Int. J. Open Problems Compt. Math., Vol. 4, No. 4, December 2011
ISSN 1998-6262;
www.i-csrs.org

Optimal Choice of the Simple Multipole Coefficients By Minimizing the Condition Number in Linear Elasticity

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Abstract

In this work the determination of an optimal choice of the simple multipole coefficients for an exterior Dirichlet problem in two-dimensional elastic waves is investigated. We introduce a modification of the Green's function in order to remove the lack of uniqueness for the solution of the boundary integral equation describing the problem, and to simultaneously minimize his condition number. In view of this procedure the cases of the circle and perturbations of circle are examined.

Keywords: *Multipole coefficients, Green's function, integral equations of Fredholm type, elasticity.*

MSC (2000): *45B05, 34B27, 34B30.*

1 Introduction

As is well known, the reformulation of an exterior boundary value problem to a boundary integral equation presents difficulties caused by the lack of uniqueness of its solutions. In order to remove this problem the modified Green's function technique was proposed by Jones [9] and Ursell [19] for the acoustical case. In [12, 11] Kleinman and Roach have shown that the choice of the multipole coefficients, apart from the removal of the non-uniqueness problem, can also satisfy other criteria of best modification. These include that of the best approximation to the actual Green's function and that of minimization of the norm of the modified integral operator. In [10] Kleinman and Kress have established the criterion of minimization of the condition number of the boundary integral equation for the acoustical case.

Similar arguments hold for the elastic case. The first work in linear elasticity in which this technique was introduced is due to Jones [9]. In [5, 6] results for the elastic two-dimensional case are presented by Bencheikh. In [1, 3] and [8] exterior elastic problems in IR^3 are examined and criteria for best modification are also established by Argyropoulos, Kiriaki, Roach and Gintides. In [14, 16] we have established the minimization of the norm of the modified integral operator, and in [17, 18, 15] we have also with the colaboration of Bencheikh established the minimization of the norm of the modified Green's function for the elastic two-dimensional case.

In this work we treat an open problem cited in [17, 18], the modified Green's function technique is adopted and using it, the minimization of the condition number of the boundary integral operator describing the exterior Dirichlet problem in IR^2 is established.

Many ideas of [2] are exploited, nevertheless there are noteworthy differences between the acoustic and the elastic case. In section 2 the formulation of the problem in integral form through a layer theoretic approach is given. In section 3 the criterion of minimization is established for the circular boundary. In section 4 the shapes which can be produced as perturbations of the circle are discussed.

2 Preliminary notes

Let D_- denote a bounded connected domain in IR^2 with boundary ∂D, which will be assumed closed, bounded and Lyapunov. Let $D_+ = IR^2/\ \overline{D_-}$, where $\overline{D_-} = D_- \cup \partial D$. We assume that D_+ is filled by an isotropic and homogeneous elastic medium specified by the Lamé constants $\lambda,\ \mu$ and mass density ρ.

The displacement field $U(P) \in L^2(D)$ satisfies the following equation :

$$\frac{1}{k^2}\ grad\ (div\ U(P)) - \frac{1}{K^2}\ rot\ (rot\ U(P)) + U(P) = 0 \qquad P \in D_+ \quad (2.1)$$

We also define the surface stress operator :

$$T\ U\,(p) = \lambda.\hat{n}\,(p)\ \ div\ U\,(p) + 2\mu\,\frac{\partial U\,(p)}{\partial n_P} + \mu\ \hat{n}\,(p) \times rot\ U\,(p) \qquad p \in \partial D \quad (2.2)$$

where $\hat{n}$ is the exterior unit normal on ∂D. The radiation conditions, due to Kupradze, which the displacement field must satisfy are [13] :

$$\underset{r_P\ \longrightarrow +\infty}{Lim}\ U'\,(P) = 0, \qquad \underset{r_P\ \longrightarrow +\infty}{Lim}\ U''\,(P) = 0 \qquad (2.3)$$

$$\lim_{r_P \longrightarrow +\infty} (r_p)^{\frac{1}{2}} \left\{ \frac{\partial U^{'}(P)}{\partial r_P} - i\ k\ U^{'}(P) \right\} = 0$$

$$\lim_{r_P \longrightarrow +\infty} (r_p)^{\frac{1}{2}} \left\{ \frac{\partial U^{''}(P)}{\partial r_P} - i\ K\ U^{''}(P) \right\} = 0$$

where

$$U^{'}(P) = -\frac{1}{k^2}\ grad\ \ (div\ U(P))$$

$$U^{''}(P) = -\frac{1}{K^2}\ rot\ \ (rot\ U(P))$$

and $k^2 = \frac{\rho\ \omega^2}{\lambda+2\mu}$, $K^2 = \frac{\rho\ \omega^2}{\mu}$ and ω is the angular frequency.

The exterior boundary value problems which we examine are the problems of the rigid body and the cavity. So we have to determine the displacement field which satisfies the differential equation (2.1) for D_+, the boudary condition :

$$U(p) = g \qquad p \in \partial D \tag{2.4}$$

where g is a known function, and the radiation conditions (2.3).

In order to reformulate the problem in itegral form, we can follow either the direct method, based on Betti's formulae, or the indirect method using the layer potential.

Following the layer theoretic approach, we define the double layer potential :

$$(D\ \varphi)(P) = \frac{1}{2\ \pi}\int_{\partial D} T_q G_0(P,q).\varphi(q).ds_q \qquad\qquad P \in D \tag{2.5}$$

for a density $\varphi \in L_2(\partial D)$, where $G_0(P,Q)$ is the Green's function given by [4] :

$$G_0(P,Q) = \frac{i}{4\ \mu}\left[\psi.I + \frac{1}{K^2}\ grad\ \ (grad\ \ (\psi - \phi))\right] \tag{2.6}$$

here I denotes the identity tensor, and $\psi(P,Q) = H_0^1(K\ R)$, $\phi(P,Q) = H_0^1(k\ R)$. $H_0^1(\ .\ )$ denotes the function of Hankel and R is the distance between P and Q.

Exploiting the jump relations at the boundary [13], we can see that if we seek solutions of the exterior Dirichlet problem in terms of a double layer potential of an unknown density φ, then φ is required to satisfy the boundary integral equation :

$$\left(\frac{1}{2}\ I+\overline{K_0^*}\right)\varphi\,(p)=g\,(p)\qquad p\in\partial D \tag{2.7}$$

where the integral operator appearing in (2.7) is the L_2-adjoint of K_0 given by the relation :

$$(K_0\ \varphi)(p)=\frac{1}{2\ \pi}{}_{\partial D}T_pG_0(p,q).\varphi(q).ds_q\qquad p\in\partial D \tag{2.8}$$

So K_0^* may be expressed as :

$$(K_0^*\ \varphi)(p)=\frac{1}{2\ \pi}{}_{\partial D}\overline{T_qG_0(p,q)}.\varphi(q).ds_q\qquad p\in\partial D \tag{2.9}$$

and the bar in (2.9) indicates the complex conjugate.

The above defined integral operators K_0 and K_0^* have singular kernels. In [13] a Fredholm type theory for these boundary integral equations based on a regularization procedure is established. The global regularizer, which is shown to exist, is equivalent. So the original and the regularized equations have the same solutions. Many properties of the resolvents are also presented. It is proven that the homogeneous interior Dirichlet problem has a discrete spectrum. In order to have uniqueness for the boundary integral equation describing the exterior problem we have to avoid the irregular frequencies, that is, the eigenvalues of the adjoint interior problem. To accommodate this difficulty we shall adopt the modified Green's function technique established in [4]. The modified Green's function $G_1(P,Q)$ is given by [4] :

$$G_1(P,Q)=G_0(P,Q)+\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{l=1}^{2}\left(a_m^{\sigma l}\ F_m^{\sigma l}(P)\otimes F_m^{\sigma l}(Q)\right) \tag{2.10}$$

where

$$F_m^{\sigma 1}(P)=grad\ \left(H_m^1\,(k\ R_P)\ .\ E_m^{\sigma}\,(\theta_P)\right) \tag{2.11}$$

$$F_m^{\sigma 2}(P)=rot\ \left(H_m^1\,(K\ R_P)\ .\ E_m^{\sigma}\,(\theta_P)\ \widehat{e}_3\right)$$

are the mutipole vectors, and $a_m^{\sigma l}$ are the simple multipole coefficients. $(R_P,\ \theta_P)$ are the polar coordinates of the point P, and $E_m^{\sigma}\,(\theta_P)=\sqrt{\varepsilon_m}\ .\{.\cos\,(m\ \theta_P)\qquad \sigma=1\sin\,(m\ \theta_P)\qquad c$, with $\varepsilon_m=\{.1\qquad m\ =\ 02\qquad m\ >\ 0$

In view of this multipole vectors system the Green's function $G_0(P,Q)$ admits the representation [4] :

$$G_0(P,Q)=\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{l=1}^{2}\left(F_m^{\sigma l}(P)\otimes\widehat{F}_m^{\sigma l}(Q)\right) \tag{2.12}$$

where $\widehat{F}_m^{\sigma l}$ is obtained by changing $H_m^1(\,.)$ by the function of Bessel $J_m^1(\,.)$

In [4] sufficient conditions on the simple multipole coefficients of the modification for unique solvability of the boundary integral equation are established. These conditions are given by :

$$\left|a_m^{\sigma l} + \tfrac{1}{2}\right|^2 - \tfrac{1}{4} < 0, \qquad (\forall\ m = 0 : \infty,\ \forall\ \sigma, l = 1 : 2) \tag{2.13}$$

In [17, 18] we have proposed a criterion for minimizing the norm of the kernel of the modified integral operator.

3 Mains results

3.1 Minimization of the condition number (circular case)

If we introduce the operator :

$$M\ :\ L_2(\partial D) \longrightarrow L_2(\partial D)$$

where $M = \frac{1}{2}\ I + \overline{K_1^*}$, then (2.7) may be written as :

$$M\ (\varphi) = g$$

we also introduce his L_2-adjoint M^*.

As is well known, the condition number which is given by the relation [7] :

$$Cond\,(M) = \|M\|\,.\,\left\|M^{-1}\right\|$$

with respect to the L_2-norm can be expressed as :

$$Cond\,(M) = \sqrt{\frac{\lambda_{\max}^M}{\lambda_{\min}^M}}$$

where $\lambda_{\max}^M$ and $\lambda_{\min}^M$ denote the largest and the smallest spectral value of the self-adjoint operator M^*M.

As is shown in [12], it is extremely difficult to get explicit results for the multipole coefficients which minimize the operator norms for arbitrary boundaries ∂D. Similar discussions are given in [10] for the condition number of integral equations in acoustics and in [2] for the elastic three-dimensional case. Neverthless, the special result for minimizing the condition number when ∂D is a circle serves as a guide to an explicit coefficient choice which leads to well conditioned integral equations for perturbations of circular domains. So we examine first the circular case. It is easily proved [14] that the following relations hold for a circle centered at the origin with radius a :

$$F_m^{\sigma 1}(p) = (k\ H'_m\,(k\ a))\, P_m^{\sigma}\,(\theta_P) + \left(\frac{m}{a}\ H_m\,(k\ a)\right) Q_m^{\sigma}\,(\theta_P) \tag{3.1}$$

$$(-1)^{\sigma}\ \ F_m^{\sigma 2}(p) = \left(\frac{m}{a}\ H_m\,(K\ a)\right) P_m^{(3-\sigma)}\,(\theta_P) + (K\ H'_m\,(K\ a))\, Q_m^{(3-\sigma)}\,(\theta_P)$$

where $P_m^{\sigma}\,(\theta_P) = E_m^{\sigma}\,(\theta_P)\ \ \overrightarrow{r}$ and $Q_m^{\sigma}\,(\theta_P) = (-1)^{\sigma}\, E_m^{(3-\sigma)}\,(\theta_P)\ \ \overrightarrow{\theta}$

$$T_p G_0\,(p,\ q) = [T_q G_0\,(p,\ q)]^t \qquad \text{and} \qquad T_p H\,(p,\ q) = [T_q H\,(p,\ q)]^t \tag{3.2}$$

where the superscript t indicates the transpose matrix. In view of (3.2) we conclude that $K_1 = \overline{K_1^*}$. We now consider the modified single layer potentials with densities given by P_n^{ν} and Q_n^{ν} :

$$V_n^{\nu}\,(p) = \frac{1}{2\ \pi}{}_{R_q\ =\ a} G_1(p,q).P_n^{\nu}\,(\theta_q)\,.ds_q \tag{3.3}$$

$$U_n^{\nu}\,(p) = \frac{1}{2\ \pi}{}_{R_q\ =\ a} G_1(p,q).Q_n^{\nu}\,(\theta_q)\,.ds_q \tag{3.4}$$

exploiting the orthogonality relations for the vectors P_n^{ν} and Q_n^{ν}, we obtain :

$$\frac{1}{2\ \pi}{}_0^{2\ \pi} F_m^{\sigma 1}(q).P_n^{\nu}\,(\theta_q)\,.a\ .d\theta_q = k\ a\ H'_m\,(k\ a)\,\delta_{m\ n}\ \delta_{\sigma\ \nu} \tag{3.5}$$

$$\frac{1}{2\ \pi}{}_0^{2\ \pi} F_m^{\sigma 2}(q).P_n^{\nu}\,(\theta_q)\,.a\ .d\theta_q = (-1)^{\sigma}\ \ m\ H_m\,(K\ a)\,\delta_{m\ n}\ \delta_{(3-\sigma)\ \nu} \tag{3.6}$$

$$\frac{1}{2\ \pi}{}_0^{2\ \pi} F_m^{\sigma 1}(q).Q_n^{\nu}\,(\theta_q)\,.a\ .d\theta_q = m\ H_m\,(k\ a)\,\delta_{m\ n}\ \delta_{\sigma\ \nu} \tag{3.7}$$

$$\frac{1}{2\ \pi}{}_0^{2\ \pi} F_m^{\sigma 2}(q).Q_n^{\nu}\,(\theta_q)\,.a\ .d\theta_q = (-1)^{(3-\nu)}\ \ K\ a\ H'_m\,(K\ a)\,\delta_{m\ n}\ \delta_{\sigma\ (3-\nu)} \tag{3.8}$$

Substituting the expressions (3.5)-(3.8) in (3.3) and (3.4), we obtain the relations :

$$V_n^{\nu}\,(p) = \frac{i}{4\mu K^2}\ k\ a\ H'_n\,(k\ a)\left[\widehat{F}_n^{\nu 1}(p) + a_n^{\nu 1}\ F_n^{\nu 1}(p)\right] \tag{3.9}$$

$$+\frac{i}{4\mu K^2}\ \ (-1)^{3-\nu}\, n\ H_n\,(K\ a)\left[\widehat{F}_n^{(3-\nu)\ 2}(p) + a_n^{(3-\nu)\ 2}\ F_n^{(3-\nu)\ 2}(p)\right]$$

$$U_n^{\nu}\,(p) = \frac{i}{4\mu K^2}\ n\ H_n\,(k\ a)\left[\widehat{F}_n^{\nu 1}(p) + a_n^{\nu 1}\ F_n^{\nu 1}(p)\right] \tag{3.10}$$

$$+\frac{i}{4\mu K^2}\ (-1)^{3-\nu}\, K\ a\ H_n'\,(K\ a)\left[\widehat{F}_n^{(3-\nu)\ 2}(p)+a_n^{(3-\nu)\ 2}\ F_n^{(3-\nu)\ 2}(p)\right]$$

In order to find the eigenvalues of M, the following relation has to be satisfied :

$$MU\,(p)=\lambda U\,(p)\qquad\qquad R_p=a \tag{3.11}$$

Taking into account that $\{P_n^\nu,\ Q_n^\nu\}$ is a basis in $(L_2\,(\partial D))^2$. We can express $U\,(p)$ as a linear combination of these vectors, So :

$$U\,(p)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\left(\alpha_m^\sigma\ P_m^\sigma\,(p)+\beta_m^\sigma\ Q_m^\sigma\,(p)\right) \tag{3.12}$$

To calculate $MU\,(p)$, we must calculate $MP_n^\nu\,(p)=\left(\frac{1}{2}\ I+\overline{K_1^*}\right)P_n^\nu\,(p)$ and $MQ_n^\nu\,(p)=\left(\frac{1}{2}\ I+\overline{K_1^*}\right)Q_n^\nu\,(p)$. If we apply the Neumann boundary condition to V_n^ν, we obtain [18] :

$$TV_n^\nu\,(p)=\left(\frac{1}{2}\ I+K_1\right)P_n^\nu\,(p)\qquad\qquad R_p=a \tag{3.13}$$

using $K_1=\overline{K_1^*}$, (3.13) becomes :

$$MP_n^\nu\,(p)=\left(\frac{1}{2}\ I+\overline{K_1^*}\right)P_n^\nu\,(p)\ \ =TV_n^\nu\,(p)\qquad\qquad R_p=a \tag{3.14}$$

in the same way, we can obtain :

$$MQ_n^\nu\,(p)=\left(\frac{1}{2}\ I+\overline{K_1^*}\right)Q_n^\nu\,(p)\ \ =TU_n^\nu\,(p)\qquad\qquad R_p=a \tag{3.15}$$

Exploiting the following relations [14] :

$$TF_n^{\nu 1}(p)=k^2\left(2\mu\ H_n''\,(k\ a)-\lambda\ H_n\,(k\ a)\right)P_n^\nu\,(\theta_P) \tag{3.16}$$

$$+\left(\frac{2\mu\ n}{a}\ \left(k\ H_n'\,(k\ a)-\frac{H_n\,(k\ a)}{a}\right)\right)Q_n^\nu\,(\theta_P)$$

$$TF_n^{\nu 2}(p)=\mu\ K^2\left(2\ H_n''\,(K\ a)+H_n\,(K\ a)\right)(-1)^\nu\,P_n^{(3-\nu)}\,(\theta_P) \tag{3.17}$$

$$+\frac{2\mu\ n}{a}\left(K\ H_n'\,(K\ a)-\frac{H_n\,(K\ a)}{a}\right)(-1)^\nu\,Q_n^{(3-\nu)}\,(\theta_P)$$

$$T\widehat{F}_n^{\nu 1}(p) = k^2 \left(2\mu\ J_n''(k\ a) - \lambda\ J_n(k\ a)\right) P_n^\nu(\theta_P) \tag{3.18}$$
$$+ \left(\frac{2\mu\ n}{a}\left(k\ J_n'(k\ a) - \frac{J_n(k\ a)}{a}\right)\right) Q_n^\nu(\theta_P)$$

$$T\widehat{F}_n^{\nu 2}(p) = \mu\ K^2 \left(2\ J_n''(K\ a) + J_n(K\ a)\right)(-1)^\nu P_n^{(3-\nu)}(\theta_P) \tag{3.19}$$
$$+\frac{2\mu\ n}{a}\left(K\ J_n'(K\ a) - \frac{J_n(K\ a)}{a}\right)(-1)^\nu Q_n^{(3-\nu)}(\theta_P)$$

and using the notations :

$$\begin{aligned} b_n^{\nu 1} &= k^2 \left(2\mu\ J_n''(k\ a) - \lambda\ J_n(k\ a)\right) 3.20 \\ &+a_n^{\nu 1}\ k^2 \left(2\mu\ H_n''(k\ a) - \lambda\ H_n(k\ a)\right) \end{aligned} \tag{1}$$

$$\begin{aligned} b_n^{\nu 2} &= \mu\ K^2 \left(2\ J_n''(K\ a) + J_n(K\ a)\right) 3.21 \\ &+a_n^{(3-\nu)\ 2}\ \mu\ K^2 \left(2\ H_n''(K\ a) + H_n(K\ a)\right) \end{aligned} \tag{2}$$

$$b_n^{\nu 3} = \frac{2\mu\ n}{a}\left(k\ J_n'(k\ a) - \frac{J_n(k\ a)}{a}\right) + a_n^{\nu 1}\ \frac{2\mu\ n}{a}\left(k\ H_n'(k\ a) - \frac{H_n(k\ a)}{a}\right) \tag{3.22}$$

$$b_n^{\nu 4} = \frac{2\mu\ n}{a}\left(K\ J_n'(K\ a) - \frac{J_n(K\ a)}{a}\right) + a_n^{(3-\nu)\ 2}\ \frac{2\mu\ n}{a}\left(K\ H_n'(K\ a) - \frac{H_n(K\ a)}{a}\right) \tag{3.23}$$

Then we obtain :

$$\begin{aligned} TV_n^\nu(p) &= \frac{i}{4\mu K^2}\ k\ a\ H_n'(k\ a)\left[b_n^{\nu 1}\ P_n^\nu(\theta_P) + b_n^{\nu 3}\ Q_n^\nu(\theta_P)\right] 3.24 \\ &-\frac{i}{4\mu K^2}\ n\ H_n(K\ a)\left[b_n^{\nu 2}\ P_n^\nu(\theta_P) + b_n^{\nu 4}\ Q_n^\nu(\theta_P)\right] \end{aligned} \tag{3}$$

$$\begin{aligned} TU_n^\nu(p) &= \frac{i}{4\mu K^2}\ n\ H_n(k\ a)\left[b_n^{\nu 1}\ P_n^\nu(\theta_P) + b_n^{\nu 3}\ Q_n^\nu(\theta_P)\right] 3.25 \\ &+\frac{i}{4\mu K^2}\ K\ a\ H_n'(K\ a)\left[b_n^{\nu 2}\ P_n^\nu(\theta_P) + b_n^{\nu 4}\ Q_n^\nu(\theta_P)\right] \end{aligned} \tag{4}$$

In view of (3.14), (3.15), (3.24) and (3.25). (3.11) leads to the equations :

$$\left(\lambda - \frac{i}{4\mu K^2}\left[k\ a\ H_n'(k\ a)\ b_n^{\nu 1}\ +\ n\ H_n(K\ a)\ b_n^{\nu 2}\right]\right)\ \alpha_n^{\nu} \tag{3.26}$$

$$-\ \frac{i}{4\mu K^2}\left[n\ H_n(k\ a)\ b_n^{\nu 1}\ +\ K\ a\ H_n'(K\ a)\ b_n^{\nu 2}\right]\ \beta_n^{\nu}\ =\ 0$$

$$-\ \frac{i}{4\mu K^2}\left[k\ a\ H_n'(k\ a)\ b_n^{\nu 3}\ +\ n\ H_n(K\ a)\ b_n^{\nu 4}\right]\ \alpha_n^{\nu}$$

$$+\ \left(\lambda - \frac{i}{4\mu K^2}\left[n\ H_n(k\ a)\ b_n^{\nu 3}\ +\ K\ a\ H_n'(K\ a)\ b_n^{\nu 4}\right]\right)\ \beta_n^{\nu}\ =\ 0$$

If we use the following notations :

$$A_n^{\nu 1} = k\ a\ H_n'(k\ a)\ b_n^{\nu 1}\ +\ n\ H_n(K\ a)\ b_n^{\nu 2}$$

$$A_n^{\nu 2} = k\ a\ H_n'(k\ a)\ b_n^{\nu 3}\ +\ n\ H_n(K\ a)\ b_n^{\nu 4}$$

$$A_n^{\nu 3} = n\ H_n(k\ a)\ b_n^{\nu 1}\ +\ K\ a\ H_n'(K\ a)\ b_n^{\nu 2}$$

$$A_n^{\nu 4} = n\ H_n(k\ a)\ b_n^{\nu 3}\ +\ K\ a\ H_n'(K\ a)\ b_n^{\nu 4}$$

then (3.26) can be rewritten as follows :

$$\left(\lambda - \frac{i}{4\mu K^2}\ A_n^{\nu 1}\right)\ \alpha_n^{\nu} - \frac{i}{4\mu K^2}\ A_n^{\nu 3}\ \beta_n^{\nu}\ =\ 0 \tag{3.27}$$

$$-\ \frac{i}{4\mu K^2}\ A_n^{\nu 2}\ \alpha_n^{\nu} + \left(\lambda - \frac{i}{4\mu K^2}\ A_n^{\nu 4}\right)\ \beta_n^{\nu}\ =\ 0$$

In order that the solution of the above system be non-trivial, its determinant must vanish. So we arrive at the following relation which the eigenvalues must satisfy :

$$\lambda^2 - \frac{i}{4\mu K^2}\ \left(A_n^{\nu 4} + A_n^{\nu 1}\right)\ \lambda + \left(\frac{i}{4\mu K^2}\right)^2 \left(A_n^{\nu 1}\ A_n^{\nu 4} - A_n^{\nu 2}\ A_n^{\nu 3}\right) = 0 \tag{3.28}$$

Obviously, in order to minimize the condition number we have to choose the multipole coefficients, in such a way that all eigenvalues become 1. Then the condition number is 1. From (3.28), using the same technique developed in [2], in order that all the eigenvalues be equal to 1, we obtain the relations

$$\left(\frac{i}{4\mu K^2}\right)^2 \left(A_n^{\nu 1}\ A_n^{\nu 4} - A_n^{\nu 2}\ A_n^{\nu 3}\right) = e^{2\ i\ \theta_n^\nu} \tag{3.29}$$

$$-\frac{i}{4\mu K^2}\ \left(A_n^{\nu 4} + A_n^{\nu 1}\right)\ = \rho_n^\nu\ e^{i\ \theta_n^\nu}$$

where $\rho_n^\nu,\ \theta_n^\nu$ are arbitrary real numbers satisfying the inequalities :

$$0 \leq \rho_n^\nu \leq 2 \quad \text{and} \quad 0 \leq \theta_n^\nu < 2\ \pi.$$

Indeed, using (3.25), (3.24) becomes :

$$\lambda^2 + \left(\rho_n^\nu\ e^{i\ \theta_n^\nu}\ \right)\ \lambda + e^{2\ i\ \theta_n^\nu}\ = 0 \tag{3.30}$$

which admits the solutions :

$$\lambda_1 = \frac{-\left(\rho_n^\nu\ e^{i\ \theta_n^\nu}\ \right) + i\ e^{i\ \theta_n^\nu}\ \sqrt{4 - (\rho_n^\nu)^2}}{2}$$

$$\lambda_2 = \frac{-\left(\rho_n^\nu\ e^{i\ \theta_n^\nu}\ \right) - i\ e^{i\ \theta_n^\nu}\ \sqrt{4 - (\rho_n^\nu)^2}}{2}$$

note here that we have : $|\lambda_1|\ = |\lambda_2| = 1$.

Obviously there are infinitely many choices of simple mulitipole coefficients $a_m^{\sigma\ l}$, which satisfy the imposed conditions. If we choose $a_m^{\sigma\ l}$ as the coefficients which minimize the norm of the modified integral operator [14], after some calculations we obtain :

$$A_n^{\nu 1}\ = A_n^{\nu 4} = -\ 4\ i\ \mu\ K^2 \quad \text{and} \quad A_n^{\nu 2}\ = A_n^{\nu 3} = 0 \tag{3.31}$$

For these values (3.31) and for $\rho_n^\nu = 2,\ \theta_n^\nu = \pi$, we find that (3.28) has a double root $\lambda = 1$.

The above choice of the multipole coefficients does not satisfy the inequalities (2.13) imposed on the coefficients by the uniqueness theorem [4]. But, as in [14], it has been proved that with this choice, the norm of the modified integral operator equals zero. So the boundary integral equation is uniquely solvable.

3.2 Minimization of the condition number (the perturbation of the circle)

As in [11, 10, 3, 14] we can consider a family of non-circular boundaries given parametrically by the relation :

$$R_\varepsilon = a + \varepsilon\ \varphi\left(\theta_P\right) \qquad 0 \le \theta_P \le 2\pi \tag{3.32}$$

where φ and $\frac{\partial \varphi}{\partial \theta}$ are all bounded. We will use the estimates for the multipole vectors given in [14] :

$$F_m^{\sigma l}\left(P_\varepsilon\right) = F_m^{\sigma l}\left(p_a\right) + O\left(\varepsilon\right) \tag{3.33}$$

$$TF_m^{\sigma l}\left(P_\varepsilon\right) = TF_m^{\sigma l}\left(p_a\right) + O\left(\varepsilon\right) \tag{3.34}$$

where P_ε is a point in the perturbed circle while p_a describes points on the circle of radius a. In [14] it has been proved that the boundary integral operator K_1^ε is a perturbation of the boundary integral operator K_1^a defined on the circle :

$$K_1^\varepsilon = K_1^a + O\left(\varepsilon\right) \tag{3.35}$$

In view of these estimates it is straightforward that the eigenvalues of the perturbed operator M_ε are perturbations of the eigenvalues of the original operator M. So :

$$Cond\left(M_\varepsilon\right) = Cond\left(M\right) + O\left(\varepsilon\right) = 1 + O\left(\varepsilon\right) \tag{3.36}$$

4 Open problems

1- Investigate an other special cases.

2- Investigate an other criterion of optimization choosing the cross multipole coefficients of the modification, that of the minimization of the condition number of the integral equation (in the case of three dimensions, see [10] for acoustical case and [2] for elastical case).

3- Investigate an other criterion of optimization choosing the simple and cross multipole coefficients of the modification, that of the minimization of the traction of the modified Green's function $\|TG_1\|$.

4- Establish the numerical results for this work (for numerical results see [5] and [6]).

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