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Some Open Problems On a Class of Finite Groups

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Abstract

In this note we introduce and study a class of finite groups for which the exponents of subgroups satisfy a certain inequality. It is closely connected to some well-known arithmetic classes of natural numbers.

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1 Introduction

Let n be a natural number and $\sigma(n)$ be the sum of all divisors of n. We say that n is a *deficient number* if $\sigma(n) < 2n$ and a *perfect number* if $\sigma(n) = 2n$ (for more details on these numbers, see [3]). Thus, the set consisting of both the deficient numbers and the perfect numbers can be characterized by the inequality

$$\sum_{d \in L_n} d \le 2n\,,$$

where $L_n = \{d \in \mathbb{N} \mid d|n\}$.

Now, let G be a finite group. Then the set $L(G)$ of all subgroups of G forms a complete lattice with respect to set inclusion, called the *subgroup lattice* of G. Two remarkable subsets of $L(G)$ are constituted by all cyclic subgroups and by all normal subgroups of G, respectively. They are called the *poset of cyclic subgroups* and the *normal subgroup lattice* of G and will be denoted by $C(G)$ and $N(G)$. Notice that $N(G)$ is a sublattice of $L(G)$, while $C(G)$ is in general only a subposet of $L(G)$. If the group G is cyclic of order n, then

$$L(G) = C(G) = N(G)$$

and these are isomorphic to the lattice L_n. So, the fact that n is deficient or perfect can be written in the following equivalent three ways:

$$\sum_{H \in L(G)} |H| \leq 2|G|\,, \tag{1}$$

$$\sum_{H \in C(G)} |H| \leq 2|G|\,, \tag{2}$$

$$\sum_{H \in N(G)} |H| \leq 2|G|\,. \tag{3}$$

In [1] and [8] we have studied the classes $\mathcal{C}_i$, $i = 1, 2, 3$, consisting of all finite groups G which satisfy the above inequalities, respectively. Recall only that the unique groups contained in $\mathcal{C}_1$ are the cyclic groups of deficient or perfect orders and that $\mathcal{C}_1$ is properly included in $\mathcal{C}_2$.

The starting point for our discussion is given by the remark that (1)-(3) also equivalent for finite cyclic groups to the inequality

$$\sum_{H \in L(G)} \exp(H) \leq 2 \exp(G)\,, \tag{4}$$

where $\exp(H)$ denotes the *exponent* of the subgroup H of G, that is the lower common multiple of the orders of elements of G. Hence, in a similar manner, one can introduce the class $\mathcal{C}$ consisting of all finite groups G which satisfy (4). Its investigation is the main goal of our paper.

The paper is organized as follows. In Section 2 we study some basic properties of $\mathcal{C}$ and the connections with $\mathcal{C}_i$, $i = 1, 2, 3$. The containment of some important classes of finite groups, as p-groups, nilpotent groups, dihedral groups, ZM-groups and symmetric groups to $\mathcal{C}$ is also discussed. In the final section we propose several problems with respect to this subject.

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [2] and [5]. For subgroup lattice concepts we refer the reader to [4] and [6].

2 Main results

For a finite group G let us denote

$$\sigma(G) = \frac{1}{\exp(G)} \sum_{H \in L(G)} \exp(H)\,.$$

In this way, $\mathcal{C}$ is the class of all finite groups G for which $\sigma(G) \leq 2$. First of all, we remark that σ is a multiplicative function, that is if G and G' are two finite groups satisfying $\gcd(|G|, |G'|) = 1$, then

$$\sigma(G \times G') = \sigma(G)\sigma(G')\,.$$

By a standard induction argument, it follows that if G_i, $i = 1, 2, \ldots, k$, are finite groups of coprime orders, then

$$\sigma(\bigtimes_{i=1}^{k} G_i) = \prod_{i=1}^{k} \sigma(G_i)\,.$$

Obviously, $\mathcal{C}$ contains the finite cyclic groups of prime order. On the other hand, we easily obtain

$$\sigma(\mathbb{Z}_p \times \mathbb{Z}_p) = \frac{1 + 2p + p^2}{p} > 2,\ \text{for any prime}\ p\,.$$

This relation shows that $\mathcal{C}$ is not closed under direct products or extensions.

Observe next that for every finite group G we have

$$\sigma(G) \geq \frac{1}{\exp(G)} \sum_{H \in C(G)} \exp(H) \geq \frac{1}{|G|} \sum_{H \in C(G)} |H|$$

and thus if G does not belong to $\mathcal{C}_2$, then $\sigma(G) > 2$. Consequently, the following result holds.

Theorem 1. *The class $\mathcal{C}$ is contained in the class $\mathcal{C}_2$.*

In the following we will focus on characterizing some particular classes of groups in $\mathcal{C}$. The simplest case is constituted by finite p-groups.

Lemma 2. *A finite p-group is contained in $\mathcal{C}$ if and only if it is cyclic.*

Proof. Let G be a finite p-group of order p^n which is contained in $\mathcal{C}$, set $p^m = \exp(G)$ and suppose that G is not cyclic. Then $m < n$. It is well-known that G possesses at least an element a of order p^m. One obtains that $\exp(\langle a \rangle) = p^m$ and therefore

$$\sigma(G) \geq \frac{\exp(1) + \exp(\langle a \rangle) + \exp(G)}{p^m} = \frac{1 + 2p^m}{p^m} > 2.$$

This shows that G does not belong to $\mathcal{C}$, a contradiction.

Conversely, for a finite cyclic p-group G of order p^n, we obviously have

$$\sigma(G) = \frac{1 + p + \ldots + p^n}{p^n} = \frac{p^{n+1} - 1}{p^{n+1} - p^n} \leq 2.$$

$\square$

The above lemma leads to a precise characterization of finite nilpotent groups contained in $\mathcal{C}$. It shows that the finite cyclic groups of deficient or perfect order are in fact the unique such groups.

Theorem 3. *Let G be a finite nilpotent group. Then G is contained in $\mathcal{C}$ if and only if it is cyclic and its order is a deficient or perfect number.*

Proof. Assume that G belongs to $\mathcal{C}$ and let $\bigtimes_{i=1}^{k} G_i$ be its decomposition as a direct product of Sylow subgroups. Since G_i, $i = 1, 2, ..., k$, are of coprime orders, one obtains

$$\sigma(G) = \prod_{i=1}^{k} \sigma(G_i) \leq 2.$$

This inequality implies that $\sigma(G_i) \leq 2$, for all $i = \overline{1,k}$. So, each G_i is contained in $\mathcal{C}$ and it must be cyclic, by Lemma 2. Therefore G itself is cyclic and $|\, G \,|$ is a deficient or perfect number.

The converse is obvious. □

Theorem 3 shows that in order to find examples of noncyclic groups contained in $\mathcal{C}$, we must look to some classes of nonnilpotent groups. Three such classes are investigated in the following theorem.

Theorem 4. *The following finite groups are not contained in $\mathcal{C}$:*

1. *dihedral groups;*
2. ZM-*groups;*
3. *symmetric groups.*

Proof. 1. The exponent of the dihedral group

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yxy = x^{-1} \rangle,\ n \geq 2,$$

can be easily computed:

$$\exp(D_{2n}) = \begin{cases} 2n, & n \equiv 1 \pmod 2 \\ \\ n, & n \equiv 0 \pmod 2 . \end{cases}$$

For every divisor d of n, D_{2n} has a unique cyclic subgroup of order d and $\frac{n}{d}$ subgroups isomorphic to D_{2d}. Denote by $\tau(n)$ the number of divisors of n. We infer that for n odd we have

$$\sigma(D_{2n}) = \frac{1}{2n}\left(\sum_{d|n} d + \sum_{d|n} \frac{n}{d}\, 2d\right) = \frac{1}{2n}\left(\sigma(n) + 2n\tau(n)\right) > \tau(n) \geq 2,$$

while for n even we have

$$\sigma(D_{2n}) \geq \frac{1}{n}\left(\sum_{d|n} d + \sum_{d|n} \frac{n}{d}\, d\right) = \frac{1}{n}\left(\sigma(n) + n\tau(n)\right) > \tau(n) \geq 2.$$

Hence D_{2n} does not belong to $\mathcal{C}$.

2. Recall that a ZM-group is a finite nonabelian group with all Sylow subgroups cyclic. By [2], such a group is of type

$$\mathrm{ZM}(m,n,r) = \langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r \rangle,$$

where the triple (m,n,r) satisfies the conditions

$$\gcd(m,n) = \gcd(m, r-1) = 1 \quad \text{and} \quad r^n \equiv 1 \pmod{m}.$$

Obviously, we have

$$\exp(\mathrm{ZM}(m,n,r) = mn.$$

Set

$$L = \left\{(m_1, n_1, s) \in \mathbb{N}^3 \ \mid\ m_1 \mid m,\ n_1 \mid n,\ s < m_1,\ m_1 \mid s\frac{r^n - 1}{r^{n_1} - 1}\right\}.$$

Then it is well-known that there is a bijection between L and the subgroup lattice of $\mathrm{ZM}(m,n,r)$, namely the function that maps a triple $(m_1, n_1, s) \in L$ into the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)} = \bigcup_{k=1}^{\frac{n}{n_1}} \alpha(n_1, s)^k \langle a^{m_1} \rangle = \langle a^{m_1}, \alpha(n_1, s) \rangle,$$

where $\alpha(x,y) = b^x a^y$, for all $0 \leq x < n$ and $0 \leq y < m$. Since $\exp(H_{(m_1,n_1,s)}) = \frac{mn}{m_1 n_1}$, we infer that

$$\sigma(\mathrm{ZM}(m,n,r)) = \frac{1}{mn} \sum_{m_1 |\, m} \sum_{n_1 |\, n} \frac{mn}{m_1 n_1} \gcd\left(m_1, \frac{r^n - 1}{r^{n_1} - 1}\right) \geq$$

$$\geq \frac{1}{mn}\left(x + y\right),$$

where

$$x = \sum_{n_1 |\, n} \frac{mn}{n_1} \gcd\left(1, \frac{r^n - 1}{r^{n_1} - 1}\right) = m\,\sigma(n)$$

and

$$y = \sum_{n_1 |\, n} \frac{n}{n_1} \gcd\left(m, \frac{r^n - 1}{r^{n_1} - 1}\right) \geq mn.$$

The above inequalities lead to

$$\sigma(\mathrm{ZM}(m,n,r)) \geq \frac{1}{mn}\,(m\,\sigma(n)+mn) > \frac{1}{mn}\,(mn+mn)=2,$$

which proves that $\mathrm{ZM}(m,n,r)$ is not contained in $\mathcal{C}$.

3. The exponent of the symmetric group S_n is given by

$$\exp(S_n) = \mathrm{lcm}(1,2,\ldots,n)\,.$$

Clearly, we have

$$\exp(S_n) = \mathrm{lcm}(\exp(S_{n-1}),n) \leq n\exp(S_{n-1})\,.$$

On the other hand, S_n has at least n subgroups isomorphic to S_{n-1}, namely $H_i = \{\sigma \in S_n \mid \sigma(i)=i\}$, $i=1,2,...,n$. One obtains

$$\begin{aligned}\sigma(S_n) &\geq \frac{1}{\exp(S_n)}\left(1+\sum_{i=1}^{n}\exp(H_i)+\exp(S_n)\right) = \\ &= \frac{1}{\exp(S_n)}\,(1+n\exp(S_{n-1})+\exp(S_n)) \geq \\ &\geq \frac{1}{\exp(S_n)}\,(1+2\exp(S_n)) > 2,\end{aligned}$$

implying that S_n does not belong to $\mathcal{C}$.

□

The containment to $\mathcal{C}$ can be also studied for other important classes of finite groups. Unfortunately, for the moment we are not able to decide whether $\mathcal{C}$ consists only of cyclic groups of deficient or perfect orders.

3 Open problems

Problem 1. Determine all finite groups contained in the class $\mathcal{C}$. Is it true that $\mathcal{C}=\mathcal{C}_1$?

Problem 2. Study if $\mathcal{C}$ is closed under subgroups and homomorphic images, i.e. if all subgroups and all quotients of a group in $\mathcal{C}$ also belong to $\mathcal{C}$.

Problem 3. Investigate the class $\mathcal{C}'$ consisting of all finite groups G for which

$$\sigma'(G) = \frac{1}{\exp(G)}\sum_{H\in N(G)}\exp(H) \leq 2\,.$$

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