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Note on a nonlocal boundary value problem with solutions positive on an interval

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Abstract

We reconsider the boundary value problem studied in [1] and prove the existence of sign changing solutions under more general conditions on the nonlinear term.

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1 Introduction

The following problem is studied in [1]

$$u'' + g(t)f\left(u(t)\right) = 0, \quad 0 < t < 1 \tag{1}$$

$$u\left(0\right) = \alpha u'\left(0\right), \quad u\left(1\right) = \beta u'\left(\eta\right) \tag{2}$$

where $\eta \in (0,1)$, $g \in C\left([0,1], [0,\infty)\right)$, $f \in C\left(\mathbb{R}, [0,\infty)\right)$. The parameters α and β are such that $\alpha > 0$, $\beta > 0$ and $1+\alpha \neq \beta$. In a personal communication, Prof. J.R.L. Webb remarked that proper account was not taken of the fact that G is both discontinuous and changes sign, and, in fact, positive solutions may not exist under the given conditions, the main result in [1] is therefore not valid. We give a correction here and provide new results on existence of solutions that are positive on a sub-interval of $[0,1]$.

This type of problem has been studied by Infante-Webb in [2,3]. The case $\alpha = 0$, $0 \leq \beta < 1 - \eta$ was studied in detail in [2]. They established, using fixed point index theory, the existence of multiple nonzero solutions that are positive on a subinterval of $[0,1]$ but can change sign. Similar problems were studied in [3] and in the interesting paper [4] which studied the model of a thermostat. In [4] the authors proved that there is loss of positivity as the parameter decreases, and proved a uniqueness result. For more results on this subject we refer to [3,5,6].

2 Preliminaries

Let $E = C[0,1]$, with the supremum norm $||y|| = \sup\{|y(t)|, t \in [0,1]\}$. To study the problem (1)-(2), we write it as an equivalent fixed point problem for the Hammerstein integral operator

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds$$

where the Green's function is defined by

$$G(t,s) = \frac{(t+\alpha)(1-s)}{1+\alpha-\beta} - \begin{cases} \beta\frac{t+\alpha}{1+\alpha-\beta}, s \leq \eta \\ 0, s > \eta \end{cases} - \begin{cases} (t-s), s \leq t \\ 0, s > t \end{cases}$$

We give a correction to [1] here and provide new results for the case for $\alpha > 0$ and $0 < \beta < 1 - \eta$. We replace the previous assumptions that f is either sublinear or superlinear by more general conditions and prove existence of sign changing solutions. The method used is to apply the theory of [2,3] which is based on the fixed point index for the compact map T defined on a cone in the Banach space E.

3 Main Results

Following the theory of [2,3], an important step is to show that

$$|G(t,s)| \leq \Phi(s), \forall (t,s) \in [0,1] \times [0,1], \tag{3}$$

$$G(t,s) \geq c\Phi(s), \ \forall t \in [0,b], \forall s \in [0,1]. \tag{4}$$

Theorem 3.1 *If $0 < \beta < 1 - \eta$ and $\alpha > 0$, then there exists a continuous function $\Phi(s) = \frac{(1+\alpha)}{1+\alpha-\beta}(1-s)$ on $[0,1]$, a real number $b \in (0, 1-\beta)$ and a constant $c = \frac{\alpha \min((1-b-\beta),(1-\eta-\beta))}{1+\alpha} \in (0,1)$ such that inequalities (3) and (4) hold.*

Proof. We find upper and lower bounds of G.
Upper bounds. Case 1. $s > \eta$. If $t < s$ then $G(t,s) \geq 0$ and

$$G(t,s) = \frac{(t+\alpha)(1-s)}{1+\alpha-\beta} \leq \frac{(1+\alpha)}{1+\alpha-\beta}(1-s) = \Phi(s)$$

If $t \geq s$ then $G(t,s) \geq 0$ since

$$G(t,s) = \frac{(t+\alpha)(1-s)}{1+\alpha-\beta} - (t-s) = \frac{(s+\alpha)(1-t)+\beta(t-s)}{1+\alpha-\beta} \geq 0$$

and we have

$$\begin{aligned} G(t,s) &= \frac{(t+\alpha)(1-s)}{1+\alpha-\beta} - (t-s) \leq \frac{(t+\alpha)(1-s)}{1+\alpha-\beta} \\ &\leq \frac{1+\alpha}{1+\alpha-\beta}(1-s) = \Phi(s) \end{aligned}$$

Case 2. $s < \eta$. If $t < s$, then $G(t,s) = \frac{(t+\alpha)(1-s-\beta)}{1+\alpha-\beta}$ and the function G is positive because we are taking $\beta < 1-\eta$. Consequently we have

$$G(t,s) = \frac{(t+\alpha)(1-s-\beta)}{1+\alpha-\beta} \leq \frac{(1+\alpha)}{1+\alpha-\beta}(1-s) = \Phi(s),$$

(the case $\eta > s > 1-\beta$ is impossible since by hypothesis we have $\beta < 1-\eta$.)

If $t \geq s$ then

$$G(t,s) = \frac{(t+\alpha)(1-s-\beta)}{1+\alpha-\beta} - (t-s) = \frac{(s+\alpha)(1-t-\beta)}{1+\alpha-\beta}$$

the function G is positive if $t \leq 1-\beta$, in this case we get

$$G(t,s) = \frac{(s+\alpha)(1-t-\beta)}{1+\alpha-\beta} \leq \frac{(1+\alpha)}{1+\alpha-\beta}(1-s) = \Phi(s)$$

if $t > 1-\beta$ then $G(t,s) \leq 0$ and

$$\begin{aligned} -G(t,s) &= \frac{(s+\alpha)(-1+t+\beta)}{1+\alpha-\beta} \leq \frac{(1+\alpha)}{1+\alpha-\beta}(1-\eta) \\ &\leq \frac{(1+\alpha)}{1+\alpha-\beta}(1-s) = \Phi(s) \end{aligned}$$

therefore $|G(t,s)| \leq \frac{1+\alpha}{1+\alpha-\beta}(1-s) = \Phi(s)$.

Lower bounds. Let $b \in (0, 1-\beta)$ and $\alpha > 0$, then for $t \in [0,b]$ and $s \in [0,1]$ it yields: **Case 1.** $s > \eta$, if $t < s$ then

$$G(t,s) = \frac{(t+\alpha)(1-s)}{1+\alpha-\beta} \geq \frac{\alpha(1-s)}{1+\alpha-\beta} = \frac{\alpha}{1+\alpha}\Phi(s)$$

If $t \geq s$ then

$$\begin{aligned} G(t,s) &= \frac{(s+\alpha)(1-t)+\beta(t-s)}{1+\alpha-\beta} \geq \frac{\alpha(1-t)}{1+\alpha-\beta} \\ &\geq \frac{\alpha(1-b)}{1+\alpha-\beta}(1-s) = \frac{\alpha(1-b)}{1+\alpha}\Phi(s) \end{aligned}$$

Case 2. $s < \eta$. If $t < s$ then

$$G(t,s) = \frac{(t+\alpha)(1-s-\beta)}{1+\alpha-\beta} \geq \frac{\alpha(1-\eta-\beta)}{1+\alpha-\beta}(1-s) = \frac{\alpha(1-\eta-\beta)}{1+\alpha}\Phi(s)$$

If $t \geq s$ then

$$G(t,s) = \frac{(s+\alpha)(1-t-\beta)}{1+\alpha-\beta} \geq \frac{\alpha(1-b-\beta)}{1+\alpha}\Phi(s).\square$$

Define the operator $T : E \to E$ by $Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds$.

Notations. Let K be the cone $K = \left\{ u \in C[0,1] : \min\limits_{t\in[0,b]} u(t) \geq c\,||u|| \right\}$ and define the following subsets of K,
$K_r = \{u \in K : \|u\| < r\}$, $\overline{K}_r = \{u \in K : \|u\| \leq r\}$,
$\overline{K}_{\rho,r} = \{u \in K : \rho \leq \|u\| \leq r\}$, where $0 < \rho < r < \infty$.
We also let

$$f^0 = \limsup_{u\to 0}\frac{f(u)}{|u|},\quad f^\infty = \limsup_{|u|\to\infty}\frac{f(u)}{|u|},\quad f_0 = \liminf_{u\to 0+}\frac{f(u)}{u},\quad f_\infty = \liminf_{u\to\infty}\frac{f(u)}{u},$$

and

$$f^{-\rho,\rho} = \sup_{u\in[-\rho,\rho]}\frac{f(u)}{\rho},\quad f_{c\rho,\rho} = \inf_{u\in[c\rho,\rho]}\frac{f(u)}{\rho}.$$

Let $m = \left(\max_{0\leq t\leq 1}\int_0^1 |G(t,s)|\, g(s)ds\right)^{-1}$ and $M = \left(\min\limits_{t\in[0,b]}\int_0^b G(t,s)g(s)ds\right)^{-1}$.

As in [5] define the continuous function $q : E \to \mathbb{R}$, $q(u) = \min\limits_{t\in[0,b]} u(t)$ and the set $\Omega_\rho = \{u \in K : q(u) < c\rho\}$. It is clear that if $u \in \partial\Omega_\rho$, (the boundary relative to K) then $c\rho \leq u(t) \leq \rho$ for all $t \in [0,b]$. The set Ω_ρ was introduced in [5] for cones of positive functions, the case here was used in [2], for further properties of Ω_ρ see [5].

Now we state the existence results from [2] specialized to our case.

Theorem 3.2 *Assume that* $0 < \beta < 1-\eta$*, and let* $b \in (0, 1-\beta)$ *and suppose that* $\int_0^b \Phi(s)g(s)ds > 0$*. Then for* $\alpha > 0$*, the problem (1)-(2) has at least one nonzero solution, positive on* $[0,b]$*, if either*

H1) $0 \leq f^0 < m$ *and* $M < f_\infty \leq \infty$*, or*

H2) $0 \leq f^\infty < m$ *and* $M < f_0 \leq \infty$*,*

and has two nonzero solutions, positive on $[0,b]$*, if there is* $\rho > 0$ *such that either*

S1) $0 \leq f^0 < m$*,* $f_{c\rho,\rho} \geq cM$*,* $u \neq Tu$ *for* $u \in \partial\Omega_\rho$ *and* $0 \leq f^\infty < m$ *or*

S2) $M < f_0 \leq \infty$*,* $f^{-\rho,\rho} \leq m$*,* $u \neq Tu$ *for* $u \in \partial K_\rho$ *and* $M < f_\infty \leq \infty$*.*

Example 3.3 *Consider the following BVP*

$$u'' + 1 = 0,\ t \in (0,1),\ \ u(0) = u'(0),\ u(1) = \frac{1}{3}u'(\frac{1}{2}). \tag{5}$$

Here $\alpha = 1$, $\eta = 1/2$, $\beta = 1/3$, $g(t) = 1$ *on* $[0,1]$ *and* $f = 1$ *on* $\mathbb{R}$. *Then* $f_0 = \infty$, $f^\infty = 0$ *so this is a sublinear case. Choosing* $b = \frac{3}{5} < 2/3$, *then* $\int_0^b \Phi(s)g(s)ds > 0$, *(H2) holds and the Theorem gives that the BVP has at least one solution which is positive on* $[0, 3/5]$. *In fact, the solution is easily found to be* $u(t) = \frac{1}{5} + \frac{1}{5}t - \frac{t^2}{2}$ *which is positive on* $[0, (1+\sqrt{11})/5] \approx [0, 0.863]$.

Example 3.4 *Let* $f(u) = \begin{cases} 10^3u^2, |u| < 1 \\ -999u + 1999, 1 \leq |u| < 2 \\ 1, |u| \geq 2 \end{cases}$

Choosing $g = 1$ $\alpha = 1$ $\eta = 1/2$ $\beta = 1/3$ *and* $b = 3/5$ *as in Example 2.4, then* $f^0 = 0$. $f^\infty = 0$, $c = \frac{1}{30}$, $M = \frac{1250}{79} \approx 15.823$. *Let* $\rho = \frac{1}{2}$ *then* $f_{c\rho,\rho} = \frac{10}{18} \approx 0.55556 \geq cM = \frac{1250}{79\times30} \approx 0.52743$. *Assume that there exists* $u \in \partial\Omega_\rho$ *such that* $u = Tu$, *then for* $t \in [0, \frac{3}{5}]$ *it yields by property of* Ω_ρ,

$$\begin{aligned} u\,(t) = Tu\,(t) &= \int_0^1 G(t,s)f\,(u(s))\,ds \geq \int_0^{\frac{3}{5}} G(t,s)f\,(u(s))\,ds \\ &= 10^3 \int_0^{\frac{3}{5}} G(t,s)u^2(s)ds \geq 10^3c^2\rho^2 \int_0^{\frac{3}{5}} G(t,s)ds. \end{aligned}$$

Taking the minimum over $t \in \left[0, \frac{3}{5}\right]$, *we get*

$$c\rho \geq 10^3c^2\rho^2/M,\ \textit{that is}\ \ c\rho \leq M/10^3.$$

Since $c\rho = 1/60 > M/10^3$, *this is a contradiction, consequently (S1) holds and the BVP has at least two nonzero solutions positive on* $[0, \frac{3}{5}]$.

4 Open problem

In the present note we have established the existence of nonzero solutions changing sign and positive on a subinterval of $[0,1]$, in the case $\alpha > 0$, $0 < \beta < 1 - \eta$ and under more general conditions on f. The existence of nonzero solutions for problem (1)-(2) could be investigated for other cases such as $\alpha > 0$ and $\beta < 0$.

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