Polar Bergman Polynomials on Domain with Corners

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Abstract

In this paper we present a new class named polar of monic orthogonal polynomials with respect to the area measure supported on G where G is a bounded simply-connected domain in the complex plane ℂ. We analyze some open questions and discuss some ideas properties related to solving asymptotic behavior of polar Bergman polynomials over domains with corners and asymptotic behavior of modified Bergman polynomials by affine transforms in variable and polar modified Bergman polynomials by affine transforms in variable. We show that uniform asymptotic of Bergman polynomials over domains with corners and by Pritsker's theorem imply uniform asymptotic for all their derivatives.

Keywords: Bergman orthogonal polynomials, polar orthogonal polynomials, asymptotic behavior, Faber polynomials.

1 Introduction

We consider $G$ be a bounded simply-connected domain in the complex plane $\mathbb{C}$ whose boundary $E = \partial G$ is a Jordan curve. Let $\sigma$ be a positive finite area measure defined on the Borel set $B(\mathbb{C}) \cap G$ with compact support $G$. The $L_2$ extremal polynomials $(L_n)_{n=0,1,2,...}$ associated to the measure $\sigma$ and the support $G$ are defined as
the solutions of the extremal problems in the space $L^2(E, d\sigma)$. Let $m_{n,2}(\sigma)$ ($n = 0, 1, 2, \ldots$) denotes the extremal constants associated with $\sigma$ and $G$. $m_{n,2}(\sigma)$ satisfy

$$m_{n,2}(\sigma) = \min_{Q_n \in L^2(E, d\sigma)} \{ Q_n(z) = z^n + q_1z^{n-1} + \ldots + q_n \}$$

(1)

Let us remark that the polynomials $\left(L_n\right)_{n=0, 1, 2, \ldots}$ coincide exactly with the monic orthogonal polynomials, and satisfy

$$L_n(z) = z^n + \text{lower degree terms}$$

And

$$\int_G L_n(z) \overline{L_m(z)} d\sigma = \delta_{m,n} \left\| L_n \right\|_{L^2(E, d\sigma)}^2, \quad n, m = 0, 1, 2, \ldots$$

(2)

Where $d\sigma_z$ stands for the area measure. In this case $\left\| L_n \right\|_{L^2(E, d\sigma)} = \sqrt{\int_G \left| L_n(z) \right|^2 d\sigma}$.

Let $\left(\varphi_n(z)\right)_{n=0, 1, 2, \ldots}$ be the sequence of corresponding orthonormal Bergman polynomials such that $\varphi_n(z) = \lambda_n z^n + \text{lower degree terms} \ (\lambda_n > 0)$. And

$$\int_G \varphi_n(z) \overline{\varphi_m(z)} d\sigma = \delta_{m,n}, \quad n, m = 0, 1, 2, \ldots$$

(3)

Let $\Omega = \overline{C} - \overline{D}$ and let $E$ be a Jordan closed curve, $\Omega = \text{Ext} (E)$, $H = \{w, |w| > 1\}$. Let $w = \phi(z)$ be a function which maps $\Omega$ conformally on $H$ in a such manner that

$$\lim_{z \to \infty} \frac{\phi(z)}{z} > 0 \quad \text{and} \quad \phi(\infty) = \infty.$$ In fact this limit is equal to $\gamma$, where $\gamma^{-1}$ is the logarithmic capacity of $E$. Let $\psi : H \to \Omega$ be the inverse function of $\phi$. Let us remark that $\phi$ can be represented by the Laurent expansion at infinity

$$\phi(z) = \frac{1}{\text{cap}(E)} z + v_0 + \sum_{i=1}^{\infty} v_i z^i$$

The two functions $\phi(z)$ and $\psi(w)$ have a continuous extension to $E$ and on the unit circle respectively (Carathéodory's theorem). Their derivatives $\phi'$ and $\psi'$ have no zeros in $\Omega$ and $H$ and have limit values on $E$ and on the unit circle almost everywhere (with respect to the Lebesgue measure). The functions $\phi'(z)$ and $\psi'(w)$ are defined and integrable on $E$ and on the unit circle.

We introduce a set of monic polynomials associated to this monic Bergman polynomials on domain with corners. This class of Bergman polynomials is studied in ([11]) by D. Khavinson and N. Stylianopoulos. We study their relevant recurrence relations (polynomial) and show that they satisfy mixed recurrence relations which are similar to those of the Bergman polynomials case. We also indicate some questions concerning the asymptotic behavior of Bergman polynomials.
-ls over domains with corners, in short BPDC and discuss some theorems speaking the asymptotic behavior of polar BPDC, and giving some compatible equations between them. It was shown that some of polar’s characterizations can be represented in terms of the corresponding BPDC. We investigate this results for showing some relations concerning the asymptotic behavior of modified Bergman polynomials by affine transforms in variable and Polar modified Bergman polynomials by affine transforms in variable and by comparison equations we conclude the next’s asymptotic behavior for derivatives of polar BPDC. We also gave some algebraic tools for working with this polar BPDC.

The following theorem concerning the finite recurrence relation of Bergman polynomials over domains with corners is the main basic result of these works.

**Theorem 1.1**. The Bergman polynomials \( \varphi_n(z) \) satisfy an \( (s+1) \)-terms recurrence relation, for any \( n \geq s-1 \).

\[
z \varphi_n(z) = a_{n,s} \varphi_{n+1}(z) + a_{n,s-1} \varphi_{n+s}(z) + \ldots \]

Assume that \( E \) is piecewise analytic without cusps. If the Bergman polynomials \( \varphi_n(z) \) satisfy an \( (s+1) \)-terms recurrence relation, with some \( s \geq 2 \). Then \( s=2 \) and \( E \) is an ellipse.

**Proof.** For more details see ([1]).

The modified Bergman polynomials by affine transforms in variable is one of the marquable orthogonal polynomials associated Bergman Polynomials.

**Definition 1.2.** If we denote for brevity \( l_n(z) = a^{-n} L_n(az+b), n = 0,1,2 \ldots \) These polynomials are called monic modified Bergman polynomials by affine transform -s in variable.

## 2 Problem Formulations

### 2.1 BPDC and Recurrence relations

**Proposition 2.1.** The modified BPODC satisfy

\[
z l_n(z) = b_{n+1,n} l_{n+1}(z) + b_{n,n} l_{n}(z) + b_{n,n-1} l_{n-1}(z)
\]

Where

\[
b_{n+1,n} = \frac{\lambda_{n+1}}{\lambda_n} a_{n+1,n} , \quad b_{n,n} = \frac{a_{n,n} - b}{a} , \quad b_{n,n-1} = \frac{\lambda_{n-1}}{a^2 \lambda_n} a_{n-1,n}
\]

**Proof.** From the know three terms \( \varphi_n \)'s recurrence relations we have for a such that \( a \neq 0 \).

\[
z l_n(z) = \frac{\lambda_{n+1}}{\lambda_n} a_{n+1,n} L_{n+1}(z) + a_{n,n} L_n(z) + \frac{\lambda_{n-1}}{\lambda_n} a_{n-1,n} L_{n-1}(z)
\]
Thus we get
\[(az + b)L_n(az + b) = \frac{\lambda_{n+1}}{\lambda_n} a_{n+1,n} L_{n+1}(az + b) + a_{n,n} L_n(az + b) + \frac{\lambda_{n-1}}{\lambda_n} a_{n-1,n} L_{n-1}(az + b)\]

After cancellations we readily conclude that (5). And this completes the proof.

**Proposition 2.2** Notice that
\[\sum_{n=0}^{\infty} \frac{|a|^{2n}}{\|L_n\|^2} \int_{\mathbb{D}} \left( z \frac{L_n(z)}{L_n(z_0)} \right)^2 \, d\mu(z) = \frac{1}{\pi} \phi'(az + b) \phi'(az_0 + b)\]  \hspace{1cm} (6)

Moreover If $|\phi| = 1$ we conclude that
\[\sum_{n=0}^{\infty} \frac{l_n(z_0)}{\|L_n\|^2} = \frac{1}{\pi} \phi'(e^{i\tau} z + b) \phi'(e^{i\tau} z_0 + b)\]  \hspace{1cm} (7)

Where $\lambda_n = \|L_n\|^2_{L(E,\phi)}$.

**Proof.** The well known bilinear series formula with taking into account that $\phi'(z_0) >> 0$ is
\[\sum_{n=0}^{\infty} \frac{L_n(z)L_n(z_0)}{\|L_n\|^2} = \frac{1}{\pi} \phi'(z) \phi'(z_0), \quad |z| < 1, \text{ and } |z_0| < 1.\]

Applying this equation we readily conclude that (6), (7).

**Example 2.3.** For a system of polynomials orthonormal on the unit disc the bilinear series of $\varphi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$ , $n = 0, 1, 2, \ldots$ is follows
\[\sum_{n=0}^{\infty} \frac{L_n(z)L_n(z_0)}{\|L_n\|^2} = \frac{1}{\pi} \frac{1}{(1 - z \overline{z_0})^\tau}, \quad |z| < 1, \text{ and } |z_0| < 1\]  \hspace{1cm} (8)

Let us now put $l_n(z) = a^{-n} L_n(az + b)$ for $n = 0, 1, 2, \ldots$ accordingly we get for $z$ such that: $|z| < 1, \text{ and } |z_0| < 1$.
\[\sum_{n=0}^{\infty} (n+1)(az + b)^n \left( \overline{az_0 + b} \right)^\tau = \frac{1}{(1 - (\overline{az_0 + b})(az + b))^{\tau/2}}\]

### 2.2 Polar BPODC and Recurrence relations

**Definition 2.4.** For a fixed complex number $\xi$ we are going to define $P_n(z)$ as a monic polynomials such that
\[(z - \xi)P_n(z) = (n+1)\int_\xi^z L_n(t) \, dt\]  \hspace{1cm} (9)
Obviously the following calculus shows that the pole of $P_n(z)$ do not have to be irregular.

$$\lim_{z \to \xi} L_n(z) = (n+1) \lim_{z \to \xi} \frac{z - \xi}{z - \xi} = (n+1)L_n(\xi).$$

More information on the history and applications of this concept may be found in ([5]).

The $L_n$ orthogonality can be written as

$$\int_G L_n(z) \overline{L_n} \, d\sigma = \delta_{\alpha,\nu} \|L_n\|^2_{L^2(E,d\sigma)}, \quad \nu = 0, 1, 2, ..., n - 1.$$  

Analogously to (7) we derive for $\nu = 0, 1, 2, ..., n - 1$.

$$\int_G (P_n(z) + (z - \xi)P_n'(z))L_n(\xi) \, d\sigma = \delta_{\alpha,\nu} \|L_n\|^2_{L^2(E,d\sigma)},$$

**Proposition 2.5.** The recurrence relation of Polar can be written in the form

$$zP_n(z) = d_{n+1,n}P_{n+1}(z) + d_{n,n}P_n(z) + d_{n-1,n}P_{n-1}(z)$$

$$+ (z - \xi)(d_{n+1,n}P_n' + d_{n,n}P_n' + d_{n-1,n}P_{n-1}')$$

(10)

Where

$$d_{n+1,n} = \frac{n + 1}{n + 2} \frac{\lambda_{n+1}a_{n+1,n}}{\lambda_n}, \quad d_{n,n} = a_{n,n}, \quad d_{n-1,n} = \frac{n + 1}{n} \frac{\lambda_{n-1}a_{n-1,n}}{\lambda_n}$$

**Proof.** Integrating both sides of the three term's recurrence relation we get for $z \neq \xi$

$$(n+1)(z - \xi)^{-1} \int_{\xi}^{z} tL_n(t) \, dt =$$

$$\frac{n + 1}{n + 2} \frac{\lambda_{n+1}a_{n+1,n}P_{n+1}(z)}{\lambda_n} + a_{n,n}P_n(z) + \frac{n + 1}{n + 2} \frac{\lambda_{n+1}a_{n+1,n}P_{n+1}(z)}{\lambda_n} + \frac{n + 1}{n} \frac{\lambda_{n-1}a_{n-1,n}P_{n-1}(z)}{\lambda_n}$$

In other hand . Let us remark

$$(n+1)(z - \xi)^{-1} \int_{\xi}^{z} tL_n(t) \, dt = (z - \xi)^{-1} \int_{\xi}^{z} \left[tP_n(t)\right] \, dt =$$

$$\frac{1}{(z - \xi)^{-1}} \left[z(z - \xi)P_n(z) - \int_{\xi}^{z} (t - \xi)P_n(t) \, dt \right] =$$

Therefore
\[-(z - \xi)^{-1} \int_\xi (t - \xi) P_n(t) dt =
\]
\[
\frac{n+1}{n+2} \frac{\lambda_{n+1}}{\lambda_n} a_{n+1,n} P_{n+1}(z) - \left( z - a_{n,n} \right) P_n(z) + \frac{n+1}{n} \frac{\lambda_{n}}{\lambda_{n-1}} a_{n-1,n} P_{n-1}(z)
\]

Differentiating the two sides of this equality we get,
\[-(z - \xi) P_n(z) =
\]
\[
\frac{n+1}{n+2} \frac{\lambda_{n+1}}{\lambda_n} a_{n+1,n} P_{n+1}(z) - \left( z - a_{n,n} \right) P_n(z) + \frac{n+1}{n} \frac{\lambda_{n}}{\lambda_{n-1}} a_{n-1,n} P_{n-1}(z) +
\]
\[
(z - \xi) \left( \frac{n+1}{n+2} \frac{\lambda_{n+1}}{\lambda_n} a_{n+1,n} P'_{n+1}(z) - \left( z - a_{n,n} \right) P'_n(z) - P_n(z) + \frac{n+1}{n} \frac{\lambda_{n}}{\lambda_{n-1}} a_{n-1,n} P'_n(z) \right)
\]

Which leads to (10).

**Example 2.6.** The Bergman Polynomials on Cassini Ovals ([2]).

Let us consider the curve \( z \to \{ z - 1 \| z + 1 \| = \alpha \} \) if \( \alpha \leq 1 \) the curve define two bounded domains, let us denote either of them by \( B_\alpha \) the function \( \frac{1}{\alpha} (z^2 - 1) \) maps \( B_\alpha \) one to one on to the disc \( |w| < 1 \). The orthonormal Bergman Polynomials on Cassini Ovals in short BPCO can be written in the form:

\[ \varphi_{2n+1}(z) = 2^{\frac{n+1}{2\sqrt{\pi \alpha}}} z^{(z^2 - 1)^n} \quad n = 0,1,2,\ldots \]

is also a closed orthonormalised system in the class \( H^2(B_\alpha, d\sigma) \).

The corresponding monic orthogonal polynomials can be written in the form

\[ L_{2n+1}(z) = 2z(z^2 - 1)^n \quad n = 0,1,2,\ldots \]

They satisfy for \( \nu = 0,1,2,\ldots, 2n \)

\[ \int_{B_\alpha} L_{2n+1}(z) \bar{z}^\nu d\sigma = \delta_{2n+1,\nu} \frac{\alpha^{2n+2}}{(n+1)!} \]

Define \( (P_{2n+1}(z))_n^{\infty} \) as a monic polynomials polar BPCO. In the next we prove the following assertions

\[ (z - \xi) P_{2n+1}(z) = (z^2 - 1)^n + (\xi^2 - 1)^n \]

\[ P_{2n+1}(1) = (\xi - 1)^n (\xi + 1)^n, \quad \text{and} \quad P_{2n+1}(-1) = (\xi - 1)^n (\xi + 1)^n. \]  \( (11) \)

\[ (z - \xi) P_{2n+1}(z) = 2 \int_\xi (t^2 - 1)^n d\xi = (z^2 - 1)^n + (\xi^2 - 1)^n. \]  \( (12) \)

Clearly we have

Thus, the results (11), (12) are proved. Discovering that the pole of \( P_n(z) \) do not have to be irregular.
\[ \lim_{z \to \infty} P_{2n+1}(z) = (n+1)2 \xi (\xi^2 - 1)^n. \]

Next we use two excellent important theorems due to Pritsker.I.E, ([3]).

**Theorem 2.7.** ([3]). Let us put \( \partial \Omega = E \) an analytic curve so that \( \varphi \) is conformal map from \( \Omega \) to \( G \) with properties: Suppose that \( \partial \Omega = E \) is an analytic curve so that \( \varphi \) is conformal map from \( \Omega \) to \( G \) with properties:

\[ \varphi (\infty) = \infty, \quad \lim_{z \to \infty} \varphi(z) = \frac{1}{\text{cap}(E)} = \nu \neq 0 \]

can be used (by Carathéodory’s theorem ([10],[8]), \( |\varphi(\xi)| = 1, \ \forall \xi \in E \). In addition \( \varphi(E) = T = \{ w, |w| = 1 \} \) conformally. Suppose that : \( \psi : G \to \Omega \), its inverse map . Let us put

\[ (\varphi(z))^n = \left( \frac{1}{\text{cap}(E)} z + v_0 + \sum_{i=1}^{\infty} v_i z^i \right)^n = F_n(z) + E_n \left( \frac{1}{z} \right) \quad (13) \]

Where \( E_n \left( \frac{1}{z} \right) \to 0 \) as \( n \to \infty \). Then

\[ F_n^{(k)}(z) = \frac{d^k}{dz^k} [ (\varphi(z))^n + O(r^k) ] , \quad r \in [0,1], \text{as } n \to \infty, \quad z \in \overline{\Omega} \]

And as \( n \to \infty \),

\[ F_n^{(k)}(z_0) = \frac{\alpha_k! n^{\alpha_k} w_0^n}{(a_\alpha w_0^n) \Gamma(1 + \alpha_k)} + o(n^{\alpha_k}) \quad (15) \]

Where \( z_0 \) is fixed point, \( \varphi(z_0) = w_0 \). And

\[ \psi(w) = \psi(w_0) + a_\alpha (w - w_0)^\alpha + \ldots \quad a_\alpha \neq 0 \]

**Theorem 2.8.** ([3]). Let \( \Gamma \) be a rectifiable Jordan curve and let \( g(z) \) belongs to \( H^\alpha(\Omega) \). For \( z_0 \in \Gamma \), assume that \( \Gamma \) has an exterior angle of opening \( \alpha \pi \) at the point \( z_0, 0 \leq \alpha \leq 2\pi \) formed by \( C^\alpha \) arcs, where \( \omega(x) \) satisfies the Alper condition:

\[ \int_0^\xi \omega(x) \log \left( \frac{1}{x} \right) dx < \infty \]

Suppose that the boundary values of \( g(z) \) satisfy a Dini condition at the point \( z_0 \)

\[ |g(z) - g(z_0)| \leq \lambda(z - z_0) \}, \quad |z - z_0| \leq \delta \}

With \( \delta > 0 \) and a modulus of continuity \( \lambda(x) \) such that

\[ \int_0^\delta \lambda(x) dx < \infty \]

Then, the generalized Faber polynomials satisfy

\[ Q_n(z_0) = a_g(z_0) \Phi^s(z_0) + o(1) \quad \text{as } n \to \infty. \]
Under condition of convexity and απ is the largest exterior angle at the boundary Γ, and for the case \( g(z) = 1 \) then:

\[
\text{Max} F_n(z) \rightarrow \alpha \quad \text{as} \quad n \rightarrow \infty.
\]

Moreover under more general conditions we have

\[
|F_n(z_0)| \rightarrow \alpha \quad \text{as} \quad n \rightarrow \infty.
\]

**Proof.** For more details concerning the proofs of two theorems see ([3],[6]).

### 3 Main Results

#### 3.1 Asymptotic behavior of BPODC and their derivatives.

**Theorem 3.1** ([1]). Assume that \( E \) is piecewise analytic without cusps. Then, for any \( n \in \mathbb{N} \)

\[
\lim_{n \rightarrow \infty} \frac{\lambda_{n+1}}{\lambda_n} = \nu, \quad \lim_{n \rightarrow \infty} \frac{\pi}{n+1} \nu^{n+1} = 1
\]

(18)

Where \( \gamma^{-1} = \text{Cap} (E) \). In addition uniformly for compacts subsets of \( \Omega \)

\[
\lim_{n \rightarrow \infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = \varphi(z)
\]

Also uniformly for compacts subsets of \( \Omega \)

\[
\lim_{n \rightarrow \infty} \frac{\pi}{n+1} \varphi_n(z) = \varphi'(z) \lim_{n \rightarrow \infty} F_n(z)
\]

(19)

**Proof.** For more details ([1]). Under the same conditions as in theorem 2.7 we can only need to show that (19).

**Corollary 3.2.** Under same conditions as in theorem 2.7 follows asymptotic formulas holds

\[
\lim_{n \rightarrow \infty} \nu^{n+1} L_n(z) = \varphi'(z) \lim_{n \rightarrow \infty} F_n(z).
\]

(20)

Uniformly for compacts subsets of \( \Omega \). Hence uniformly for compacts subsets of \( \overline{\Omega} \)

\[
\lim_{n \rightarrow \infty} L_{n+1}(z) = \nu^{-1} \varphi(z).
\]

(21)

Moreover uniformly for compacts subsets of \( \overline{\Omega} \)

\[
\lim_{n \rightarrow \infty} \nu^{n+1} L_n^{(k)}(z) = \sum_{j=0}^{k} C_j^k \varphi^{(k-j)}(z) \lim_{n \rightarrow \infty} F_n^{(j)}(z)
\]

(22)

Under more general conditions we have

\[
\lim_{n \rightarrow \infty} \nu^{n+1} L_n(z_0) = \alpha \varphi'(z_0)
\]

(23)
Where \( z_0 \) and \( \alpha \) are defined by Pritsker's theorem ([3]).

**Proof.** Notice that
\[
\lim_{n \to \infty} \phi_{n+1}(z) = \phi(z) \Rightarrow \lim_{n \to \infty} \frac{L_{n+1}(z)}{L_n(z)} = \nu^{-1} \phi(z)
\]

By (18) and (19) we get (20) and (21) uniformly for compacts subsets of \( \Omega \). By Pritsker's theorem ([3]) and differentiating \( k \)-times the both sides of (20) we get assertion (22). With help of more general conditions and by Pritsker's formula we get \( F_n(z_0) \to \alpha \). As \( n \to \infty \). Thus property (23) is proved.

The following three theorems are the main result of this work.

### 3.2 Relative asymptotic formula of polar BPODC.

**Theorems 3.3.** We have uniformly for compacts subsets of \( \Omega \)
\[
\lim_{n \to \infty} \frac{L_n(z)}{P_n(z)} = (z - \xi)F(z)
\]
(24)

Where uniformly for compacts subsets of \( \Omega \).

\[
F(z) = \lim_{n \to \infty} \frac{P'_n(z)}{nP_n(z)}
\]
(25)

Hence uniformly for compacts subsets of \( \Omega \).

\[
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = \nu^{-1} \phi(z)
\]
(26)

**Proof.** To prove (24) let us remark from this equality \((z - \xi)P_n(z))' = (n+1)L_n(z)\)

We get uniformly for compacts subsets of \( \Omega \).

\[
\lim_{n \to \infty} \frac{L_n(z)}{P_n(z)} = (z - \xi)\lim_{n \to \infty} \frac{P'_n(z)}{nP_n(z)}
\]

Now use (24) we obtain \( \lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = \lim_{n \to \infty} \frac{L_{n+1}(z)}{L_n(z)} \). Accordingly to (21) we readily conclude that (26).

**Theorem 3.4** For each \( a \in \Omega \) we have uniformly for compacts subsets of domain \( \Omega \) \( - \{a\} \)

\[
\lim_{n \to \infty} \left( \frac{P_{n+1}(z)}{P_n(z)} \left( \frac{P_{n+1}(a)}{P_n(a)} \right)^{-1} = \exp \left( \int_{\xi}^{z} F(t) dt \right) \right) = \exp \left( \int_{\xi}^{z} F(t) dt \right)
\]
(27)

Where \( F \) is defined by (25). This gives for every complex number \( a \).

\[
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = \nu^{-1} \phi(a) \exp \left( \int_{\xi}^{z} F(t) dt \right)
\]
(28)

Thus uniformly for compacts subsets of \( \Omega \).
\[ \lim_{n \to \infty} \frac{P'_n(z)}{n P_n(z)} = \frac{\varphi'(z)}{\varphi(z)} \]  

(29)

**Proof.** We can put
\[
\frac{(z - \xi)P'_n(z)}{(z - \xi)P_n(z)} = (n + 1) \frac{L_n(z)}{(z - \xi)P_n(z)}
\]

Taking exponential yields:
\[
\frac{(z - \xi)P_n(z)}{(a - \xi)P_n(a)} = \exp(n + 1) \int_{a}^{z} \frac{L_n(t)dt}{(t - \xi)P_n(t)}.
\]

Thus we get
\[
P_n(z) = P_n(a) \frac{a - \xi}{z - \xi} \exp(n + 1) \int_{a}^{z} \frac{L_n(t)dt}{(t - \xi)P_n(t)}
\]

Which leads to prove (27) if we taking into account that
\[
\exp\left( (n + 2) \int_{a}^{z} \frac{L_n(t)dt}{(t - \xi)P_n(t)} - (n + 1) \int_{a}^{z} \frac{L_n(t)dt}{(t - \xi)P_n(t)} \right) \to \exp\left( \int_{a}^{z} \frac{L_n(t)dt}{(t - \xi)P_n(t)} \right).
\]

In order to complete proving the rest. By (27) with help of (26) we deduce (28). We can state for each \( z \neq a \)

\[
\exp\left( \int_{a}^{z} F(t)dt \right) = \frac{\varphi(z)}{\varphi(a)}
\]

Differentiating both sides of this equation yields to (29).

### 3.3 Polar modified BPODC by affine transforms in variable

Next we denote by \( R_n(z) \) the sequence of the monic polynomial of degree \( n \) polar of \( l_n(z) \). i-e

\[
(z - \xi)R_n(z) = (n + 1) \int_{\xi}^{z} l_n(t)dt \quad n = 0, 1, 2, ..., \]

(30)

**Theorem 3.5.** For each \( \nu \)

\[
(z - \xi)R_n(z) - (\nu - \xi)R_n(\nu) = a^{-n}[ (az + b - \xi)P_n(az + b) - (a \nu + b - \xi)P_n(a \nu + b)]
\]

Hence

\[
-\xi R_n(0) - (\nu - \xi)R_n(\nu) = a^{-n}[ (b - \xi)P_n(b) - (a \nu + b - \xi)P_n(a \nu + b)]
\]

Moreover

\[
(z - \xi)R'_n(z) + R_n(z) = a^{-n}[ (az + b - \xi)P'_n(az + b) + P_n'(az + b)]
\]

In addition

\[
-\xi^2 R'_n(0) - (\nu - \xi)R_n(\nu) = a^{-n-1}[ (b - \xi)P_n(b) - (a \nu + b - \xi)P_n(a \nu + b)] + a^{-n-2}[ (b - \xi)P'_n(b) + P'_n(b)]
\]

Therefore for \( z \neq \xi \)
\[ a^{-n}(az+b-\xi)P_n(az+b) = \]
\[ a((z-\xi)R_n(z) - (v-\xi)R_n(v)) + a^{-n}(av+b-\xi)P_n(az+b) \]  \hspace{1cm} (31)

**Proof.** When \( l_n \) is replaced by \( a^{-n}L_n(at+b) \) in equation (30) we get
\[ (z-\xi)R_n(z) - (v-\xi)R_n(v) = a^{-n}(n+1) \int_{at+b}^{az+b} L_n(t)dt \]
Hence the first above assertion is proved. Replacing \( z = 0 \) in it we readily conclude that the second assertion. Differentiation of \((z-\xi)R_n(z)\) yields to the third assertion. Taking \( z = 0 \) in it and use the first assertion we get the fourth one. Thus to the first assertion we readily conclude that (31).

**Proposition 3.6.** We have uniformly for compacts subsets of \( \Omega - \{\xi\} \)
\[ G(z) = \left(1 + \frac{\xi^2 - \xi + b}{\xi(z-\xi)}\right)\sigma(z)aF(az+b) \]  \hspace{1cm} (32)

Where \( G(z) = \lim_{n \to \infty} \frac{R_n(z)}{nR_n(z)} \) and \( \sigma(z) = \lim_{n \to \infty} \frac{R_n(z-b)}{aR_n(z)} \) uniformly for compacts subsets of \( \Omega - \{\xi\} \).

If \( \sigma(z) = 0 \). Then uniformly for compacts subsets of \( \Omega - \{\xi\} \)
\[ G(z) = a \frac{\phi'(az+b)}{\phi(az+b)} \]  \hspace{1cm} (33)

**Proof.** Notice that
\[ (z-\xi)G(z) = \lim_{n \to \infty} \frac{L_n(z)}{nR_n(z)} \] and \( (z-\xi)F(z) = \lim_{n \to \infty} \frac{L_n(z)}{P_n(z)} \)
Hence
\[ F(az+b) = \lim_{n \to \infty} \frac{a^{-n}L_n(az+b)}{a^{-n}(az+b-\xi)P_n(az+b)} = \lim_{n \to \infty} \frac{\frac{L_n(z)}{a(z-\xi)R_n(z)+a^{-n}(a(\xi-1)+b)}P_n(a\xi+b)}{a^{-n}(az+b-\xi)P_n(az+b)} \]
Therefore
\[ F(az+b) = \lim_{n \to \infty} \frac{L_n(z)}{a(z-\xi)R_n(z)+a^{-n}(a(\xi-1)+b)}P_n(a\xi+b) \]
Let remark now
\[ (a\xi+b-\xi)P_n(\xi) = a^{n+1} \int_{\xi}^{az+b} L_n(t)dt = a^n(n+1) \int_{\xi}^{az+b} L_n(\frac{t-b}{a})dt \]
\[ = a^{n+1} \int_{\xi-b/a}^{a\xi+b/a} L_n(t)dt = -a^{n+1} \int_{\xi-b/a}^{a\xi+b/a} L_n(t)dt + \int_{a\xi+b/a}^{a\xi+b} L_n(t)dt \]
We arrive at the representation
\[ F(az + b) = \lim_{n \to \infty} \frac{l_n(z)}{R_n(z)} - a(z - \xi)R_n(z) - a\left(\frac{\xi - b}{a} - \xi\right)R_n\left(\frac{\xi - b}{a}\right) \]

Hence

\[ F(az + b) = \lim_{n \to \infty} \frac{l_n(z)}{R_n(z)} - a(z - \xi) - a\left(\frac{\xi - b}{a} - \xi\right)R_n\left(\frac{\xi - b}{a}\right) \]

We readily conclude that (32). Thus (33) is confirmed.

**Corollary 3.7. (Asymptotic formulas)**

\[ \lim_{n \to \infty} \frac{P_n(az + b)P_n(z)}{a^n L_n(z) R_n(z)} = \frac{a}{(az + b - \xi) \phi'(az + b)}. \quad (34) \]

Uniformly for compacts subsets of $\Omega$.

Hence uniformly for compacts subsets of $\overline{\Omega}$.

\[ \lim_{n \to \infty} \frac{a^n L_n(z)}{P_n(az + b)} = (az + b - \xi) \phi'(az + b). \quad (35) \]

Analogously to this equation we derive

\[ \lim_{n \to \infty} \frac{a^n R_n(z)}{P_n(az + b)} = (az + b - \xi). \quad (36) \]

Uniformly for compacts subsets of $\overline{\Omega}$.

**Proof.** In order to prove (34) applying the first assertion of theorem 3.5 to (9). With help of (24) and taking into account that

\[ \lim_{n \to \infty} \frac{L_n(az + b)}{P_n(az + b)} = (az + b - \xi) \frac{\varphi'(az + b)}{\varphi(az + b)}. \]

It follows that

\[ \frac{G(z)}{F(z)} = \lim_{n \to \infty} \frac{l_n(z) P_n(z)}{R_n(z) L_n(z)} = \lim_{n \to \infty} \frac{a^{-n} L_n(az + b)}{P_n(az + b) R_n(z)} \frac{P_n(az + b) P_n(z)}{L_n(z)}. \]

Thus

\[ \frac{G(z)}{(az + b - \xi) F(z)} \frac{\varphi'(az + b)}{\varphi'(az + b)} = \lim_{n \to \infty} \frac{a^{-n}}{R_n(z)} \frac{P_n(az + b) P_n(z)}{L_n(z)}. \]

Applying (33) we get

\[ \frac{a}{(az + b - \xi)} \frac{\varphi(z)}{\varphi'(z)} = \lim_{n \to \infty} \frac{a^{-n}}{R_n(z)} \frac{P_n(az + b) P_n(z)}{L_n(z)}. \]

Of course

\[ \lim_{n \to \infty} \frac{P_n(z)}{R_n(z)} \frac{G(z)}{F(z)} = a \frac{F(az + b)}{F(z)} = a \frac{\varphi(z)}{\varphi(az + b)} \frac{\varphi'(az + b)}{\varphi'(z)}. \]
Taking into account that
\[
\frac{(az + b - \xi) \varphi'(z)}{a} = \lim_{n \to \infty} \frac{a^n L_n(z) R_n(z)}{P_n(az + b) P_n(z)}
\]
Becomes to (35). Thus (36) is established.

4 Open Problem

Let \( G \) be a bounded simply-connected domain in the complex plane \( \mathbb{C} \), whose boundary \( E = \partial G \) is a Jordan curve. Let \( \sigma \) be a positive finite area measure defined on the Borel set \( B(\mathbb{C}) \cap G \) with compact support \( G \). The \( L^2 \) extremal polynomials \((L_{n,m})_{n=0,1,2,...}\) associated to the measure \( \sigma \) and the support \( G \) are defined as the solutions of the extremal problems in the space \( L^\infty(E,d\sigma) \). Let \( \mu_{n,m}(\sigma) \) \((n = 0,1,2,3,4,......)\) denotes the extremal constants associated with \( \sigma \) and \( G \).

\[
\mu_{n,m}(\sigma) = \min_{Q_n} \left\{ \|Q_n\|_{L^\infty(E,d\sigma)}, Q_n = z^n + q_1 z^{n-1} + q_2 z^{n-2} + \ldots q_n \right\}
\]

Where
\[
\|Q_n\|_{L^\infty(E,d\sigma)} = \left\{ \int_G \|L_n(z)\|^m d\sigma(z) \right\}^{\frac{1}{m}}.
\]

For a fixed complex number \( \xi \) we are going to define \( (P_{n,m}(z))_{n=0}^\infty \) as a monic polynomials such that
\[
(z - \xi)P_{n,m}(z) = (n + 1) \int_{\xi}^{z} L_{n,m}(t) dt.
\]

How it is possible to characterize the asymptotic formulas of \( (P_{n,m}(z))_{n=0}^\infty \) in terms of the asymptotic properties of the corresponding extremal polynomials.

ACKNOWLEDGEMENTS.
The authors want to thank the referee for kind suggestion to submit a paper to this volume and for frequent useful communications.

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