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On Generalized Vector Variational-Like Inequality Problem

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Abstract

In this paper, we introduce the concepts of relaxed M-η-α-$P_{\_}$-pseudomonotonicity and relaxed M-η-α-$P_{\_}$-pseudomonoton icity-type mappings. Using the KKM techniques, we obtain the existence of solutions for generalized vector variational-like inequalities with relaxed M-η-α-$P_{\_}$-pseudomonotone-type mappings in reflexive Banach spaces. The results presented in this paper generalize, unify and improve a number of previously known results.

Keywords: *KKM mapping, Minty's-type lemma, M-η-hemicontinuous mapping, relaxed M-η-α-$P_{\_}$-pseudomonotone-type mapping.*
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1 Introduction

Vector variational inequalities were initially introduced and considered by Giannessi [5] in a finite-dimensional Euclidean space in 1980, which is generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Later on vector variational inequalities and their generalizations have been investigated and applied in various directions; see for example [1,2,7,9,10,12,13] and references therein. In recent years, many authors proposed several important generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity, relaxed η-α-monotonicity, quasi-monotonicity and semimonotonicity and applied to establishing existence results for vector variational inequality problems; see for example [2,4,6,8,17].

Recently in 1997, Verma [17] studied a class of variational inequalities with relaxed monotone operators. Very recently in 2003, Fang *et al.* [4] introduced a new concept of relaxed η-α-monotone mappings and obtained the existence of solutions for variational-like inequalities with relaxed η-α-monotone mappings in reflexive Banach spaces.

Inspired and motivated by Verma [17] and Fang *et al.* [4], in this paper we introduce the concept of relaxed M-η-α-$P_{\_}$-pseudomonotone and relaxed M-η-α-$P_{\_}$-pseudomonotone-type set-valued mappings. Further, we consider generalized vector variational-like inequality problem involving relaxed M-η-α-$P_{\_}$-pseudomonotone-type set-valued mappings. Furthermore, by using the KKM techniques, we established some existence results for this generalized vector variational-like inequalities involving relaxed M-η-α-$P_{\_}$-pseudomonotone-type mappings in reflexive Banach spaces. Our results are the generalization of many existing works of [4,11,16-17].

2 Problem Formulations

Throughout the paper unless otherwise specified, let X and Y are two Banach spaces and let K be a nonempty subset of X and N a nonempty subset of $L(X, Y)$, where $L(X, Y)$ denotes the space of all linear continuous mappings from X into Y. Let $P : K \to 2^Y$ be a set-valued mapping such that for each $x \in K$, $P(x)$ is closed, pointed and convex cone with $\operatorname{int} P(x) \neq \emptyset$. An ordered Banach space (Y, P) is a real Banach space Y with an ordering defined by a cone $P \subseteq Y$ with an apex at the origin in the form of

$$x \leq y \;\Leftrightarrow\; y - x \in P$$

Let $M : K \times N \to L(X,Y)$, $\eta : K \times K \to X$ and $f : K \times K \to Y$ are bi-mappings and $T : K \to 2^N$ be a set-valued mapping. In this paper we consider following *generalized vector variational-like inequality problem* (in short, GVVLIP): Find $x \in K$ such that for all $y \in K$ there exists an $u \in T(x)$ satisfying that

$$\langle M(x, u), \eta(y, x)\rangle \; + f(y, x) \notin -\operatorname{int} P(x).$$

Some special cases of (GVVLIP)

(I) If f is zero mapping, then (GVVLIP) reduces to the problem of finding $x \in K$ such that for all $y \in K$, $\exists\, u \in T(x)$ such that

$$\langle M(x,u), \eta(y,x)\rangle \ \notin -\mathrm{int}\, P(x).$$

which was introduced and studied by Ansari *et al.* [1], that generalizes some kinds of vector variational inequalities considered by many authors; see for details [1, 9-11,13].

(II) If K=N and $M(x,u)$ =Au, where $A : K \to L(X,Y)$ then (GVVLIP) reduces to the problem of finding $x \in K$, such that for all $y \in K$, $\exists\, u \in T(x)$ such that

$$\langle Au, \eta(y,x)\rangle \ + f(y,x) \notin -\mathrm{int}\, P(x),$$

which has been studied by Usman *et al.* [16].

(III) If f is zero mapping, K =N, $M(x,u)$=u, and $\eta(y,x) = y - x$, $\forall x, y \in K$, then (GVVLIP) reduces to the problem of finding $x \in K$ such that for all $y \in K$, $\exists\, u \in T(x)$ such that

$$\langle u, y-x\rangle \ \notin -\mathrm{int}\, P(x),$$

which has been studied by Lee *et al.* [12].

(IV) If f is zero mapping, let $K = \mathbf{R}^n$, $N = \mathbf{R}^m$, $Y = \mathbf{R}^l$ and let $L : K \times K \to \mathbf{R}^l$ be such that $M(x,u) = L'(x,u)$, $\forall (x,u) \in K \times K$, where L' denotes the Frechet derivative of L at x and let $T : K \to 2^N$ is defined by $T(x) := \ \{y \in N : L(x,z) - L(x,y) \notin -\mathrm{int}\, \mathbf{R}_+{}^l, \ \ \forall z \in N\}$ then above (GVVLIP) reduces to problem of finding $x \in K$ such that for all $y \in K$, $\exists\, u \in T(x)$ such that

$$\langle L(x,u), \eta(y,x)\rangle \notin -\mathrm{int}\, \mathbf{R}_+{}^l,$$

which has been studied by Kazmi [7] in finding out the weak saddle point of non convex mapping L.

Throughout the paper, unless otherwise specified, let $P_- = \bigcap_{x \in K} P(x)$ is a closed, convex, solid and pointed cone. Now we recall the following concepts and results which are needed in the sequel.

Definition 2.1 *A mapping $f : K \times K \to Y$ is said to be* *(a)* P_-*-convex in first argument, if for all* $\alpha \in [0,1]$ *and* $x_1, x_2 \in K$,

$$f(\alpha x_1 + (1-\alpha)x_2,\ y) \leq_{P_-} \alpha f(x_1, y) + (1-\alpha) f(x_2, y);$$

(b) P_-*-concave, if* $-f$ *is* P_-*-convex.*

Definition 2.2 [**3**] *Let* K *be a subset of a topological vector space* X. *A mapping* $T : K \to 2^X$ *is called Knaster-Kuratowski-Mazurkiewieg mapping (KKM mapping), if for each nonempty finite subset* $\{x_1, x_2, \cdots, x_n\} \subset K$, *we have* $Co\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i)$.

Lemma 2.3 [**2**] *Let* (Y, P) *be an ordered Banach space with a closed, pointed and convex cone* P *with* $int P \neq \emptyset$. *Then* $\forall x, y, z \in Y$, *we have*

(i) $y - z \in int P$ *and* $y \notin int P \Rightarrow z \notin int P$;

(ii) $y - z \in -P$ *and* $y \notin -int P \Rightarrow z \notin -int P$.

Theorem 2.4 (KKM-Fan Theorem) [**3**] *Let* K *be a subset of a topological vector space* X *and let* $F : K \to 2^X$ *be a KKM mapping. If for each* $x \in K$, $F(x)$ *is closed and for atleast one* $x \in K$, $F(x)$ *is compact, then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

We have following fixed point theorem which play an important role in establishing existing theorem for (GVVLIP).

Theorem 2.5 [**15**] *Let* K *be a nonempty convex subset of a Hausdorff topological vector space* X *and let* $S : K \to 2^K$ *be a set-valued mapping such that*

(i) *for each* $x \in K$, $S(x)$ *is a nonempty convex subset of* K;

(ii) *for each* $y \in K$, $S^{-1}(y) := \{x \in K : y \in S(x)\}$ *contains an open subset* O_y *of* K, *where* O_y *may be empty;*

(iii) $\bigcup_{y \in K} O_y = K$;

(iv) K *contains a nonempty subset* K_0 *contained in a compact subset* K_1 *of* K *such that the set* $D = \bigcap_{y \in K_0} O_y^c$ *is compact, where* D *may be empty and* O_y^c *denotes complement of* O_y *in* K_0.

Then $\exists\, x_0 \in K$ *such that* $x_0 \in S(x_0)$.

3 Existence results for (GVVLIP)

First, we define the following concepts.

Definition 3.1 *Let $M : K \times N \to L(X,Y)$, $f : K \times K \to Y$ and $\eta : K \times K \to X$ be mappings, let $T : K \to 2^N$ are the set-valued mapping and let $\alpha : X \to Y$ be a mapping such that $\alpha(tz) = t^p \alpha(z)$, $\forall z \in X$ for all $t > 0$ and a constant $p > 1$. Then T is said to be*

(a) relaxed M-η-α-P_--pseudomonotone, if for every pair of points $x, y \in K$ and for all $u \in T(x)$, $v \in T(y)$, we have

$$\langle M(x,u), \eta(y,x)\rangle + f(y,x) \notin -int\, P(x) \quad \text{implies}$$

$$\langle M(y,v), \eta(y,x)\rangle + f(y,x) - \alpha(y-x) \notin -int\, P(x);$$

(b) relaxed M-η-α-P_--pseudomonotone-type, if for every pair of points $x, y \in K$ and for all $u \in T(x)$, we have

$$\langle M(x,u), \eta(y,x)\rangle + f(y,x) \notin -int\, P(x) \quad \text{implies}$$

$$\langle M(y,v), \eta(y,x)\rangle + f(y,x) - \alpha(y-x) \notin -int\, P(x), \quad \text{for some} \quad v \in T(y)$$

Remark 3.2 *(I) (a) implies (b)) but not conversely.*

(II) If $\alpha \equiv 0$, $f(y,x) = f(y) - f(x)$, $M(x,u) = u$ and $\eta(y,x) = y - x$, $\forall x, y \in K$, then we obtain Definition 2.1 (iii) and (vi) in [9], respectively.

(III) If $\alpha \equiv 0$, $f(y,x) = f(y) - f(x)$, $L(X,Y) = X^*$, $Y = \mathbf{R}$ and $P(x) = \mathbf{R}^+$, $\forall x \in K$, then we obtain Definition 2.1 (i) in [14].

Definition 3.3 *Let $M : K \times N \to L(X,Y)$, $f : K \times K \to Y$ and $\eta : K \times K \to X$ are bi-mappings and let $T : K \to 2^N$ be a set-valued mapping. Then T is said to be M-η-hemicontinuous if, for any $x, y \in K$, $u_n \in T(x+ny)$, $\exists u_0 \in T(x)$ such that*

$$\langle M(x+ny, u_n), \eta(y,x)\rangle + f(y,x) \to \langle M(x,u_0), \eta(y,x)\rangle + f(y,x) \quad \text{as } n \to 0^+.$$

Now, we give Minty's-type lemma for (GVVLIP).

Lemma 3.4 *Let X be real reflexive Banach space and Y be a Banach space. Let $K \subset X$ be a nonempty, closed and convex subset of X and N a nonempty subset of $L(X,Y)$. Let $P : K \to 2^Y$ be such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $int\, P \neq \emptyset$. Let $M : K \times N \to L(X,Y)$ be a mapping and $f : K \times K \to Y$ is P_--convex in first argument with $f(x,x) = 0, \forall x \in K$. Suppose following conditions hold*

(i) *$\eta : K \times K \to X$ is a mapping such that $\eta(x, x) = 0$, $\forall x \in K$;*

(ii) *for any fixed $x \in K$, $u \in T(x)$ the mapping $y \to \langle M(x, u), \eta(y, x)\rangle$ is $P_{\_}$-convex;*

(iii) *$T : K \to 2^N$ be M-η-hemicontinuous and relaxed M-η-α-$P_{\_}$-pseudomonotone-type mapping.*

Then following two problems are equivalent:

(A) *Find $x \in K$ such that for all $y \in K$, there exists an $u \in T(x)$ satisfying that*
$$\langle M(x, u), \eta(y, x)\rangle + f(y, x) \notin -int\, P(x). \tag{3.1}$$

(B) *Find $x \in K$ such that for all $y \in K$, there exists an $v \in T(y)$ satisfying that*
$$\langle M(y, v), \eta(y, x)\rangle + f(y, x) - \alpha(y - x) \notin -int\, P(x). \tag{3.2}$$

Proof. Let $x \in K$ be a solution of problem (3.1), therefore there exists $u \in T(x)$ such that

$$\langle M(x, u), \eta(y, x)\rangle + f(y, x) \notin -\mathrm{int}\, P(x).$$

Since T is relaxed M-η-α-$P_{\_}$-pseudomonotone-type, which implies that there exists $v \in T(y)$ such that

$$\langle M(y, v), \eta(y, x)\rangle + f(y, x) - \alpha(y - x) \notin -\mathrm{int}\, P(x).$$

Conversely, suppose that there exists $x \in K$ such that

$$\langle M(y, v), \eta(y, x)\rangle + f(y, x) - \alpha(y - x) \notin -\mathrm{int}\, P(x) \quad \forall\, y \in K, \quad v \in T(y).$$

For any given $y \in K$, we know that $y_t := (1 - t)x + ty \in K$, $\forall t \in (0, 1)$, as K is convex.
Since $x \in K$ is a solution of problem (3.2), so for each $v_t \in T(y_t)$ it follows that
$$\langle M(y_t, v_t), \eta(y_t, x)\rangle + f(y_t, x) - \alpha(y_t - x) \notin -\mathrm{int}\, P(x). \tag{3.3}$$
$$\langle M(y_t, v_t), \eta((1 - t)x + ty, x)\rangle + f((1 - t)x + ty, x) - \alpha(t(y - x)) \notin -\mathrm{int}\, P(x).$$

As f is $P_{\_}$-convex in first argument, we have

$$f((1 - t)x + ty, x) \leq_{P(x)} (1 - t)f(x, x) + tf(y, x) = tf(y, x). \tag{3.4}$$

By using the conditions (i) and (ii) on η, it follows

$$\langle M(y_t, v_t), \eta(y_t, x)\rangle = \langle M(y_t, v_t), \eta((1 - t)x + ty, x)\rangle$$

$$\leq_{P(x)} (1-t)\langle M(y_t,v_t),\eta(x,x)\rangle + t\langle M(y_t,v_t),\eta(y,x)\rangle$$
$$\leq_{P(x)} t\langle M(y_t,v_t),\eta(y,x)\rangle \tag{3.5}$$

It follows from inclusions (3.3)-(3.5) and Lemma 2.3, that for $t>0$ and $p>1$

$$t\langle M(y_t,v_t),\eta(y,x)\rangle + tf(y,x) - t^p\alpha(y-x) \notin -\mathrm{int}\,P(x).$$

$$\langle M(y_t,v_t),\eta(y,x)\rangle + f(y,x) - t^{p-1}\alpha(y-x) \notin -\mathrm{int}\,P(x).$$

Since T is M-η-hemicontinuous and $p>1$, there exists $u \in T(x)$ such that

$$\langle M(x,u),\eta(y,x)\rangle + f(y,x) \notin -\mathrm{int}\,P(x).$$

as $t \to 0^+$. This completes the proof.

Now, we have following existence theorem for (GVVLIP).

Theorem 3.5 *Let X be real reflexive Banach space and Y be a Banach space. Let $K \subset X$ be a nonempty, bounded, closed and convex subset of X and N a nonempty subset of $L(X,Y)$. Let $P : K \to 2^Y$ be such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $int P \neq \emptyset$. Let $M : K \times N \to L(X,Y)$ be a mapping, $\alpha : X \to Y$ is weakly lower semicontinuous and P_--convex mapping. Suppose following conditions hold:*

(i) The set-valued mapping $W : K \to 2^Y$ defined as $W(x) = Y\backslash\{-int\,P(x)\}$ such that graph of W is weakly closed in $X \times Y$;

(ii) $\eta : K \times K \to X$ is continuous in second argument such that $\eta(x,x) = 0$, $\forall x \in K$;

(iii) $f : K \times K \to Y$ is lower semicontinuous and P_--convex in second and first arguments, respectively, with $f(x,x) = 0$, $\forall x \in K$;

(iv) for any fixed $x \in K$ and $u \in T(x)$, the mapping $y \to \langle M(x,u),\eta(y,x)\rangle$ is P_--convex;

(v) $T : K \to 2^N$ be M-η-hemicontinuous and relaxed M-η-α-P_--pseudomonotone-type mapping with compact-values.

Then (GVVLIP) is solvable.

Proof. Let $F_1, F_2 : K \to 2^X$ be two set-valued mappings such that for any $y \in K$,

$$F_1(y) = \{x \in K : \exists\, u \in T(x) \text{ such that } \langle M(x,u),\eta(y,x)\rangle + f(y,x) \notin -\mathrm{int}\,P(x)\}.$$

$$F_2(y) = \{x \in K : \exists\, v \in T(y) \text{ such that } \langle M(y,v),\eta(y,x)\rangle + f(y,x) - \alpha(y-x) \notin -\mathrm{int}\,P(x)\}.$$

We claim that F_1 is KKM mapping. Indeed, let $\alpha_i \geq 0$, $1 \leq i \leq n$, with $\sum_{i=1}^{n} \alpha_i = 1$. Suppose that $x = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_1(x_i)$. Then, for any $u \in T(x)$,

$$\langle M(x,u), \eta(x_i,x)\rangle + f(x_i,x) \in -\text{int } P(x), \qquad i = 1,2,...,n.$$

We have

$$\begin{aligned} 0 &= \langle M(x,u), \eta(x,x)\rangle + f(x,x) \\ &= \langle M(x,u), \eta(\sum_{i=1}^{n} \alpha_i x_i, x)\rangle \ + f(\sum_{i=1}^{n} \alpha_i x_i, x) \\ &\leq_{P_-} \sum_{i=1}^{n} \alpha_i [\langle M(x,u), \eta(x_i,x)\rangle \ + f(x_i,x)] \end{aligned}$$

i.e., $0 \in -\text{int } P(x)$, which is not possible for a pointed cone and thus our claim is verified.

Next, we prove that $F_1(y) \subset F_2(y)$ for each $y \in K$. For any given $y \in K$, let $x \in F_1(y)$ then there exists $u \in T(x)$ such that

$$\langle M(x,u), \eta(y,x)\rangle + f(y,x) \notin -\text{int } P(x).$$

Since T is relaxed M-η-α-P_--pseudomonotone-type, we have

$$\langle M(y,v), \eta(y,x)\rangle + f(y,x) - \alpha(y-x) \notin -\text{int } P(x).$$

i.e., $x \in F_2(y)$. It follows that $F_1(y) \subset F_2(y)$ for each $y \in K$. Hence F_2 is also a KKM mapping.

Now, we claim that $F_2(y)$ is weakly closed in K for each $y \in K$. Indeed, let $\{x_n\} \subset F_2(y)$ such that $x_n \to x_0 \in K$. Since $x_n \in F_2(y)$, there exists $v_n \in T(y)$ such that

$$\langle M(y,v_n), \eta(y,x_n)\rangle \ + f(y,x_n) - \alpha(y-x_n) \notin -\text{int } P(x_n),$$

i.e., $\quad \langle M(y,v_n), \eta(y,x_n)\rangle + f(y,x_n) - \alpha(y-x_n) \in Y \backslash \{-\text{int } P(x_n)\} \in W(x_n).$

Since $T(y)$ is compact, $\{v_n\}$ has a convergent subsequence in $T(y)$ without loss of generality, we can assume that there exists $v_0 \in T(y)$ such that $v_n \to v_0$. Since graph of W is weakly closed, T is continuous, f and α are lower semicontinuous, it follows that

$$\begin{aligned} &\langle M(y,v_n), \eta(y,x_n)\rangle + f(y,x_n) - \alpha(y-x_n) \to \\ &\qquad\qquad \langle M(y,v_0), \eta(y,x_0)\rangle + f(y,x_0) - \alpha(y-x_0) \in W(x_0) \end{aligned}$$

i.e, $x_0 \in F_2(y)$ and hence $F_2(y)$ is closed. Since K is closed, bounded and convex subset of a reflexive Banach space X, then K is weakly compact. $F_2(y)$

is also weakly compact because $F_2(y) \in K$. Hence by KKM-Fan Theorem 2.4, we have

$$\bigcap_{y\in K} F_2(y) \neq \emptyset.$$

By Lemma 3.4, we have

$$\bigcap_{y\in K} F_1(y) \neq \emptyset.$$

Consequently, there exists $x_0 \in K$ such that for each $y \in K$ and $u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0)\rangle \ + f(y, x_0) \notin -\text{int } P(x_0).$$

This completes the proof.

Theorem 3.6 *Let X be real reflexive Banach space and Y be a Banach space. Let $K \subset X$ be a nonempty, bounded, closed and convex subset of X and N a nonempty subset of $L(X, Y)$. Let $P : K \to 2^Y$ be such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $int P \neq \emptyset$. Let $M : K \times N \to L(X, Y)$ be a mapping, $\alpha : X \to Y$ is weakly lower semicontinuous and P_--convex mapping. Let the conditions (i)-(v) of Theorem 3.5 hold and also the following conditions hold:*

(vi) For each $x \in K$, $\exists\, x_0 \in K$, such that $u_0 \in T(x_0)$ and $\langle M(x_0, u_0), \eta(x_0, x)\rangle + f(x_0, x) - \alpha(x_0 - x) \notin -int\ P(x)$;

(vii) There exists a nonempty set K_0 contained in a compact and convex subset K_1 of K such that

$$D := \bigcap_{x_0\in K_0} \bigcap_{u_0\in T(x_0)} \{x \in K : \langle M(x_0, u_0), \eta(x_0, x)\rangle \ +f(x_0, x)-\alpha(x_0 - x) \in W(x)\}.$$

Then (GVVLIP) is solvable.

Proof. Suppose on contrary that (GVVLIP) admits no solution, then for each $x_0 \in K$, there exists $u_0 \in T(x_0)$ and $x \in K$ such that

$$\langle M(x_0, u_0), \eta(x, x_0)\rangle + f(x, x_0) \in -\text{int } P(x_0)$$

then the set

$$F(x_0) := \{x \in K : \exists u_0 \in T(x_0) \text{ such that } \langle M(x_0, u_0), \eta(x, x_0)\rangle + f(x, x_0)) \in -\text{int } P(x_0)\},$$

is nonempty. We claim that the set $F(x_0)$ is convex. Indeed, let $x_1, x_2 \in F(x_0)$ and let $m, n \geq 0$ be such that $m + n = 1$ then $\exists u_0 \in T(x_0)$ such that

$$m[\langle M(x_0, u_0), \eta(x_1, x_0)\rangle + f(x_1, x_0)] \in m(-\text{int } P(x_0)) = -\text{int}P(x_0)$$

$$n[\langle M(x_0,u_0),\eta(x_2,x_0)\rangle + f(x_2,x_0)] \in n(-\text{int } P(x_0)) = -\text{int}P(x_0)$$

Since $\eta(.,x_0)$ and $f(.,x)$ are P_--convex, then from preceding two inclusions, we have $mx_1 + nx_2 \in F(x_0)$, i.e., the set $F(x_0)$ is convex for each $x_0 \in K$. Thus $F : K \to 2^K$ is a nonempty and convex set-valued mapping. Now $F^{-1}(x_0) := \{x \in K : x_0 \in F(x_0)\}$

$$= \{x \in K : \exists u \in T(x) \text{ such that } \langle M(x,u),\eta(x_0,x)\rangle + f(x_0,x) \in -\text{int } P(x)\}$$

$$[F^{-1}(x_0)]^c = \{x \in K : \exists u \in T(x),\ \langle M(x,u),\eta(x_0,x)\rangle + f(x_0,x) \notin -\text{int } P(x)\}$$

Since T is relaxed M-η-α-P_--pseudomonotone-type mapping, therefore above inclusion implies that

$$\subseteq \{x \in K : \exists u_0 \in T(x_0), \langle M(x_0,u_0),\eta(x_0,x)\rangle + f(x_0,x) - \alpha(x_0-x) \notin -\text{int } P(x)\}$$

$$= \{x \in K : \exists u_0 \in T(x_0), \langle M(x_0,u_0),\eta(x_0,x)\rangle + f(x_0,x) - \alpha(x_0-x) \in Y\backslash(-\text{int } P(x))\}$$

$=: B(x_0) \subseteq K$.

Since α, $f(.,x)$ are P_--convex and $\eta(.,x)$ is affine, we can easily show that $B(x_0)$ is convex. Also lower semicontinuity of $f(.,x)$, continuity of $\eta(x_0,.)$ and closeness of $Y\backslash(-\text{int } P(x))$ yield the relatively closeness of $B(x_0)$.

Hence, for each $x_0 \in K$, $O_{x_0} := [B(x_0)]^c$ is a relatively open subset of K. Now, by assumption (vi), it follows that $\bigcup_{x_0\in K} O_{x_0}$. Finally from assumption (vii)

$$D = \bigcap_{x_0\in K_0} \bigcap_{u_0\in T(x_0)} B(x_0) \;=\; \bigcap_{x_0\in K_0} \bigcap_{u_0\in T(x_0)} O^c(x_0)$$

is compact or empty. Hence from fixed point Theorem 2.5, there exists $x_0 \in K$ such that $x_0 \in F(x_0)$, i.e., $0 \in -\text{int } P(x)$, which is not possible for a pointed cone. Hence (GVVLIP) admits a solution. This completes the proof.

4 Open Problem

It is of further research effort to study and establish existence results for the strong generalized vector variational-like inequality problem, i.e., to find $x \in K$ such that for all $y \in K$ there exists an $u \in T(x)$ satisfying that

$$\langle M(x,u),\eta(y,x)\rangle \;+ f(y,x) \notin -P(x)\backslash\{0\}.$$

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