneously distributed, has been ignored by all who worked on Bertrand’s paradox. Take a look at the phrases for Bertrand-chords used in the literature. Bertrand [2], Jaynes [9], Shackel [19], Marinoff [11], Sorensen [21], and the Oxford Philosophy Dictionary [15] spoke of a Bertrand-chord as “drawing a chord at random” or “a randomly drawn chord”. On the other hand, Sainsbury [18], Vujicic [23] and Virtual Laboratories [22] worded it as: “a random chord”.

These phrases were simply accepted as synonyms by all authors. No one has ever cast serious doubt on this or even thought to explore the issue.

We can now see that the phrase "chords drawn at random" is an incorrect term for Bertrand-chords since randomly drawn chords are not necessarily homogeneously distributed and therefore not necessarily Bertrand-chords. No scholar who has written on Bertrand's paradox, including J. Bertrand himself, has ever pointed out the misrepresentation; and no one has ever tried to distinguish "a chord drawn at random" from "a random chord". Actually, the term "random chords" is an acceptable wording for Bertrand-chords if it means "truly random chords", because "truly random chords" would be homogeneously distributed, as "truly random points" would be uniformly distributed. But those who happened to use this acceptable wording, such as Sainsbury [18] and Vujicic [23], did not realize their "random chords" must be truly random without any subsequent transformations on top of the randomness.

The above two stumbling blocks set obstacles to fully resolving the paradox. Because of their existence plus the *lack of recognition* of their very existence, the claimed "solutions" to Bertrand's paradox so far in the literature are invariably unconvincing. As the result, Bertrand's paradox has obstinately stayed in the list of unsolved paradoxes.

7 Bertrand's Paradox Is Dissolved

In general, a paradox, according to Sainsbury, is an absurd situation in which "an unacceptable conclusion is derived by apparently acceptable reasoning from apparently acceptable premise" [18]. Thus, a paradox is resolved, or more exactly, dissolved, if the conflict between the acceptableness of reasoning process and un-acceptableness of its conclusion is eliminated, either by showing that the conclusion is acceptable or the reasoning process is unacceptable. We have showed that the reasoning processes for deriving two of the three purported solutions are unacceptable. Hence, the conflict existing in Bertrand's paradox is cleared away, and the paradox is therefore dissolved.

In particular, for this paradox, recall the three "must-do tasks" to resolve it as discussed in Section 1. For Task One, we showed in Section 2 that Bertrand's problem has one and only one solution. For Task Two, we derived the correct solution, and showed why the other two solutions were not correct, in Sections 4 and 5. For Task Three, we identified in Section 6 the stumbling blocks that generated the perplexity. We have accomplished all three must-do tasks for resolving Bertrand's paradox. Therefore, we claim that this paradox is resolved.

8 Reflections and Clarifications

This article would end here if it were solving an ordinary problem. However, Bertrand’s paradox has been a particularly intransigent puzzle, due in large part to the many entangled recessive elements arising from sloppy assumptions, misunderstandings, careless conclusions, and misconceptions. In this last section we will lay out our reflections and clarifications on the subtleties of the puzzle, on the soundness and completeness of our method, and on the prevalent misunderstanding and false assumptions, so as to help fully cleanse the perplexities of the paradox which have haunted people for a long time. At some places, we have to point out candidly, but with respect, errors and misconceptions that have existed in the literature on Bertrand’s paradox in order to get the perplexities cleared away. In our experience in working on this project and sharing the findings with our colleagues, we have found a number of criticisms and questions are in many cases the same question in different guises. As a result, we have to sometime reiterate the arguments we have made earlier, although we hope our efforts constitute an enrichment of these arguments not simply mere repetition.

8.1. “Bertrand’s paradox is not a well-recognized paradox, since the random chords can be generated in different ways which cause different solutions.”

This comment was from a colleague after he read the draft of this article. It is incorrect in both its conclusion and its argument.

Bertrand’s paradox has been recognized by mathematicians and philosophers for 120 years. It was cited in books of philosophy and probability theory [21] [18] [2]. It is listed in important current dictionary of philosophy [15]. It has been viewed as an evidence against the principle of indifference and its extension, the principle of maximum entropy [1] [7]. It has been debated in major journals such as *Philosophy of Science* [19] [14], *Foundation of Physics* [11] [9], *The Mathematical Intelligencer* [7] [5], and *Analysis* [1]. The conclusion of the above statement, “Bertrand’s paradox is not a well-recognized paradox”, is therefore groundless. The debate on the paradox is clearly not over at all. To make the matters worse, some recent articles, as [19], not only failed to resolve the paradox but also strayed farther from the progress Jaynes had made in 1973 [9]. So, Bertrand paradox has been, and is, a well-recognized paradox in philosophy and probability theory and has not yet been resolved, at least up until now.

The argument of the above, “since the random chords can be generated in different ways which cause different solutions”, is also flawed. Random chords mean truly random chords just as random points mean truly random points. Truly random chords are homogeneously distributed. Homogeneous distribution has only one distribution pattern that is homogeneity. Such a pattern would result in just one solution for Bertrand's problem, even though its definition can be worded differently. That is, homogeneity has just one meaning and interpretation, though it can be worded differently. Since Bertrand-chords are homogeneous, there is just one solution for Bertrand's problem, no matter how many versions of definitions of homogeneity. This type of argument reflects a quick and simple attempt to get rid of the perplexity caused by Bertrand's paradox. Unfortunately, it is superficial and fallacious. It reveals a basic misconception about the nature of random chords, which is a prime reason why Bertrand's problem has been seen as a paradox for 120 years. It is precisely misconceptions such as this one that we have been at pains to bring to light in this article.

8.2 "Where did it all go wrong and why has it not been resolved until now?"

We have proved that Bertrand's problem *ought* to have only one solution and that it *does* have one solution. So, it is no longer a paradox. In what follows we share our reflections with our readers on how such a seemingly uncomplicated problem turned into a paradox that has puzzled scholars for so long. To do this we "decompose" the occurrence of the paradox in stages, thereby to reveal what we believe to be the thought process that generated the perplexity. Although we readily admit that we do not have any special insight of what happened to Mr. Bertrand 120 years ago and the scholars thereafter, we believe that the following represents a plausible picture of the thought process that occurs with many scholars when confronting the puzzle, which leads to viewing Bertrand's problem as a paradox.

The thought process leading people to viewing Bertrand's problem as a paradox may be composed of three stages.

Stage 1. One is confronted with a problem, the *original* Bertrand's problem as it were, in his mind. At this stage, the understanding he has of the problem is primarily a *graphical one*. He is imagining the random chords as those in Figure 1, back in Section 2. To the extent that he is articulating the problem in his head it would be something like, "What percent of these lines are longer than the side of an inscribed equilateral triangle?" As such, he thinks the problem obviously has one and only one solution.

Stage 2. The problem is put down into words. "Drawing a chord at random in a circle, what is the probability that the chord is longer than a side of the inscribed equilateral triangle?" This is thought to exactly reflect the original problem in our minds at Stage 1.

Stage 3. He tries to solve the original problem at stage 1 based on the problem described at Stage 2. He ends up with three solutions, 1/4, 1/3, and 1/2, each of which is obviously on a solid ground. He double-checks every piece of the arguments and finds nothing wrong. It turns out that a problem that ought to have only one solution ends up with three solutions. – A paradox!

Of the three stages, which stage(s) is(are) wrong? The arguments in this paper have already showed:

(i) There is nothing wrong at Stage 1. The graphic image of the chords in people's minds (and in Mr. Bertrand's mind, and in everyone else' mind) are homogeneously distributed, as those in Fig. 1. It is correct that the problem ought to have one solution only.

(ii) The incorrectness starts at Stage 2 because the wording about the "chords" at this stage does not reflect the "chords" at Stage 1. That is, the problem addressed at Stage 2 is not the problem at Stage 1, because chords drawn at random are not necessarily uniformly distributed therefore not necessarily the chords in our minds at Stage 1.

(iii) The three solutions derived at Stage 3 are all correct solutions to the problem at Stage 2. But they are not all correct solutions to the problem at Stage 1, only 1/2 is.

As addressed in Section 6, there are two significant stumbling blocks with Bertrand's paradox. The first is the mistaken assumption that the perplexity is incurred by different pictures of Bertrand-chords in people's minds. The falsity of the assumption was shown in Section 2. Uniform distribution is the unstated but common understanding of Bertrand-chords, which has been either explicitly acknowledged [9] or tacitly accepted [2] [11] [19]. There is no difference among people on what Bertrand-chords are like, as no one would take the non-uniformly distributed chords in Fig. 2 for Bertrand-chords. Assuming otherwise would be digressing and result in irrelevant arguments.

The second stumbling block is the false assumption that chords drawn at random are homogeneously distributed. The falsity of this assumption was proved in Sections 4 and 5 and further discussed in Section 6. This mistaken assumption resulted in three consequences: (i) An incorrect wording was laid out at Stage 2 to represent the problem at Stage 1; (ii) People incorrectly took for granted that the problem of Stage 2 had only one solution; (iii) People incorrectly took the solutions to the problem at Stage 2 as the solutions to the original problem at Stage 1.

Given the way these confusions multiply upon each other, it is not surprising that Bertrand's paradox has been so recalcitrant.

8.3 "It seems that your arguments are self-fulfilling: - You gave a definition of Bertrand-chords according to your understanding, then proved your conclusion based on your definition. Is that right?"

No, it is not right, our arguments are not self-fulfilling. Our solution is based on people's common understanding of Bertrand-chords, as addressed in Sections 2 and 8.2, rather than an idiosyncratic vision tailored to our arguments. The definition of Bertrand-chords given in Section 3 is consistent with definitions of random lines in literatures [12] [6] but in non-technical terms.

Having identified the fact that everyone agrees that Bertrand-chords are homogeneously distributed, we developed the definition of homogeneously distributed chords with plain wording. Based on this, we then proved that 1/2 was the correct solution, while 1/3 and 1/4 are not; therefore resolving the paradox. There is no self-fulfillment loop existing in this solution process.

8.4 "Is Bertrand's paradox well posed or ill posed?"

The issue of well-posedness or ill-posedness has divided scholars for years in their pursuit of the solution to Bertrand's paradox [9] [11] [19] [7]. Whether a problem is well posed or ill posed refers to the way the problem is presented. If a problem is presented verbally or written down, then we examine the "wording" of the problem. A problem is well posed if its wording correctly represents the problem and has only one interpretation. A problem is ill posed, on the other hand, if its wording either incorrectly or vaguely represents the problem. In Bertrand's problem, Bertrand-chords were verbally posed as "chords drawn at random". As we know now, chords drawn at random are not necessarily Bertrand-chords. So, Bertrand's problem is not well posed. It is ill posed, because *its wording does not correctly reflect the original problem* as we addressed in 8.1.

But those scholars who thought that Bertrand's problem was ill posed have based their arguments on the wrong reason: - they insisted that it was ill posed because the wording "chords drawn at random" did not tell the method of drawing, and that therefore it was vague. This is a misconception due to the entanglement of their misunderstanding of Bertrand-chords and homogeneously distributed lines. They mistakenly conflated Bertrand-chords and chords drawn at random, and took

Bertrand-chords as whatever chords drawn at random. They concluded that Bertrand's problem was ill posed because the vagueness of the method of drawing chords. They failed to recognize that (i) chords drawn at random are not necessarily Bertrand-chards, (ii) there is only one distribution pattern of Bertrand-chords, homogeneity, and (iii) the wording "chords drawn at random" does not correctly represent Bertrand-chords at all. Let us take an example to illustrate the misconception.

Suppose we intend to find out how many horses a farmer has, and we ask him, "how many animals with four legs do you have?" Obviously, this is an ill-posed question, because animals with four legs are not necessarily horses, hence it does not correctly reflect what we had intended to find out. But imagine that, due to our ignorance about the nature of horses, we made an additional error in that we conflated the concepts "horse" and "animals with four legs." Then, we would have the misconception that any animal with four legs, including a dog, a cat, a mouse, was a "horse." In such a scenario we would not only be unable to grasp the mistake of using "animal with four legs" to represent "horse", but also make an additional mistake by thinking that the difficulty we encounter when we argue with each other about the number of "horses" is because of the vagueness of the question. Under the circumstances, we would think that the question was ill posed because it did not tell what 'kind' of horse (e.g., dog-horse, cat-horse, mouse-horse, etc.) it refers to, and, of course, there would be many answers to this vague question due to the many 'kinds' of horses; and we would fail to realize that the question was ill posed because we did not know what a horse was in the first place.

In some articles, "random chords" were used to represent Bertrand-chords, which is a correct representation. But the scholars who happened to have used the correct wording "random chords" would not care substituting it with the wording "chords drawn at random". They took "random chords", "chords drawn at random", and "Bertrand-chords" as interchangeable wordings. So, their inadvertently correct wording did not show they had known how to correctly pose Bertrand's problem, because to them, "random chords", which is a correct wording, refers to "chords drawn at random", which is an incorrect wording.

In short, our answer to this question is: *Bertrand's problem was ill posed, because the wording incorrectly represented the original problem, rather than because of the vague wording*. In this sense, instead of ill posed, we would rather say *Bertrand's problem was incorrectly posed*.

8.5 "What is the relationship between Bertrand's paradox and the principle of indifference?"

There is no relationship between Bertrand's paradox and the principle of indifference.

The principle of indifference (PI) says, if n possibilities are indistinguishable except for their names, then each possibility should be assigned a probability equal to 1/n.

From the very beginning, Bertrand's paradox was tied to the principle of indifference. J. Bertrand took this paradox to show that the principle of indifference might not be true when there were infinitely many possibilities [2]. Many researchers thereafter explored the paradox with the similar assumption that Bertrand's paradox posed a "devastating objection" [1] to the principle of indifference [8] [11] [19]. Those contentions, no matter whether favorable to PI or not, were based on a same assumption: Bertrand's paradox has something to do with PI. They were all wrong on this point.

We have seen that the contradiction in this paradox is due to the incorrect understanding of chords drawn at random and homogeneously distributed chords. As we have shown, chords associated with the "solutions" 1/4 and 1/3 are not homogeneously distributed. With the non-homogeneous distribution, chord density in the central area is lower than that in the peripheral area. In circumstances, "n possibilities" would obviously *not* be "indistinguishable except for their names" since probabilities at central area are distinct from those at peripheral area. Therefore, the chords associated with solutions 1/4 and 1/3 do not meet the antecedent of PI. It is wrong to blame PI for the absurdity of Bertrand's paradox and Bertrand's paradox does not at all pose a devastating objection to PI. PI is innocent and safe; it has nothing to do with Bertrand's paradox.

8.6. "Do 'random chords' or 'Bertrand-chords' have an univocal meaning, as you argued in this article, that their empirical distribution will fill the circle with equal density as in Figure 1?"

This question was from a colleague who read the draft of this article.

Our answer to this question is: Yes. Bertrand-chords have just one meaning and it is univocal. They are homogeneously distributed on the circle. We argued in Section 2 that Bertrand-chords must be truly random, therefore they must be homogeneous. The random lines have exact definition in theory of geometry probability [12]. So, the meaning of Bertrand-chords or random chords is unique, exact, and univocal.

This question reflects a widespread confusion among scholars regarding random chords. Many people take random chords to be equivalent to randomly drawn chords. That is a key stumbling block causing the puzzle of Bertrand’s paradox. We have showed in Section 4 that chords generated in the ways of Chord-Drawing-Method-1 and Chord-Drawing-Mathod-2 are not ‘random chords’ nor ‘Bertrand-chords’. Chords generated by the two methods have lower density at the central area than at the peripheral area, as those in Fig. 2.

Random chords should be understood as truly random chords, just as random points are understood as purely random points. When generating “random chords”, the chords must be drawn at *purely* random without any transformation on top of the randomness. “Purely random” means “random without any restriction”. “Transformation” necessarily involves ‘restrictions”. In the one dimensional case, suppose among 10,000 points in [0, 1] there are 5,000 points less than 0.25, the other 5,000 points greater than 0.25. Do we assume that the 10,000 points were randomly drawn from [0, 1]? Of course not, because they do not look uniformly distributed over [0, 1]. Actually, the points were indeed drawn at random at first. But after a point x is randomly drawn, x is transformed to $y = x^2$. The 10,000 points contains just y’s, not x’s. Although points x’s are truly random and uniformly distributed, y’s are not due to the transformation $y=x^2$. Examine Chord-Drawing-Method-2 as an example for two dimensions. It generates a chord in two steps: - selecting at purely random an angle first, then randomly select a point on the circumference. Note that the second step is *not* “purely random” because it has restriction of “on circumference”. The chords thus generated would be like those in Fig. 2, which are not be “truly random chords”, not homogeneously distributed, and not Bertrand-chords.

8.7. “Homogeneity and randomness can be defined in various ways, which explains the three solutions in Bertrand’s paradox.”

We have refuted such arguments in 8.1 and 8.6. There is only one pattern for homogeneity. The only possible difference between two homogeneous distributions of lines is the density. Homogeneity can be defined with different ways, but all the definitions are semantically identical. Non-homogeneities, on the other hand, are semantically different from each other. Since Bertrand-chords are homogeneous, there are *not* “various ways” of definitions of homogeneity that lead to multiple solutions.

9 Conclusion

The main result of this paper is: The 120 years-old Bertrand's paradox is completely solved. We first showed two facts, the random chords referred to in Bertrand's paradox were homogeneously distributed and that there was no disagreement among people on this point. Based on these facts, we rigorously proved that two of the three alleged answers were not sound because of their false assumptions and that only one solution was correct. We also revealed two significant stumbling blocks that had persistently caused puzzles with regard to this paradox: 1) misunderstanding of the nature of homogeneously distributed chords, and 2) a disparity between the problem as originally conceived in the mind and its subsequent representation. The paradox is therefore no longer paradoxical.

A paradox typically shows a limit or a flaw in some aspect of our knowledge. It stimulates our curiosity, arouses deeper thoughts about the subject, forces us to clarify confusions and sloppy assumptions, and, in the best of all possible worlds, leads us to new knowledge. Bertrand's paradox, as we now see, has led us to the flaw in our understanding of uniformly distributed lines. In the process of pursuing the solution to the paradox, we have achieved a better understanding of the distribution of lines in a two-dimensional system and advanced our knowledge. As such, Bertrand's paradox has done its job, and now we believe it should be retired from the family of paradoxes.

10 Open Problems

Some open problems emerge as the ramifications of the Bertrand's problem, such as calculating Bertrand probability given a way of drawing chords at random other than those three cited in Bertrand paradox, and extending our discovery from random chords to random curves, that is, characterizing and differentiating the uniformly distributed curves and the curves drawn at random.

References

[1] Bangu, S., On Bertrand's Paradox, *Analysis*, Vol. 70, Issue 1, 2010, p.30-35

[2] Bertrand, J., *Calcul des probabilités*, Gauthier-Villars, Paris, 1889, pp.4-5

[3] Diaconis, P. and Keller, J. B., Fair Dice, *The American Mathematical Monthly*, 96(4), 1989, p.337-339

[4] Garwood, F. and Holroyd, E. M., The distance of a "random chord" of a circle from the centre. *Mathematical Gazette*, 50, 1966, pp.283-286

[5] Guiasu, S. and Shenitzer, A., The principle of maximum entropy, *The Mathematical Intelligencer,* 7(1), 1985

[6] Hall, Peter, On the distribution of random lines, *Journal of Applied Probability*, 18, 1981, pp.606-616

[7] Holbrook, John and Kim, Sung Soo, Bertrand's paradox revisited, *The Mathematical Intelligencer*, 22, 2000, pp.16-19

[8] Jaynes, E. T., Information Theory and Statistical Mechanics, *Physical Review*, Vol. 106, 4, 1957, p.620-630

[9] Jaynes, E. T., The Well-posed Problem, *Foundation of Physics*, 3, 1973, p. 477-493

[10] Klain, D. A., and Rota, Gian-Carlo, *Introduction to Geometric Probability*, Cambridge University Press, 1997

[11] Marinoff, L., A Resolution of Bertrand's Paradox, *Philosophy of Science,* 61, 1994, pp.1-24.

[12] Miles, R. E., Random polygons determined by random lines in a plane. *Proceedings of the National Academy of Sciences of the U.S.A.* Vol. 52, 1964, pp.901-907

[13] Moran, P. A. P., A second note on recent research in geometrical probability. *Advances in Applied Probability*, Vol. 1, No. 1, Spring 1969, pp. 73-89

[14] Nathan, Amos, The fallacy of Intrinsic Distributions, *Philosophy of Science*, 51, 1984, pp.677-684

[15] Oxford philosophy dictionary, Oxford University Press, Bertrand’s paradox, *http://www.answers.com/topic/bertrand-s-paradox?cat=technology*, 2010

[16] Richards, Paul I., Averages for polygons formed by random lines. *Proceedings of the National Academy of Sciences of the U.S.A.* Vol. 52, No. 5 (Nov. 15, 1964), pp.1160-1164

[17] Ross, S. M., *Introduction to Probability Models*, 3rd edition, Academic Press, Inc., 1985

[18] Sainsbury, R. M., *Paradoxes*, 2nd Edition, Cambridge University Press, 1995

[19] Shackel, N., Bertrand's paradox and the principle of indifference, *Philosophy of Science*, 74, April 2007

[20] Solomon, H., Geometric Probability, *CBMS-NSF Regional Conference Series in Applied Mathematics,* Society for Industrial Mathematics, 1987

[21] Sorensen, R., *A Brief History of the Paradox*, Philosophy and the Labyrinths of the Mind. pp.246, Oxford University Press, 2003

[22] Virtual Laboratories, *Geometric Models. http://www.ds.unifi.it/VL/VL_EN/buffon/buffon3.html*, 2008

[23] Vujicic, B., Is it a paradox or a blunder? http://www.bertrands-paradox.com, 2008

[24] Wang, J., Bertrand’s Paradox and Distribution of Lines, *Proceeding of DSI Annual Conference*, Nov. 2008, Baltimore, Maryland, p.3031-3036.