Even-order Magic Squares
with Special Properties

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Abstract

In this paper we introduce new concepts in the field of magic squares. We focus on special types of magic squares of order six. We list enumerations of the squares of this type. We prove that the nullity of some even-order magic squares is at least one.

Keywords: Magic Squares, nullspace.

1 Introduction

A magic square is a square matrix, where the sum of all entries in each row, column or both main diagonals yields the same number. This number is called the magic constant. A natural magic square of order $n$ is a matrix such that its entries consist of all integers from one to $n^2$. The magic constant in this case is $\frac{n(n^2+1)}{2}$. A pandiagonal magic square is a magic square such that the sum of all entries in all broken diagonals equals the magic constant. A symmetric magic square is a natural magic square of order $n$ such that the sum of all opposite entries equals $n^2 + 1$, i.e. the following relations hold

$$a_{ij} + a_{n+1-i,n+1-j} = n^2 + 1$$

for all $1 \leq i, j \leq n$

Example 1 The following square is a natural symmetric magic square
The sum of all opposite entries is 26.

We emphasize here that a symmetric magic square is not a symmetric matrix in the sense that its identical to its transpose. A natural magic square can not be a symmetric matrix since all entries are distinct.

The number of natural magic squares of order five is known. Schroeppel computed this number in 1971 (see [11]). It is well-known that there are pandiagonal magic squares and symmetric squares of order five (see [6]). The number of natural magic squares of order six is til now unknown. We give here the number of a subset of such squares. One possible form of a symmetric magic square with magic sum 3s is

\[
\begin{bmatrix}
  a & f & C & s-F & s-B & s-A \\
  b & g & k & s-j & s-p & s-E \\
  h & r & o & s-v & s-D & s-q \\
  q & D & v & s-o & s-r & s-h \\
  E & p & j & s-k & s-g & s-b \\
  A & B & F & s-C & s-f & s-a
\end{bmatrix}
\]

where

\[
A = j - 2b - g - h - a - k + p - q + 3s, \\
B = q - g - h - o - p - f - 2r + 3s + v, \\
C = \frac{9}{2}s - b - f - g - h - k - o - r - a, \\
D = h + o - q + r - v, \\
E = b + g - j + k - p, \\
F = a + b + f + g + h - j + r - \frac{3}{2}s - v.
\]

We obtain this form by solving the equations resulting from the definition of such squares. Due to the fraction in the assignment for C we see that noninteger numbers might appear. In particular, a natural symmetric magic square can not exist since the magic constant for natural magic squares is 111. The same problem occurs when we solve the equations resulting from the definition of the pandiagonal magic square. Hence, there are also no natural pandiagonal magic squares.

It is well-known that the following structure
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A B C 2s - A - B - C
E 2s - A - B - E A + E - C B + C - E
s - C A + B + C - s s - A s - B
s - A - E + C s - B - C + E s - E A + B + E - s

is the general structure of the pandiagonal magic square 4 × 4 (see [11]). Here, the magic constant is 2s. We note that the sum in each pair of antipodals is s, i.e.

\[ a_{ij} + a_{i+2,j+2} = s, \text{ for all } 1 \leq i, j \leq 2 \]
\[ a_{ij} + a_{i+2,j-2} = s, \text{ for all } 1 \leq i \leq 2, 3 \leq j \leq 4 \]

We define next classes of magic squares of order six, which possesses similar properties for the antipodals.

2 Quasi Pandiagonal Magic Squares

We can generalize idea of the structure of 4 × 4 pandiagonal magic squares in order to obtain new types of magic squares of even order. For example, the 6 × 6 square having the structure

\[
\begin{bmatrix}
  a & b & c & d & i & j \\
  e & f & g & h & k & l \\
  m & n & o & p & q & r \\
  s - d & s - i & s - j & s - a & s - b & s - c \\
  s - h & s - k & s - l & s - e & s - f & s - g \\
  s - p & s - q & s - r & s - m & s - n & s - o \\
\end{bmatrix}
\]

with the following restrictions

\[ a + b + c + d + i + j = 3s, e + f + g + h + k + l = 3s, m + n + o + p + q + r = 3s, \]
\[ a + e + m = d + h + p, b + f + n = i + k + q, c + g + o = j + l + r, \]

is a magic square with magic constant 3s. This structure is called a quasi pandiagonal magic square of order six. The previous system of linear equations has the following augmented coefficient matrix:
It has the following squared submatrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
-1 & -1 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

of determinant two. Hence, the general solution of these equations can be given as:

\[
j = 3s - a - b - c - d - e,
\]

\[
l = 3s - g - h - i - j - k,
\]

\[
n = c + d + g + h + i + k - m - r - \frac{3}{2}s,
\]

\[
o = 6s - a - b - 2c - d - e - f - 2g - h - i - k + r,
\]

\[
p = a - d + e - h + m,
\]

\[
q = b + c + d + f + g + h - m - r - \frac{3}{2}s.
\]

**Lemma 1** Let \( \Lambda \) be a quasi pandiagonal magic square of order six. Then, the nullity of \( \Lambda \) is at least one.

**Proof.** Note that the square \( \Lambda \) has the form

\[
\begin{bmatrix}
A & B \\
S - B & S - A
\end{bmatrix}
\]
where all entries of the matrix $S$ are $s$. We seek the vectors in the nullspace of this matrix having the form
\[
\begin{pmatrix} x \\ -x \end{pmatrix}
\]
for some $x \in \mathbb{R}^3$

The product between the square $\Lambda$ and this vector leads
\[
\begin{pmatrix} Ax - Bx \\ -Bx + Ax \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]
and, hence, we look for vectors $x \in \mathbb{R}^3$ satisfying the following equation:
\[
Ax - Bx = 0
\]
(1)

This equation has nonzero vectors as solution because the matrix $A - B$ does not have full rank. This is due to the following reason: replacing the last row with the summation of all rows yields the following row in the matrix:
\[
(a + e + m - d - h - p, b + f + n - i - k - q, c + g + o - j - l - r)
\]
According to our requirement this is a row of zeros. Hence, the rank of $A - B$ is two and, there exists a nontrivial solution of (1).

**Remark 1** In general the rank of this type of squares is five. The following square is an example:

\[
\begin{bmatrix}
1 & 2 & 0 & 4 & 1 & -2 \\
0 & 0 & 1 & 1 & 0 & 4 \\
1 & 3 & 1 & -3 & 4 & 0 \\
-2 & 1 & 4 & 1 & 0 & 2 \\
1 & 2 & -2 & 2 & 1 & 1 \\
5 & -2 & 2 & 1 & -1 & 1
\end{bmatrix}
\]

We can define certain types of $\Theta \times \Theta$ magic squares, where $\Theta$ is an even natural number greater than three. For example, let us consider the case $\Theta = 8$. The square has the following structure:

\[
\begin{bmatrix}
a & b & c & d & A & B & C & D \\
e & f & g & h & E & F & G & H \\
i & j & k & l & I & J & K & L \\
m & n & o & p & M & N & O & P \\
s - A & s - B & s - C & s - D & s - a & s - b & s - c & s - d \\
s - E & s - F & s - G & s - H & s - e & s - f & s - g & s - h \\
s - I & s - J & s - K & s - L & s - i & s - j & s - k & s - l \\
s - M & s - N & s - O & s - P & s - m & s - n & s - o & s - p
\end{bmatrix}
\]
where

\[ a + b + c + d + A + B + C + D = 4s, \]
\[ e + f + g + h + E + F + G + H = 4s, \]
\[ i + j + k + l + I + J + K + L = 4s, \]
\[ m + n + o + p + M + N + O + P = 4s, \]
\[ a + e + i + m = A + E + I + M, b + f + j + n = B + F + J + N, \]
\[ c + g + k + o = C + G + K + O, d + h + l + p = D + H + L + P. \]

The previous system of linear equations has the following squared submatrix in the augmented coefficient matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

of determinant different than zero. Hence, the system of linear equations possess a nontrivial solution. The generated square is called quasi magic square. We can generalize these results.

**Definition 1** Let \( \theta = \frac{\Theta}{2} \). The magic square having the following structure

\[
\begin{bmatrix}
a_{11} & \cdots & a_{1\theta} & a_{1(\theta+1)} & \cdots & a_{1\Theta} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
a_{\theta 1} & \cdots & a_{\theta\theta} & a_{\theta(\theta+1)} & \cdots & a_{\theta\Theta} \\
s - a_{1(\theta+1)} & \cdots & s - a_{1\Theta} & s - a_{11} & \cdots & s - a_{1\theta} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
s - a_{\theta(\theta+1)} & \cdots & s - a_{\theta\Theta} & s - a_{\theta 1} & \cdots & s - a_{\theta\theta}
\end{bmatrix}
\]
is called a quasi pandiagonal magic square of order $\Theta$.

The requirement that the matrix is a magic square yields to a solvable linear system. This is because we can following the above reasoning to find a nonsingular submatrix in the augmented coefficient matrix of the system. Hence, there exists quasi pandiagonal magic squares of order $\Theta$, where $\Theta$ is an even natural number greater than three.

**Proposition 2** Let $\Lambda$ be a quasi pandiagonal magic square of order $\Theta$. Then, the nullity of $\Lambda$ is at least one.

**Proof.** Recall that the square $\Lambda$ of order $\Theta$ can be rewritten as

$$
\begin{bmatrix}
A & B \\
S - B & S - A
\end{bmatrix}
$$

where $A$, $B$ and $S$ are square matrices of order $\theta$. Now, we seek the vectors in the nullspace of this matrix having the form

$$
\begin{pmatrix}
x \\
-x
\end{pmatrix}
$$

for some $x \in \mathbb{R}^\theta$

The product between the square $\Lambda$ and this vector leads

$$
\begin{pmatrix}
Ax - Bx \\
-Bx + Ax
\end{pmatrix} = \begin{pmatrix} 0 \\
\vdots \\
0 \end{pmatrix}
$$

The matrix $A - B$ does not have full rank. This is due to the following reason: replacing the last row with the summation of all rows yields the following row in the matrix:

$$(a_{11} + \ldots + a_{\theta 1} - a_{1(\theta + 1)} - \ldots - a_{\theta(\theta + 1)}; \ldots, a_{1\theta} + \ldots + a_{\theta \theta} - a_{1\theta} - \ldots - a_{\theta \theta})$$

Since the square is magic, we can assure that this row is a row of zeros. Hence, there exists a nontrivial solution of the equation

$$Ax - Bx = 0$$

Using this nontrivial solution we determine a nonzero vector in the nullspace of $\Lambda$. ■
3 Four-corner Magic Squares

Definition 2 A four-corner magic square of order 6 is a magic square \((a_{ij})_{i=1,...,6} \) \((j=1,...,6)\) with magic constant 3s such that

\[ a_{33} + a_{44} + a_{34} + a_{43} = 2s \] \[ a_{i,j} + a_{(i+3),(j+3)} + a_{i,(j+3)} + a_{(i+3),j} = 2s \text{ for all } i = 1, 2, 3 \text{ and } j = 1, 2, 3. \]

The entries of a four-corner magic square of order 6 satisfy

\[ a_{14} + a_{25} + a_{36} + a_{41} + a_{52} + a_{63} = 3s, \]
\[ a_{13} + a_{22} + a_{31} + a_{61} + a_{55} + a_{64} = 3s. \]

These two conditions represent the sum of the entries of two broken diagonals. If the magic square is pandiagonal, then we have to consider all broken diagonals. To see the validity of the first equation we know from the definition that

\[ a_{11} + a_{41} + a_{14} + a_{41} = 2s, a_{22} + a_{55} + a_{25} + a_{52} = 2s, a_{33} + a_{66} + a_{36} + a_{63} = 2s. \]

Adding up these equations and subtracting from the addition the following equation

\[ a_{11} + a_{22} + a_{33} + a_{44} + a_{55} + a_{66} = 3s \]

we obtain the desired equation.

A four-corner magic square of order 6 can be written as

\[
\begin{bmatrix}
  x & f & g & t & G & M \\
  z & h & n & j & q & Y \\
  w & H & e & a & m & J \\
  2s - b - t - x & k & 2s - b - a - e & b & D & R \\
  2s - o - j - z & p & d & o & 2s - p - q - h & T \\
  B & L & A & 3s - b - j - o - a - t & E & F
\end{bmatrix}
\]

where

\[ A = a + b - d - g - n + s, \]
\[ B = b + j + o - s + t - w, \]
\[ D = g - j - k - o - p - q + s + w + x + e, \]
\[ E = f + h + k - m + p - s, \]
\[ F = p - b + q + s - x - e, \]
\[ G = j - g - f + o + p + q + s - w - x - e, \]
\[ H = b + j + o - s + t - w, \]
\[ J = d - a + g + n - p - q + x, \]
\[ L = b + j + o - s + t - w, \]
\[ M = 2s - o - p - q - j - t + w + e, \]
\[ R = a + b - g + j + o + p + q - 2s + t - w, \]
\[ T = h - d + j + q - s + z, \]
\[ Y = 3s - j - n - q - h - z. \]

We are interested in a subclass of the four-corner magic squares.

**Definition 3** A four-corner magic squares with semi-symmetric center is a four-corner magic square of order 6 such that
\[ a_{33} + a_{43} = s \text{ and } a_{34} + a_{44} = s. \]

A four-corner magic squares with semi-symmetric center of order 6 can be written as

\[
\begin{bmatrix}
x & f & g & t & G & M \\
z & h & n & j & q & Y \\
w & H & e & a & m & J \\
s + a - t - x & k & s - e & s - a & D & R \\
2s - o - j - z & p & d & o & 2s - p - q - h & T \\
B & L & 2s - g - n - d & 2s - o - j - t & E & F
\end{bmatrix}
\]

where the capital letters are dependent variables. We can see that the square has seventeen independent variables. This can be checked with computers by solving the system of equations included in the definition of four-corner magic squares with semi-symmetric center.

### 3.1 Property preserving transformations

There are seven classical transformations, which take a magic square into another magic square. They are the composition of rotations with angles \( \frac{\pi}{2}, \pi, \frac{3\pi}{2} \) and reflection about the main diagonals. Now, a four-corner magic squares with semi-symmetric center can be transformed into other of the same class. To illustrate this we transform the square in (3) into
In order to eliminate the effect of the previous transformations we compute all natural four-corner magic squares with semi-symmetric center for which the following conditions hold:

\[ p < q, \ a < e < s - a, \ 1 \leq a \leq 17 \ldots \ldots (4) \]

The condition \( p < q \) eliminates the effect of the last mentioned transformation since it does not change the center. Hence, the total number of all natural four-corner magic squares with semi-symmetric center is the computed number multiplied with sixteen.

The four-corner magic squares with semi-symmetric center possesses another property preserving transformations besides the previous one. The square in (3) can be transformed into

\[
\begin{bmatrix}
  x & f & g & t & G & M \\
  z & 2s - p - q - h & d & o & p & Y \\
  w & m & e & a & H & J \\
  s + a - t - x & D & s - e & s - a & k & R \\
  2s - o - j - z & q & n & j & h & T \\
  B & L & 2s - g - n - d & 2s - o - j - t & E & F 
\end{bmatrix}
\]

We see that the \( 2 \times 2 \) center also remains unchanged. Besides, any square satisfying the conditions (4) will be transformed into another square satisfying (4). Hence, the number of natural four-corner magic squares with semi-symmetric center satisfying (4) will be even. Unfortunately, we can not eliminate the effect of the last transformation by requiring a comparison relation.

### 3.2 Number of squares

We used pentium IV computers core 2 duo CPU (3 GHz) to count the four-corner magic squares with semi-symmetric center. It took about three months to finish. The C code is presented in the appendix. It is based on the choice
of a specific center in each run. The values of the other independent variables will be assigned using some nested loops. The value of the dependent variables will be sequentially computed. At each stage we test the computed value for being in the range from 1 to 36 and for being different from other existing values.

The number of centers is 306. We list the number for all different values of $a$ tabulated according to $e$ in the following tables:

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The total number of squares is
Hence, the total number of natural squares is

\[ 101425060998 \times 16 = 1,622,800,975,968 \]

4 Open Problem

The number of four-corner magic squares of order 6. We want here to estimate the number of four-corner magic squares of order 6. By computing the number of squares with semi-symmetric center we noticed that the number of different possible values for \( a \) and \( e \) is

\[ 2 + 4 + 8 + \ldots + 34 = 306 \]

The average of the squares per one possible pattern for \( a \) and \( e \) is

\[ \frac{101425060998 \times 16}{306} = 5.3033 \times 10^9 \]

The number of all different possible values for \( a \), \( b \) and \( e \) by computing the number of four-corner magic squares is 3429. Hence we estimate the number of the four-corner magic squares

\[ 5.3033 \times 10^9 \times 3429 = 1.8185 \times 10^{13} \]

We noticed that the number for some possible values for \( a \), \( b \) and \( e \), which do not lead to a symmetric center is less that the largest observed number \( (398369256) \). This motivates the following

**Conjecture 3** The number of four-corner magic squares of order 6 does not exceed

\[ 398369256 \times 16 \times 3429 = 21,856,130,861,184. \]

Acknowledgement

Thanks are due to the Deanship of the Academic Research at Al-Albayt University for funding this work (Grant number 5016). We thank also K. Al Share’o for his contributions to the research, which lead to this work. Finally, Harry White contributed to preparing the source code.
References


Appendix The C-code

```
#include <assert.h>
#include <errno.h>
#include <io.h>
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <time.h>

const int N = 6; const int NN = N*N; const int Sum2 = NN - 1;
```
const int Sum4 = Sum2 + Sum2; const int Msum = Sum4 + Sum4;
struct bools {bool used[NN]};
struct bools allFree;
#define Uint unsigned int

void writeSquare(int *p, FILE *wfp) {
    char squareString[120], *s = squareString; int cells = 0;
    {int i; for (i = 0; i < NN; ++i) {int x = p[i] + 1;
        if (x < 10) { *s++ = ' '; *s++ = '0' + x; }
        else if (x < 20) { *s++ = '1'; *s++ = '0' - 10 + x; }
        else if (x < 30) { *s++ = '2'; *s++ = '0' - 20 + x; }
        else { *s++ = '3'; *s++ = '0' - 30 + x; }
    } if (++cells == N) { *s++ = '\n'; cells = 0; }
    if (1 + cells >= N) { *s++ = '\n'; cells = 0; }
    *s++ = '\n'; *s++ = '\0'; fputs(squareString, wfp);}

Uint makeSquares(int a, int b, int e, int J, FILE *wfp) {
    Uint count = 0, pcount = 0; bools v = allFree;
    int Z[NN]; Z[20] = J;
    v.used[e] = true; v.used[a] = true; v.used[b] = true; v.used[J] = true;
    {int t; for (t = 1; t < 2; ++t) if (!v.used[t]) {v.used[t] = true;
        {int x; for (x = 21; x < 22; ++x) if (!v.used[x]) {Z[18] = Sum4-b-t-x;
            if ((Z[18] < 0) || (Z[18] >= NN) || v.used[Z[18]] || (Z[18] == x)) continue;
            v.used[x] = true; v.used[Z[18]] = true;
        } if (++t == N) if (1 + t >= N) { *s++ = '\n'; t = 0; }
    } if (++t == N) if (1 + t >= N) { *s++ = '\n'; t = 0; }
    *s++ = '\n'; *s++ = '\0'; fputs(squareString, wfp);}
}

Even-order Magic Squares with Special Properties

```c
{int n; for (n = 0; n < NN; ++n) if (!v.used[n]) {Z[11]=Msum-j-z-n-q-h;
continue; v.used[n] = true; v.used[Z[11]] = true;
}
{int d; for (d = 0; d < NN; ++d) if (!v.used[d]) {
Z[29]=Msum-Z[24]-p-d-o-Z[28];
if ((Z[29] < 0) || (Z[29] >= NN) || v.used[Z[29]] || (Z[29] == d))
continue; v.used[d] = true; v.used[Z[29]] = true;
}
{int g; for (g = 0; g < NN; ++g) if (!v.used[g]) {
Z[32]=a+b-d-g-n+Sum2;
if ((Z[32] < 0) || (Z[32] >= NN) || v.used[Z[32]] || (Z[32] == g))
continue; Z[17]=d-a+g+n-p-q+x;
if ((Z[17] < 0) || (Z[17] >= NN) || v.used[Z[17]] ||
(Z[17] == g) || (Z[17] == Z[32])) continue;
Z[23]=a+b-g+j+o+p+q+t-w-Sum4;
if ((Z[23] < 0) || (Z[23] >= NN) || v.used[Z[23]] ||
(Z[23] == g) || (Z[23] == Z[32]) || (Z[23] == Z[17])) continue;
}
{int f; for (f = 34; f < 35; ++f) if (!v.used[f]) {
continue; v.used[f] = true; v.used[Z[4]] = true;
}
{int m; for (m = 0; m < NN; ++m) if (!v.used[m]) {
continue; v.used[m] = true; v.used[Z[13]] = true;
}
{int k; for (k = 0; k < NN; ++k) if (!v.used[k]) {
Z[22]=Msum-k-b-Z[18]-Z[23]-Z[20];
if ((Z[22] >= 0) && (Z[22] < NN) && !v.used[Z[22]]) {
continue; Z[34]=Msum-m-q-Z[4]-Z[22]-Z[28];
}
{int l; for (l = 0; l < NN; ++l) if (!v.used[l]) {
Z[34]=true; Z[31]=Msum-f-h-k-p-Z[13];
if ((Z[31] >= 0) && (Z[31] < NN) && !v.used[Z[31]]) {
++count; writeSquare(Z, wfp);
if (++pcount == 1000000) {
printf("count %lu\n", count);pcount = 0;
fflush(wfp);} }
}
}
```

```
v.used[Z[34]] = false; v.used[Z[22]] = false; v.used[k] = false;
v.used[Z[4]] = false;
v.used[Z[29]] = false;
v.used[Z[17]] = false; v.used[Z[23]] = false;
v.used[Z[32]] = false; v.used[d] = false; v.used[Z[29]] = false;
```

```c
v.used[n] = false; v.used[Z[11]] = false;}} v.used[h] = false;
v.used[Z[28]] = false;}}
v.used[q] = false; v.used[Z[5]] = false; v.used[Z[35]] = false;}}
v.used[p] = false;}} v.used[w] = false; v.used[Z[30]] = false;}}
v.used[z] = false; v.used[Z[24]] = false;}}
v.used[o] = false; v.used[j] = false;}}
printf("number of squares %d\n", count);return count;}
void get_rest_of_line(int c) {
if (c != '\n') do { c = getchar(); } while (c != '\n');}
void get_abe(int *a, int *b, int *e) {
int unused = scanf("%d %d %d", a, b, e);
int c = getchar(); get_rest_of_line(c);}
void getNumPatterns(int *num) {int unused = scanf("%d", num);
int c = getchar(); get_rest_of_line(c);}
bool check_abe(int a, int b, int e) {bool rv = true;if ((a < = 0) || (a > NN) || (b < = 0) || (b > NN) || (e < = 0) || (e > NN)) {
printf("\aValue range is 1 to %d.\n\n", NN);rv = false;}return rv;}
bool checkNum(int num) {
bool rv = true;if (num < = 0) {printf("\aNumber must be a positive integer\n\n\n");rv = false;}return rv;}
const int bufSize = 128;
void openOutput(int a, int b, int e, char *wfpName, FILE **wfp) {
const int defSize = 15;
char buf[bufSize], buf1[bufSize], defaultName[defSize];
if (a == 0) strcpy(defaultName, "s6_counts");
else sprintf(defaultName, "s6_a%ib%ie%i", a, b, e);
strcpy(buf, defaultName); strcat(buf, ".txt"); {int sub = 0;
do {if ((fopen(buf, "r") == NULL) && (errno == ENOENT)) {break;}
else {
strcpy(buf, defaultName); sprintf(buf1, ".%i", ++sub); strcat(buf, buf1);
strcat(buf, ".txt");} while (true); if ( (*wfp = fopen(buf, "w")) != NULL) {
printf("\n%s file is %s\n", a == 0 ? "Data" : "Squares", buf);
strcpy(wfpName, buf); } else else {strcpy(buf1, ".\a\nCan't open for write ");
strcat(buf1, buf); perror(buf1);}}
int getSquares(int a0, int b0, int e0, int num, FILE *wfpc) {
char wfpsName[bufSize]; FILE *wfps = NULL;
int linecount = 0; Uint count = 0; int patterns = 0;
{int a; for (a = – a0; a < NN; ++a) { {int b; for (b = – b0; b < NN; ++b)
if (a != b) {
{int e; for (e = –e0; e < NN; ++e) if ((a < e) && (e < b)) {
```
int J = Sum4-e-a-b; if ((J <= a) || (J == b) || (J == e) || (J >= NN)) continue;

openOutput(a+1, b+1, e+1, wfpsName, &wfps);
if (wfps != NULL) { time_t startTime; startTime = time(NULL);
count = makeSquares(a, b, e, J, wfps);
fprintf(wfpc, "a: %2d b: %2d e: %2d number of squares: %10lu time: ",
a+1, b+1, e+1, count);
{ int elapsed_t = (int)(time(NULL) - startTime);
int hr = elapsed_t/3600; elapsed_t %= 3600;
{ int min = elapsed_t/60, sec = elapsed_t%60;
{ char *fmt = "%3d:%02d:%02d\n";
printf(fmt, hr, min, sec); fprintf(wfpc, fmt, hr, min, sec);
if (++linecount == 5) { fprintf(wfpc, '\n'); linecount = 0; }
} } } } } return patterns;

int main() { int a, b, e; printf(\nInput start values for a b e: \n);
get_abe(&a, &b, &e); if (check_abe(a, b, e)) { int num = -1;
printf(\nInput number of a b e patterns to run: \n);
getNumPatterns(&num);
if (checkNum(num)) { char wfpcName[bufSize]; FILE *wfpc = NULL;
openOutput(0, 0, 0, wfpcName, &wfpc);
if (wfpc != NULL) { time_t startTime = time(NULL);
int patterns = getSquares(a, b, e, num, wfpc);
int elapsed_t = (int)(time(NULL) - startTime);
int hr = elapsed_t/3600; elapsed_t %= 3600;
{ int min = elapsed_t/60, sec = elapsed_t%60;
{ char *fmt = "%na, b, e patterns: %d elapsed time: %d:%02d:%02d\n";
printf(fmt, patterns, hr, min, sec);
fprintf(wfpc, fmt, patterns, hr, min, sec);
fclose(wfpc); } } } } { int unused = getchar(); } return 0;