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# A note on the (FMC) condition for extensions of commutative rings

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**Abstract**

*A ring extension $R \subset S$ is said to satisfy the (FMC) condition if there is a finite maximal chain of rings from $R$ to $S$ (cf. [1]). The aim of this note is to prove that if $R \subset S$ is an extension of integral domains satisfying the (FMC) condition, then $S$ is a $P$-extension of $R$. As a consequence, we establish that the polynomial extension $R[X] \subset S[X]$ does never satisfy the (FMC) condition.*

**Keywords:** *(Descending chain condition) d.c.c, Integral closure, Integral extension, Minimal ring extension, Overring, Prime ideal, Quasi-Prüfer domain, Finite maximal chain, $P$-extension.*

## 1 Introduction

We adopt the conventions that each ring considered is commutative, with unit and an inclusion (extension) of rings signifies that the smaller ring is a subring of the larger and possesses the same multiplicative identity. Let $R \subseteq S$ be a ring extension. The set of subrings of $S$ that contain $R$ is called *the set of intermediate rings in the ring extension* $R \subseteq S$. We let $[R, S]$ denote this set. For an integral domain $R$, we let $qf(R)$ denote its quotient field. An intermediate ring in the extension $R \subseteq qf(R)$ is called an *overring* of $R$.

Considerable attention has been paid over the few years to ring extensions $R \subset S$ with the following two finiteness conditions on the set of intermediate rings:

**(1)** The ring extension $R \subset S$ is said to satisfy the (FO) (or (FI)) condition if this extension has only finitely many intermediate rings.

**(2)** The ring extension $R \subset S$ is said to satisfy the (FC) (or (FCC)) condition if each chain of distinct intermediate rings in this extension is finite.

It is clear that (FO) implies (FC) and that (FC) is equivalent to the validity of both a.c.c and d.c.c in $[R, S]$.

The above two conditions have been recently introduced by Gilmer in [12] for the set of overrings of an integral domain. An integral domain is said to be (FO) (or (FC)) if the corresponding condition is satisfied for the extension $R \subset qf(R)$. Several characterizations of extensions $R \subset qf(R)$ satisfying these conditions have been established by Gilmer in [12]. Several related results can be found in [1], [3], [4], [9], [10], [14] and [15]. Several authors investigated the realization of these two conditions in the more general setting of ring extensions, where the upper ring $S$ is not necessarily the quotient field of the ring $R$ (see [1], [9], [10] and [15]).

In [1], A. Ayache introduced the following condition:

**(3)** The ring extension $R \subset S$ is said to satisfy the (FMC) condition, if there is a finite maximal chain of distinct intermediate rings from $R$ to $S$.

Clearly, a maximal chain is formed by successive minimal extension, that is by extensions with no proper intermediate rings.

This is the main finiteness condition referred in the title of this note. Notice that the author and E. Massaoud, have independently studied the (FMC) condition for the ring extension $R \subset qf(R)$ (see [17]). Ring extensions $R \subset S$ satisfying the (FMC) condition are identified in case $R$ is integrally closed in $S$ or $R \subset S$ is an integral extension (see ([1], Theorem 9 and Theorem 12) and ([17], Theorem 1)). Moreover ([8], Proposition 1.2) states that if $k \subset L$ is a field extension, then $k \subset L$ satisfies (FC) if and only if $k \subset L$ satisfies (FMC) if and only if $[L : k] < \infty$, where $[L : k]$ is the $k$-vector space dimension of $L$. However the general case is not yet answered.

In his study of some finiteness chain conditions on the set of intermediate rings in [1], A. Ayache encountered the following condition on the set of intermediate rings; we label this as (FMC)$^*$ condition:

**(4)** The ring extension $R \subset S$ is said to satisfy the (FMC)$^*$ condition, if there is a finite maximal chain $R = R_0 \subset R_1 \subset ... \subset R_n = S$ such that one of the $R_i$ is $R^*$ the integral closure of $R$ in $S$.

It follows from ([1], Theorem 24) that $R \subset S$ satisfies (FC) if and only if $R \subset S$ satisfies the (FMC)$^*$ condition. It is clear that if $R \subset S$ satisfies the (FMC)$^*$ condition, then it satisfies the (FMC) condition, but the converse is not true as it was noted in ([1], Example 25). As mentioned above, the (FMC) condition was also introduced in [17] and integrally closed domains $R$ for which $R \subset qf(R)$ satisfies the (FMC) condition are characterized (see [17], Theorem 1). Proposition 7 of [1] constitutes a generalization of this result. Lemma 1 of [17] states that if $R \subset T$ is a minimal extension such that $T$ is a quasi-Prüfer domain, then $R$ is a quasi-Prüfer domain, and Proposition 1 of [17] states that if $R \subset qf(R)$ satisfies the (FMC) condition, then $R$ is a quasi-Prüfer domain. These two results are in fact the key tools for proving ([17], Theorem 1). In this note we will generalize and improve these two results. But first recall that if $S$ is a unitary ring extension of $R$, we say that $S$ is a *$P$-extension* of $R$ (see [13]) if each element of $S$ satisfies a polynomial in $R[X]$ one of whose coefficients is a unit of $R$, or, equivalently, whose coefficients generate the unit ideal of $R$. Recall also that a ring extension $R \subset T$ is called a *residually algebraic extension* (see ([2], Definition 1.1)), if for each prime ideal $Q$ of $T$ , $T/Q$ is algebraic over $R/(Q \cap R)$. We say that $(R, S)$ is a *residually algebraic pair* (cf. ([2], Definition 2.1)), if for each $T \in [R, S]$, $R \subseteq T$ is a residually algebraic extension. It was proved that $S$ is a $P$-extension of $R$ if and only if $(R, S)$ is a residually algebraic pair (one can quote ([7], Theorem) or ([16], Lemma 2.8)). In particular $qf(R)$ is a $P$-extension of $R$ if and only if $(R, qf(R))$ is a residually algebraic pair if and only if $R$ is a quasi-Prüfer domain. We establish in Theorem 1 that if $R \subset T$ is a minimal extension of integral domains and $S$ is a $P$-extension of $T$, then $S$ is a $P$-extension of $R$. As a consequence, we prove that if $R \subset S$ is an extension of integral domains satisfying the (FMC) condition, then $S$ is a $P$-extension of $R$ (see Corollary 2). We close this note with Theorem 2 which states that for any extension of integral domains $R \subset S$, the polynomial extension $R[X] \subset S[X]$ does never satisfy the (FMC) condition.

Any unexplained terminology is standard, as in [11].

## 2 (FMC) Condition. Main Results

In what follows, we collect some facts on ring extensions satisfying the (FMC) condition. We start with the following straightforward result. We include a proof for the sake of completeness.

**Proposition 1.** *Let $R \subset S$ be a ring extension. If d.c.c holds in $[R, S]$, then there exists a ring $T \in [R, S]$ such that $R \subset T$ is a minimal extension.*

**Proof.** If $R \subset S$ is a minimal extension, then one can take $T = S$. Otherwise, there exists a ring $R_1$ such that $R \subset R_1 \subset S$. If $R \subset R_1$ is a minimal extension, we are done. If not, there is a ring $R_2$ such that $R \subset R_2 \subset R_1$. As d.c.c holds in $[R, S]$, this procedure stabilizes with a ring $T = R_n \in [R, S]$ such that $R \subset T$ is a minimal extension. $\diamondsuit$

It follows from ([1], Theorem 24) that an extension of integral domains $R \subset S$ satisfies (FC) if and only if $R \subset S$ satisfies (FMC)$^*$. So one can conclude that (FC) implies (FMC). The next corollary provides a simple proof for this fact even for non integral domains.

**Corollary 1.** *Let $R \subset S$ be a ring extension. If $R \subset S$ satisfies the (FC) condition, then $R \subset S$ satisfies the (FMC) condition.*

**Proof.** It follows from Proposition 1 that there exists a ring $T_1$ such that $R \subset T_1 \subseteq S$ and $R \subset T_1$ is a minimal extension. If $T_1 = S$, we are done, otherwise, as d.c.c holds in $[T_1, S]$ (because $T_1 \subset S$ satisfies (FC)), then it follows form Proposition 1 that there exists a ring $T_2$ such that $T_1 \subset T_2 \subseteq S$ and $T_1 \subset T_2$ is a minimal extension. If $T_2 = S$ we are done, otherwise, there exists a ring $T_3$ such that $R \subset T_1 \subset T_2 \subset T_3 \subseteq S$, and $T_2 \subset T_3$ is a minimal extension. Since a.c.c holds in $[R, S]$, this procedure stabilizes and we get a finite maximal chain $R = T_0 \subset T_1 \subset ... \subset T_n = S$. $\diamondsuit$

**Proposition 2.** *Let $R \subset S$ be a ring extension satisfying the (FMC) condition, then for each multiplicative subset $N$ of $R$, $N^{-1}R \subset N^{-1}S$ satisfies the (FMC) condition.*

**Proof.** Since $R \subset S$ satisfies the (FMC) condition, then there exists a finite maximal chain: $R_0 = R \subset R_1 \subset ... \subset R_{n-1} \subset R_n = S$ of rings going from $R$ to $S$. Hence $N^{-1}R_0 = N^{-1}R \subseteq N^{-1}R_1 \subseteq ... \subseteq N^{-1}R_{n-1} \subseteq N^{-1}R_n = N^{-1}S$ is a chain of rings going from $N^{-1}R$ to $N^{-1}S$. By refining this last chain, we get a finite maximal chain of rings from $N^{-1}R$ to $N^{-1}S$, the desired conclusion. $\diamondsuit$

Let us recall some terminology. Let $S$ be a ring, let $I$ be a nonzero ideal of $S$ and let $D$ be a subring of $E := S/I$. Consider the pullback construction of commutative rings:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ S & \longrightarrow & E := S/I \end{array}$$

Following [5], we say that $R$ is the ring of the $(S, I, D)$ construction and we set $R := (S, I, D)$.

**Proposition 3.** *If $R := (S, I, D)$ and $E := S/I$, then the following hold true:*
*$(i)$ $R \subset S$ satisfies the (FMC) condition if and only if $D \subset E$ satisfies the (FMC) condition.*
*$(ii)$ If moreover, we assume that $I$ is a maximal ideal of $S$ and that $D$ is a subfield of the field $E := S/I$, then $R \subset S$ satisfies the (FMC) condition if and only if $[E : D] < \infty$*

**Proof.** (i) Applying ([9], Lemma II.3) to the pullback $R = (S, I, D)$, we have an order-preserving and order-reflecting bijection between the set of all $R$-subalgebras of $S$ and the set of all $D$-subalgebras of $E$. The conclusion now follows readily.
(ii) It is enough to combine assertion (i) and ([8], Proposition 1.2). $\diamondsuit$

**Proposition 4.** *If $R \subset S$ is a ring extension satisfying the (FMC) condition, then there exists a subset $\{s_1, s_2, ..., s_n\}$ of $S$ such that $S = R[s_1, s_2, ..., s_n]$.*

**Proof.** There exists a finite maximal chain: $R_0 = R \subset R_1 \subset ... \subset R_{n-1} \subset R_n = S$ of rings going from $R$ to $S$. Since for each $0 \leq i \leq n-1$, $R_i \subset R_{i+1}$ is a minimal extension, then there exists $s_{i+1} \in R_{i+1}$ such that $R_{i+1} = R_i[s_{i+1}]$. Thus $S = R_{n-1}[s_n] = R_{n-2}[s_{n-1}][s_n] = R_{n-2}[s_{n-1}, s_n] = ... = R[s_1, s_2, ..., s_n]$. $\diamondsuit$

We point out that there is a relationship between Nagata rings and $P$-extensions as noticed in ([13], Proof of Theorem 4) where Gilmer and Hoffmann proved that if $R \subset T \subset S$ are rings such that $T$ is integral over $R$ and $S$ is a $P$-extension of $T$, then $S$ is a $P$-extension of $R$. Thus, for the sake of completeness, we introduce the following terminology: Let $T$ be a ring and consider the multiplicative subset $U_T = \{f \in T[X] \mid c(f) = T\}$ of $T[X]$, where $c(f)$ is the content of the polynomial $f$, that is, the ideal of $T$ generated by the coefficients of $f$. Then the ring $U_T^{-1}T[X]$ is called the *Nagata ring* in one indeterminate and with coefficients in $T$ and is denoted by $T(X)$. According to ([6], Theorem 5.4) and ([9], Theorem II.10), if $R \subset T$ is a minimal extension, then so is $R(X) \subset T(X)$ and hence $T(X) = U_R^{-1}T[X]$.

The following result improves ([17], Lemma 1). But, recall first that $S$ is a $P$-extension of $R$ if and only if $(R, S)$ is a residually algebraic pair (see ([7], Theorem) or ([16], Lemma 2.8)). Combining this result with ([2], Proposition 2.4), we can deduce that $S$ is a $P$-extension of $R$ if and only if $S_M$ is a $P$-extension of $R_M$ for each maximal ideal $M$ of $R$. Thus in the study of $P$-extensions, we can limit ourselves to the case where $R$ is local.

**Theorem 1.** *Let $R \subset T \subseteq S$ be rings. Assume that $R \subset T$ is a minimal extension, $T$ is an integral domain and $S$ is a $P$-extension of $T$, then $S$ is a $P$-extension of $R$.*

**Proof.** Without loss of generality, we can assume that $R$ is local with maximal ideal $M$. Let $s \in S$. As $S$ is a $P$-extension of $T$, then there exists a polynomial $f \in T[X]$ with some coefficient a unit in $T$ such that $f(s) = 0$. Clearly $\frac{1}{f} \in T(X) = U_R^{-1}T[X]$. Thus there exist $g \in T[X]$ and $h \in U_R = R[X] \setminus M[X]$ such that $\frac{1}{f} = \frac{g}{h}$. As $T$ is a domain, then we get $h = fg$, which implies that $h(s) = 0$. Moreover the fact that $h \in U_R = R[X] \setminus M[X]$ shows that some of the coefficients of $h$ is a unit of $R$. Therefore $S$ is a $P$-extension of $R$. $\diamondsuit$

The following result generalizes ([17], Proposition 1) and ([15], Theorem 3.9).

**Corollary 2.** *Let $R \subset T \subseteq S$ be rings such that $T$ is an integral domain. If $R \subset T$ satisfies the (FMC) condition and $S$ is a $P$-extension of $T$, then $S$ is a $P$-extension of $R$. In particular, if $R \subset S$ is an extension of integral domains, such that $R \subset S$ satisfies the (FMC) condition, then $S$ is a $P$-extension of $R$.*

Notice that even $S = R[u]$ is a minimal extension of $R$, the extension $R[X] \subset S[X]$ does never satisfy the (FC) condition. As noted in [11], there is a very simple infinite (descending) chain between $R[X]$ and $S[X]$, namely,

$$S[X] \supset R + XS[X] \supset R + XR + X^2S[X] \supset R + XR + X^2R + X^3S[X] \supset ...$$

**Theorem 2.** *Let $R \subset S$ be an extension of integral domains, then $R[X] \subset S[X]$ does never satisfy the (FMC) condition.*

**Proof.** Assume by way of contradiction that $R[X] \subset S[X]$ satisfies the (FMC) condition. Then there exists a finite maximal chain of rings from $R[X]$ to $S[X]$ of the form $\mathcal{R}_0 = R[X] \subset \mathcal{R}_1 \subset ... \subset \mathcal{R}_{n-1} \subset \mathcal{R}_n = S[X]$. Thus it follow from Corollary 2 that $S[X]$ is a $P$-extension of $R[X]$. We claim that $R \subset S$ is an integral extension. Indeed, we can assume that $R$ is local and integrally closed in $S$. Consider the ring $T = R + XS[X]$; we have $R[X] \subseteq T \subseteq S[X]$. Denote by $M$ the maximal ideal of $R$; then $\mathcal{Q} = M+XS[X]$ is a prime ideal of $T$. Let $\mathcal{P} = \mathcal{Q} \cap R[X]$. We have $\mathcal{P} = M + XR[X]$. Pick $a \in S \setminus R$. Then $aX \in T_{\mathcal{Q}}$, but $aX \notin R[X]_{\mathcal{P}}$. Indeed, if not, there exist $f \in R[X]$ and $g \in R[X] \setminus \mathcal{P}$ such that $\frac{f}{g} = aX$. Write $f = \sum_{0 \leq i \leq n} a_i X^i$ and $g = \sum_{0 \leq i \leq m} b_j X^j$. The equality $f = aXg$ shows that $n = m + 1$ and $a_1 = ab_0$. But $b_0 \in R \setminus M$. Hence $b_0$ is a unit of $R$. Therefore $a = a_1 b_0^{-1} \in R$, a contradiction. Thus $T_{\mathcal{Q}} \neq R[X]_{\mathcal{P}}$ and by ([2], Theorem 2.10), $(R[X], S[X])$ is not a residually algebraic pair. Hence $S[X]$ is not a $P$-extension of $R[X]$, absurd. Therefore, we conclude that $S = R^*$. So $S[X] = R^*[X] = (R[X])^*$ the integral closure of $R[X]$ in $S[X]$. Therefore $R[X] \subset S[X]$ satisfies the (FMC)$^*$ condition. Hence $R[X] \subset S[X]$ satisfies the (FC) condition (see ([1], Theorem 24)), a contradiction. $\diamondsuit$

# 3 Open Problem

The results of this note encourage us to ask the following question: Characterize ring extensions $R \subset S$ satisfying the (FMC) condition in the general case.

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