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# Some Extension Results in Classical Integral Inequalities

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**Abstract**

***In this paper, we give new extensions of the classical Hermite-Hadamard and Fejer integral inequalities. Some applications of these extensions are also presented.***

**Keywords:** *Convex functions, Fejer inequality, Hermite-Hadamard inequality.*

# 1 Introduction

The integral inequalities play a fundamental role in the theory of differential equations. Significant development in this area has been achieved for the last two decades. For details, we refer to [3, 10, 11]. Let us introduce now some results that have inspired our work. The first one is given in [12], where Hermite and Hadamard [7] have proved that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \qquad (1)$$

where $f$ is a convex function on $[a, b]$.
The second one is the celebrated Fejer inequality [5, 10]:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{1}{2}(f(a)+f(b))\int_a^b g(x)\,dx, \qquad (2)$$

where $f$ is a convex function on $[a,b]$ and $g$ is a non-negative function on $[a,b]$ which is integrable and symmetric with respect to $\frac{a+b}{2}$. The inequalities (1) and (2) have triggered a huge amount of interest over the years. Many mathematicians have devoted their efforts to generalize, refine, counterpart and extend them for different classes of functions such as: quasi-convex functions, Godunova-Levin class of functions, $\log -$convex and $r-$convex functions, $p-$functions, etc, see [3, 8, 9, 10, 11]. The description of best possible inequalities of Hadamard-Hermite type is due to Fink [6]. A generalization to higher-order convex functions can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to two dimensional linear space of continuous functions. Recently in [4], the authors obtained an estimate, from below and from above, for the mean value of $f : I \to R$, such that $f$ is continuous on $I$, twice differentiable on $I$ and there exist $m = \inf f''(x), x \in I$ or $M = \sup f''(x), x \in I$.
The purpose of this paper is to give new extension theorems related to (1) and (2). Some applications of our extension results are also presented.
Firstly, let us introduce the following spaces which are are used throughout this paper:

$$E([a,b]) := \left\{ f : [a,b] \to R; f(\frac{a+b}{2}) \leq \frac{f(a+b-x)+f(x)}{2} \leq \frac{f(a)+f(b)}{2} \right\} \tag{3}$$

and

$$E_{-}([a,b]) := \{ f : [a,b] \to R; -f \in E([a,b]) \}. \tag{4}$$

# 2 Main Results

In this section, we prove some new integral inequalities. These results allow us in particular to find the inequalities (1) and (2) ( see Section 1). Our first result is the following theorem:

**Theorem 2.1** *Assume that $f \in E([a,b])$ and integrable on $[a,b]$. Then the inequality (1) holds.*
*If $f \in E_{-}([a,b])$, then the inequality (1) is reversed.*

The second result is given by:

**Theorem 2.2** *Let $f \in E([a,b])$ and integrable on $[a,b]$. If $g : [a,b] \to$ is a positive integrable function on $[a,b]$ and symmetric with respect to $x_0 = \frac{a+b}{2}$, then (2) holds.*
*If $f \in E_{-}([a,b])$, then the inequality (2) is reversed.*

We give also:

**Corollary 2.3** *Let $f$ be a monotone integrable function on $[a,b]$. Then, we have*

$$0 \leq \frac{1}{b-a}\int_a^b \left| f(x) - f(\frac{a+b}{2}) \right| dx \leq \frac{1}{2}\left| f(b) - f(a) \right|. \tag{5}$$

# 3 Lemmas

In order to prove our main theorems, we need the following lemmas.

**Lemma 3.1** *Let $f$ be an integral function on $[a,b]$. Then, we have*

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx. \tag{6}$$

**Proof:** We can get (6) by the change of variable $t = a+b-x$.

The following lemma is crucial in the proof of our theorems.

**Lemma 3.2** $(a*)$ *Let $Conv([a,b])$ be the set of all convex functions on $[a,b]$. Then $Conv([a,b]) \subset E([a,b])$.*
$(b*)$ *Let $Conc_-([a,b])$ be the set of all concave functions on $[a,b]$. Then $Conc_-([a,b]) \subset E_-([a,b])$.*

Proof: $(a*)$ Let $f$ be a convex function on $[a,b]$. Then, we can write

$$f\left(\frac{a+b}{2}\right) = f(\frac{a+b-x+x}{2}) \leq \frac{1}{2}\left(f(a+b-x) + f(x)\right). \tag{7}$$

Without loss of generality, we take $x = \lambda a + (1-\lambda) b; 0 \leq \lambda \leq 1$. Then we have

$$\begin{aligned} &\frac{1}{2}\left(f(a+b-(\lambda a + (1-\lambda) b)) + f(\lambda a + (1-\lambda) b)\right) \\ &= \tfrac{1}{2}\left(f(\lambda b + (1-\lambda) a) + f(\lambda a + (1-\lambda) b)\right). \end{aligned} \tag{8}$$

Using the convexity of $f$, we get

$$\frac{1}{2}\left(f(\lambda b + (1-\lambda) a) + f(\lambda a + (1-\lambda) b)\right) \leq \frac{1}{2}(f(a) + f(b)). \tag{9}$$

Thanks to (7) and (9), we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(f(a+b-x) + f(x)\right) \leq \frac{1}{2}(f(a) + f(b)). \tag{10}$$

Thus, by (10) we have $f \in E([a,b])$.
To show that the inclusion is strict, it sufficient to take $f(x) = \cos x$ and $[a,b] = [0, 2\pi]$ and to remark that $f \in E([a,b])$ but $f \notin Conv([a,b])$.
The first part of Lemma 3.1 is thus proved.
To prove $(b*)$, we use the same arguments as in the proof of $(a*)$.

For monotone functions, we give the following lemma.

**Lemma 3.3** *Let $f$ be a monotone function on $[a,b]$. Then, we have* $\left|f(x) - f(\frac{a+b}{2})\right| \in E([a,b])$.

**Proof:** Without loss of generality we may assume that the function $f$ is increasing on $[a,b]$. Then, we proceed in two steps:
**(1.)** If $x \in \left[a, \frac{a+b}{2}\right]$, then we have

$$f(a) - f(\frac{a+b}{2}) \leq f(x) - f(\frac{a+b}{2}) \leq 0 \tag{11}$$

and

$$0 \leq f(a+b-x) - f(\frac{a+b}{2}) \leq f(b) - f(\frac{a+b}{2}). \tag{12}$$

Consequently, we get

$$\begin{aligned} 0 &\leq \left|f(x) - f(\frac{a+b}{2})\right| + \left|f(a+b-x) - f(\frac{a+b}{2})\right| \\ &\leq \left|f(a) - f(\tfrac{a+b}{2})\right| + \left|f(b) - f(\tfrac{a+b}{2})\right|. \end{aligned} \tag{13}$$

**(2.)** If $x \in \left[\frac{a+b}{2}, b\right]$, then we can write

$$0 \leq f(x) - f(\frac{a+b}{2}) \leq f(b) - f(\frac{a+b}{2}) \tag{14}$$

and

$$f(a) - f(\frac{a+b}{2}) \leq f(a+b-x) - f(\frac{a+b}{2}) \leq 0. \tag{15}$$

These imply

$$\begin{aligned} 0 &\leq \left|f(x) - f(\frac{a+b}{2})\right| + \left|f(a+b-x) - f(\frac{a+b}{2})\right| \\ &\leq \left|f(b) - f(\tfrac{a+b}{2})\right| + \left|f(a) - f(\tfrac{a+b}{2})\right|. \end{aligned} \tag{16}$$

Lemma 3.2 is thus proved.

# 4 Proof of Theorems and Corollaries

**Proof of Theorem 2.1:** Let $f \in E\left([a,b]\right)$ and integrable on $[a,b]$. Then, by definition, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a+b-x)+f\left(x\right)}{2} \leq \frac{f\left(a\right)+f\left(b\right)}{2}. \tag{17}$$

Integrating both sides of (17) with respect to $x$ over $[a,b]$, we can write

$$(b-a)\, f\left(\frac{a+b}{2}\right) \leq \int_a^b \frac{f(a+b-x)+f\left(x\right)}{2} dx \leq (b-a)\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right). \tag{18}$$

Thanks to Lemma 3.1, we obtain the desired inequality (1).
**Proof of Theorem 2.2:** Multiplying both sides of (3) by $g(x)$, we get

$$f\left(\frac{a+b}{2}\right) g(x) \leq g(x)\left(\frac{f(a+b-x)+f\left(x\right)}{2}\right) \leq g(x)\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{19}$$

Integrating both sides of (19) with respect to $x$ over $[a,b]$ and using the symmetry of $g$ with respect to $\frac{a+b}{2}$, yields

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b g(x)f(x)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}\int_a^b g(x)dx. \tag{20}$$

This ends the proof of Theorem 2.2.
**Proof of Corollary 2.3:** Let $f$ be a monotone function on $[a,b]$. Then, by Lemma 3.3, we have $\left|f\left(x\right)-f(\frac{a+b}{2})\right| \in E\left([a,b]\right)$. Using Theorem 2.1, we get

$$0 \leq \frac{1}{b-a}\int_a^b \left|f\left(x\right)-f(\frac{a+b}{2})\right| dx \tag{21}$$

$$\leq \frac{1}{2}\left(\left|f\left(b\right)-f(\frac{a+b}{2})\right| + \left|f\left(a\right)-f(\frac{a+b}{2})\right|\right). \tag{22}$$

Since $f$ is a monotone function on $[a,b]$, then we can write:

$$\left|f\left(b\right)-f(\frac{a+b}{2})\right| + \left|f\left(a\right)-f(\frac{a+b}{2})\right| = \left|f\left(b\right)-f(a)\right|. \tag{23}$$

Then the proof of Corollary 2.3 is completed.

# 5 Applications

**(i*-) The normal distribution functions:** Let $f(t) = e^{-t^2}, t \in [-x, x], x > 0$. Since $f$ is decreasing function in $[0, x]$, then for $t \in [0, x]$, we have the following estimation:

$$e^{-x^2} \leq e^{-t^2} \leq 1. \tag{24}$$

Hence, $f \in E_{-}([-x, x])$. Then, by Theorem 2.1, for $t \in [-x, x]$, we obtain the inequalities

$$xe^{-x^2} \leq \int_0^x e^{-t^2} dt \leq x, x > 0. \tag{25}$$

**(ii*-) The hyperbolic sine function:** Let $f(t) = \sinh t, t \in [a, b], a < b$. Then, by Corollary 2.3, we obtain the inequalities

$$0 \leq \frac{1}{b-a} \int_a^b \left| \sinh t - \sinh\left(\frac{a+b}{2}\right) \right| dt \leq \frac{\sinh b - \sinh a}{2}. \tag{26}$$

# 6 Open Problems

At the end, we pose the following problems:
**Open Problem 1.** Using fractional integral operator of order $\alpha$ for a function $f$ on an interval $[a, b]$, under what conditions does Theorem 2.1 hold for $f, \alpha$?
**Open Problem 2.** Is it possible to generalize Theorems 2.1 and Theorem 2.2, in fractional integration theory using two fractional parameters $\alpha$ and $\beta$?

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