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# New Inequalities For Convex Sequences With Applications

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**Abstract**

*In this paper, we will show some new inequalities for convex sequences, and we will also make a connection between them and Chebyshev's inequality, which implies the existence of new class of sequences satisfying Chebyshev's inequality. We give also some applications and generalization of Haber and Mercer's inequalities.*

**Keywords:** *Chebyshev's inequality, Convex Sequences, Symmetric sequences.*

## 1 Introduction and main results

A classic result due to Chebyshev (1882-1883) (see [2, 5, 6, 10, 11, 13]) is stated in the following theorem.

**Theorem A** *Let* $a = (a_1, a_2, \cdots, a_n)$ *and* $b = (b_1, b_2, \cdots, b_n)$ *be two sequences of real numbers monotonic in the same direction, and* $p = (p_1, p_2, \cdots, p_n)$ *be a positive sequence. Then*

$$\left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i a_i b_i\right) \geq \left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right). \tag{1.1}$$

*If* $a$ *and* $b$ *are monotonic in opposite directions, then the reverse of the inequality in* (1.1) *holds. In either case equality holds if and only if either* $a_1 = a_2 = \cdots = a_n$ *or* $b_1 = b_2 = \cdots = b_n$.

There exist several results which show that Chebyshev inequality is valid under weaker conditions, for example the condition that the sequences be monotonic can be replaced by the condition that they be similarly ordered. In this case Theorem A is a simple consequence of the following identity

$$\left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i a_i b_i\right) - \left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right)$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left(a_i - a_j\right)\left(b_i - b_j\right). \tag{1.2}$$

Note that the sequences $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ are said to be similarly ordered if

$$\left(a_i - a_j\right)\left(b_i - b_j\right) \geq 0,\ 1 \leq i, j \leq n \tag{1.3}$$

holds, and they are said to be oppositely ordered if the reverse inequality holds.

Considerable attention has been given to the study of convex sequences and their properties, and the corresponding inequalities with applications. In general, convex sequences as discrete versions of convex functions play an important role in mathematical analysis and in the theory of inequalities. Inequalities for convex sequences provided considerable interest in proving a large number of elegant results with applications (see Wu and Shi [15] , Wu and Debnath [16] and Mercer [9]). In addition, several authors including Mitrinovic and Vasic [11], Roberts and Varberg [14], and Mitrinovic et al. [10] presented a large number of major results for convex sequences and related inequalities.

The aim of this paper is to prove new type of inequalities for convex sequences, and we put a link between these inequalities and Chebyshev's inequality. Before we state our results we give the following definition.

**Definition A** ([7]) *Let $a = (a_1, a_2, \cdots, a_n)$ be a sequence of real numbers, $a$ is a convex sequence if for all $i = 1, \cdots, n-2$, we have*

$$a_i + a_{i+2} \geq 2a_{i+1}.$$

*If the above inequality reversed, then $a$ is termed concave sequence.*

We obtain the following results.

**Theorem 1.1** *Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two convex (concave) sequences, and $p = (p_1, p_2, \cdots, p_n)$ be a positive sequence symmetric about $\left[\frac{n+1}{2}\right]$ $(p_k = p_{n+1-k},$ for all $k = 1, \cdots, n)$. Then*

$$\left(\sum_{i=1}^{n} p_i a_i b_i\right) + \left(\sum_{i=1}^{n} p_i a_i b_{n+1-i}\right)$$

$$\geq \frac{2}{\left(\sum_{i=1}^{n} p_i\right)} \left(\sum_{i=1}^{n} p_i a_i\right) \left(\sum_{i=1}^{n} p_i b_i\right). \tag{1.4}$$

*If $a$ is convex (or concave) and $b$ is concave (or convex) sequences, then the inequality (1.4) is reversed. In either case equality holds if and only if either $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.*

**Corollary 1.1** *Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two convex (concave) sequences. If either $a$ or $b$ is symmetric about $\left[\frac{n+1}{2}\right]$, then*

$$\sum_{i=1}^{n} a_i b_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right). \tag{1.5}$$

*If $a$ is convex (or concave) and $b$ is concave (or convex) sequences, then the inequality (1.5) is reversed. In either case equality holds if and only if either $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.*

**Theorem 1.2** *Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two convex (or concave) sequences.*
(i) *If $a$ and $b$ are similarly ordered, then*

$$\sum_{i=1}^{n} a_i b_i \geq \frac{1}{2} \left(\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} a_{n+1-i} b_i\right) \geq \frac{1}{n} \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right). \tag{1.6}$$

(ii) *If $a$ and $b$ are oppositely ordered, then*

$$\sum_{i=1}^{n} a_{n+1-i} b_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right) \geq \sum_{i=1}^{n} a_i b_i. \tag{1.7}$$

**Theorem 1.3** *Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two sequences of real numbers where $a$ is convex sequence and $b$ decreasing for all $k = 1, \cdots, \left[\frac{n+1}{2}\right]$ and increasing for all $k = \left[\frac{n+1}{2}\right], \cdots, n$. Then the inequality (1.4) holds.*

Here we obtain the discrete version of Fejér [3] double inequality.

**Theorem 1.4** *Let $a = (a_1, a_2, \cdots, a_n)$ be a convex sequence of real numbers and $p = (p_1, p_2, \cdots, p_n)$ be a positive sequence symmetric about $\left[\frac{n+1}{2}\right]$. Then*

$$\left(\sum_{i=1}^{n} p_i\right) \frac{a_N + a_{n+1-N}}{2} \leq \sum_{i=1}^{n} p_i a_i \leq \left(\sum_{i=1}^{n} p_i\right) \frac{a_1 + a_n}{2}. \tag{1.8}$$

*If $a = (a_1, a_2, \cdots, a_n)$ is concave sequence then the inequality* (1.8) *is reversed.*

# 2 Some lemmas

**Lemma 2.1** *Let $a = (a_1, a_2, \cdots, a_n)$ be convex (or concave) sequence of real numbers. Then the sequence $c = (c_1, c_2, \cdots, c_n)$, where*

$$c_k = a_k + a_{n+1-k} \tag{2.1}$$

*is decreasing (increasing) for all $k = 1, \cdots, \left[\frac{n+1}{2}\right]$ and increasing (decreasing) for all $k = \left[\frac{n+1}{2}\right], \cdots, n$.*

*Proof.* Suppose that $a$ is convex sequence. Since $c$ is a symmetric sequence about $\left[\frac{n+1}{2}\right]$, then we need only to prove that $c$ is decreasing for all $k = 1, \cdots, \left[\frac{n+1}{2}\right]$. We have

$$c_k - c_{k+1} = (a_k + a_{n+1-k}) - (a_{k+1} + a_{n-k})$$

$$= (a_k + a_{k+1} - a_{k+1} + \cdots + a_{n-k} - a_{n-k} + a_{n+1-k})$$

$$- (a_{k+1} + a_{k+2} - a_{k+2} + \cdots + a_{n-k-1} - a_{n-k-1} + a_{n-k})$$

$$= (a_k + a_{k+2} - 2a_{k+1}) + (a_{k+1} + a_{k+3} - 2a_{k+2})$$

$$+ \cdots + (a_{n-1-k} + a_{n+1-k} - 2a_{n-k}) \tag{2.2}$$

for all $k = 1, \cdots, \left[\frac{n+1}{2}\right]$. By using mathematical induction and $(2.2)$, we obtain

$$c_k - c_{k+1} = \sum_{i=k}^{n-1-k} (a_i + a_{i+2} - 2a_{i+1}) \geq 0. \tag{2.3}$$

If $a$ is a concave sequence, then by using similar proof we obtain the result.

**Lemma 2.2** *Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two sequences of real numbers. If $a$ and $b$ are similarly ordered, then*

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{n+1-i}. \tag{2.4}$$

*If $a$ and $b$ are oppositely ordered, then the inequality* (2.4) *is reversed.*

*Proof.* Since $a$ and $b$ are similarly ordered, then we have for all $i = 1, \cdots, n$

$$(a_i - a_{n+1-i})(b_i - b_{n+1-i}) \geq 0 \tag{2.5}$$

which implies that

$$a_i b_i + a_{n+1-i} b_{n+1-i} \geq a_i b_{n+1-i} + a_{n+1-i} b_i. \tag{2.6}$$

Then

$$2\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (a_i b_i + a_{n+1-i} b_{n+1-i})$$

$$\geq \sum_{i=1}^{n} (a_i b_{n+1-i} + a_{n+1-i} b_i) = 2\sum_{i=1}^{n} a_i b_{n+1-i}. \tag{2.7}$$

It follows that

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{n+1-i}.$$

If $a$ and $b$ are oppositely ordered, then by using similar proof we obtain the result.

In the following we denote

$$\sum_{i=1}^{N}{}^{*} c_i = \begin{cases} c_1 + c_2 + \cdots + c_N, & \text{if } n \text{ is even,} \\ c_1 + c_2 + \cdots + c_{N-1} + \frac{1}{2} c_N, & \text{if } n \text{ is odd,} \end{cases}$$

where $c = (c_1, c_2, \cdots, c_n)$ and $N = \left[\frac{n+1}{2}\right]$.

**Lemma 2.3** *Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two sequences of real numbers and $p = (p_1, p_2, \cdots, p_n)$ be a positive sequence, we denote by $N = \left[\frac{n+1}{2}\right]$. If $a$ and $b$ are similarly ordered, then*

$$\left(\sum_{i=1}^{N}{}^{*} p_i\right)\left(\sum_{i=1}^{N}{}^{*} p_i a_i b_i\right) \geq \left(\sum_{i=1}^{N}{}^{*} p_i a_i\right)\left(\sum_{i=1}^{N}{}^{*} p_i b_i\right). \tag{2.8}$$

*If $a$ and $b$ are oppositely ordered, then the inequality* (2.8) *is reversed.*

*Proof.*(i) If $n$ is even, then the inequality (2.8) is equivalent to

$$\left(\sum_{i=1}^{N} p_i\right)\left(\sum_{i=1}^{N} p_i a_i b_i\right) \geq \left(\sum_{i=1}^{N} p_i a_i\right)\left(\sum_{i=1}^{N} p_i b_i\right)$$

which is Chebychev's inequality.

(ii) If $n$ is odd, we have

$$\sum_{i=1}^{N}{}^{*} c_i = c_1 + c_2 + \cdots + \frac{1}{2} c_N.$$

Since $a$ and $b$ are similarly ordered, then

$$(a_i - a_j)(b_i - b_j) \geq 0, \ 1 \leq i, j \leq n$$

which implies

$$a_i b_i + a_j b_j \geq a_i b_j + a_j b_i, \ 1 \leq i, j \leq n. \tag{2.9}$$

Multiplying both sides of inequality (2.9) by $p_i p_j$

$$p_i p_j a_i b_i + p_i p_j a_j b_j \geq p_i p_j a_i b_j + p_i p_j a_j b_i, \ 1 \leq i, j \leq n \tag{2.10}$$

which implies

$$\begin{array}{c} p_j p_1 a_1 b_1 + p_1 p_j a_j b_j \geq p_1 a_1 p_j b_j + p_1 b_1 p_j a_j, \\ p_j p_2 a_2 b_2 + p_2 p_j a_j b_j \geq p_2 a_2 p_j b_j + p_2 b_2 p_j a_j, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_j p_{N-1} a_{N-1} b_{N-1} + p_{N-1} p_j a_j b_j \geq p_{N-1} a_{N-1} p_j b_j + p_{N-1} b_{N-1} p_j a_j, \\ \frac{1}{2} p_j p_N a_N b_N + \frac{1}{2} p_N p_j a_j b_j \geq \frac{1}{2} p_N a_N p_j b_j + \frac{1}{2} p_N b_N p_j a_j, \end{array} \tag{2.11}$$

for all $1 \leq j \leq N$. Summing both sides of inequalities (2.11) with respect to $i = 1, \cdots, N$, we obtain

$$p_j \sum_{i=1}^{N}{}^{*} p_i a_i b_i + p_j a_j b_j \sum_{i=1}^{N}{}^{*} p_i \geq p_j b_j \sum_{i=1}^{N}{}^{*} p_i a_i + p_j a_j \sum_{i=1}^{N}{}^{*} p_i b_i. \tag{2.12}$$

By the same reasoning as before we have by using (2.12)

$$\left(\sum_{j=1}^{N}{}^{*} p_j\right)\left(\sum_{i=1}^{N}{}^{*} p_i a_i b_i\right) + \left(\sum_{j=1}^{N}{}^{*} p_j a_j b_j\right)\left(\sum_{i=1}^{N}{}^{*} p_i\right)$$

$$\geq \left(\sum_{j=1}^{N}{}^{*} p_j b_j\right)\left(\sum_{i=1}^{N}{}^{*} p_i a_i\right) + \left(\sum_{j=1}^{N}{}^{*} p_j a_j\right)\left(\sum_{i=1}^{N}{}^{*} p_i b_i\right) \tag{2.13}$$

which is equivalent to (2.8). If $a$ and $b$ are oppositely ordered, then by using similar proof we obtain the result.

# 3 Proof of the Theorems

**Proof of Theorem 1.1** Without loss of generality we suppose that $a$ and $b$ are convex sequences and we denote by $U$ and $V$ the following sequences

$$U_i = a_i + a_{n+1-i},\ \ V_i = b_i + b_{n+1-i}.$$

Since $a$ and $b$ are convex sequences, then by using Lemma 2.1 we deduce that $U$ and $V$ have the same direction of monotony. By applying Lemma 2.3 for all $i = 1, \cdots, N = \left[\frac{n+1}{2}\right]$, we obtain

$$\left(\sum_{i=1}^{N}{}^{*} p_i\right)\left(\sum_{i=1}^{N}{}^{*} p_i U_i V_i\right) \geq \left(\sum_{i=1}^{N}{}^{*} p_i U_i\right)\left(\sum_{i=1}^{N}{}^{*} p_i V_i\right), \tag{3.1}$$

where $p = (p_1, p_2, \cdots, p_n)$ is a positive sequence and symmetric about $\left[\frac{n+1}{2}\right]$. Then

$$\sum_{i=1}^{N}{}^{*} p_i \left(a_i b_i + a_{n+1-i} b_{n+1-i}\right) + \sum_{i=1}^{N}{}^{*} p_i \left(a_i b_{n+1-i} + a_{n+1-i} b_i\right)$$
$$\geq \frac{1}{\left(\sum\limits_{i=1}^{N}{}^{*} p_i\right)}\left(\sum_{i=1}^{N}{}^{*} p_i \left(a_i + a_{n+1-i}\right)\right)\left(\sum_{i=1}^{N}{}^{*} p_i \left(b_i + b_{n+1-i}\right)\right). \tag{3.2}$$

Using the identities

$$\sum_{i=1}^{N}{}^{*} p_i \left(a_i b_i + a_{n+1-i} b_{n+1-i}\right) = \sum_{i=1}^{n} p_i a_i b_i, \tag{3.3}$$

$$\sum_{i=1}^{N}{}^{*} p_i \left(a_i b_{n+1-i} + a_{n+1-i} b_i\right) = \sum_{i=1}^{n} p_i a_i b_{n+1-i}, \tag{3.4}$$

$$\sum_{i=1}^{N}{}^{*} p_i = \frac{1}{2}\sum_{i=1}^{n} p_i, \tag{3.5}$$

and

$$\sum_{i=1}^{N}{}^{*} p_i \left(a_i + a_{n+1-i}\right) = \sum_{i=1}^{n} p_i a_i,\ \ \sum_{i=1}^{N}{}^{*} p_i \left(b_i + b_{n+1-i}\right) = \sum_{i=1}^{n} p_i b_i. \tag{3.6}$$

By using $(3.3) - (3.6)$, we obtain from $(3.2)$

$$\sum_{i=1}^{n} p_i a_i b_i + \sum_{i=1}^{n} p_i a_i b_{n+1-i} \geq \frac{2}{\sum\limits_{i=1}^{n} p_i}\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right).$$

Now, if $a$ convex (concave) and $b$ concave ( convex) sequences, then by using similar proof as above we obtain the result.

**Proof of Theorem 1.2** (i) Since $a$ and $b$ are convex sequences and similarly ordered, then by Lemma 2.2 we have

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{n+1-i} \tag{3.7}$$

which we can write

$$2\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{n+1-i} + \sum_{i=1}^{n} a_i b_i. \tag{3.8}$$

By Theorem 1.1 and (3.8), we have

$$2\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} (a_i b_{n+1-i} + a_i b_i) \geq \frac{2}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right). \tag{3.9}$$

(ii) Since $a$ and $b$ are convex sequences, then by Theorem 1.1

$$\begin{aligned} &\sum_{i=1}^{n} a_i b_{n+1-i} - \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \\ &\geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) - \sum_{i=1}^{n} a_i b_i. \end{aligned} \tag{3.10}$$

On the other hand, we have

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \sum_{i=1}^{n} a_i b_i, \tag{3.11}$$

because $a$ and $b$ are oppositely ordered. By (3.10) and (3.11) , we get

$$\sum_{i=1}^{n} a_i b_{n+1-i} \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right). \tag{3.12}$$

Now, if $a$ and $b$ are concave sequences, then by using similar proof as above we obtain the result.

**Proof of Theorem 1.3** We denote by $U$ and $V$ the following sequences

$$U_i = a_i + a_{n+1-i}, \tag{3.13}$$

$$V_i = b_i + b_{n+1-i}. \tag{3.14}$$

Since $U$ is convex sequence, then by Lemma 2.1, $U$ is decreasing for all $i = 1, \cdots, \left[\frac{n+1}{2}\right]$ and increasing for all $i = \left[\frac{n+1}{2}\right], \cdots, n$. In order to prove (1.4) we need to prove that $V$ is decreasing for all $i = 1, \cdots, \left[\frac{n+1}{2}\right]$ and increasing for all $i = \left[\frac{n+1}{2}\right], \cdots, n$. Let $1 \leq i \leq \left[\frac{n+1}{2}\right]$, we denote by $j = n+1-i$ $\left(\left[\frac{n+1}{2}\right] \leq j \leq n\right)$. Then

$$\begin{aligned} V_i - V_{i+1} &= (b_i + b_{n+1-i}) - (b_{i+1} + b_{n-i}) \\ &= (b_i - b_{i+1}) + (b_{n+1-i} - b_{n-i}) \\ &= (b_i - b_{i+1}) + (b_j - b_{j-1}) \geq 0 \end{aligned} \tag{3.15}$$

because $b$ is decreasing for all $i = 1, \cdots, \left[\frac{n+1}{2}\right]$ and increasing for all $i = \left[\frac{n+1}{2}\right], \cdots, n$. By the same method we can prove easily that $V$ is increasing for all $i = \left[\frac{n+1}{2}\right], \cdots, n$. Then we have $U$ and $V$ having the same direction of monotony, and by applying Theorem A with $p = (p_1, p_2, \cdots, p_n)$ is a positive sequence symmetric about $\left[\frac{n+1}{2}\right]$, we obtain inequality (1.4).

**Proof of Theorem 1.4** Suppose that $a = (a_1, a_2, \cdots, a_n)$ is a convex sequence. By applying Lemma 2.1 for the sequence $v_k = a_k + a_{n+1-k}$ we obtain the following inequalities

$$v_N \leq v_k \leq v_1, \text{ for all } k = 1, \cdots, N \tag{3.16}$$

and

$$v_N \leq v_k \leq v_n, \text{ for all } k = N, \cdots, n. \tag{3.17}$$

By (3.16) and (3.17) we deduce that

$$(a_N + a_{n+1-N}) \leq a_k + a_{n+1-k} \leq (a_1 + a_n), \; k = 1, \cdots, n. \tag{3.18}$$

Multiplying inequalities (3.18) by $p_k$, we obtain for all $k = 1, \cdots, n$

$$(a_N + a_{n+1-N}) p_k \leq (a_k + a_{n+1-k}) p_k \leq (a_1 + a_n) p_k, \; k = 1, \cdots, n$$

which implies

$$(a_N + a_{n+1-N}) \left(\sum_{k=1}^{n} p_k\right) \leq \sum_{k=1}^{n} p_k a_k \leq (a_1 + a_n) \left(\sum_{k=1}^{n} p_k\right).$$

For the case of concave sequence we use similar proof.

# 4 Some Applications

In 1978, S. Haber [4] proved the following inequality:

**Theorem B** *Let $a$ and $b$ be non negative real numbers, then for every $n \geq 0$, we have*

$$\frac{1}{n+1}\left(a^n + a^{n-1}b + \cdots + b^n\right) \geq \left(\frac{a+b}{2}\right)^n.$$

Many authors are interested by this inequality (see $[1,4,8]$). It's easy to show that

$$x_k = a^{n-k}b^k \ \ (a \geq 0, b \geq 0) \ \ (k = 0, 1, \cdots, n)$$

is a convex sequence. Then by Theorem 1.4, we have for $x_k = a^{n-k}b^k$ $(a \geq 0, b \geq 0)$ $(k = 0, 1, \cdots, n)$ and $p = (1, 1, \cdots, 1)$

$$\frac{1}{n+1}\sum_{k=0}^{n} x_k \leq \frac{x_0 + x_n}{2},$$

hence

$$\frac{1}{n+1}\left(a^n + a^{n-1}b + \cdots + b^n\right) \leq \frac{a^n + b^n}{2}$$

which is the upper bound of Haber inequality, and we can state:

**Theorem 4.1** *Let $a$ and $b$ be non negative real numbers, then for every $n \geq 0$, we have*

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\left(a^n + a^{n-1}b + \cdots + b^n\right) \leq \frac{a^n + b^n}{2}.$$

A. McD. Mercer generalized Haber inequality for convex sequences and obtained the following result:

**Theorem C** ([8]) *Let $\{u\}_{i=0}^{n}$ be convex sequence of real numbers. Then*

$$\frac{1}{n+1}\sum_{i=0}^{n} u_i \geq \frac{1}{2^n}\sum_{i=0}^{n} C_n^i u_i.$$

In this section we prove that Mercer inequality can be deduced by Theorem 1.3. It's clear that the symmetric sequence about $\left[\frac{n}{2}\right]$

$$v_i = \frac{1}{n+1} - \frac{C_n^i}{2^n}$$

is decreasing for $i = 0, \cdots, \left[\frac{n}{2}\right]$ and increasing for $i = \left[\frac{n}{2}\right], \cdots, n$. Then by applying Theorem 1.3 for the sequences $u_i$, $v_i$ $(i = 0, \cdots, n)$ (where $v_i$ is a convex sequence) and $p = (1, 1, \cdots, 1)$ we obtain

$$\sum_{i=0}^{n} u_i v_i \geq \frac{1}{n+1} \left( \sum_{i=0}^{n} u_i \right) \left( \sum_{i=0}^{n} v_i \right),$$

and since $\sum\limits_{i=0}^{n} v_i = 0$, we obtain

$$\frac{1}{n+1} \sum_{i=0}^{n} u_i - \frac{1}{2^n} \sum_{i=0}^{n} C_n^i u_i \geq 0.$$

**Theorem 4.2** *Let $(a_i)_{i \in \mathbb{N}}$ be a convex and symmetric sequence of real numbers such that*

$$\sum_{i=0}^{n} a_i > 0.$$

*Then the polynomial*

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

*don't have any non negative zero.*

*Proof.* Suppose that $x \geq 0$. It's clear that $b_i = x^i$ $(i = 1, 2, \cdots, n)$ is a convex sequence for $x \geq 0$. Then by applying (1.5) we obtain for $x \neq 1$

$$P_n(x) = \sum_{i=0}^{n} a_i x^i \geq \frac{1}{n+1} \sum_{i=0}^{n} a_i \sum_{i=0}^{n} x^i$$

$$= \left( \frac{1}{n+1} \right) \left( \frac{1 - x^{n+1}}{1 - x} \right) \sum_{i=0}^{n} a_i > 0.$$

For $x = 1$ the result is trivial. This completes the proof of Theorem 4.2.

**Remark 4.1** Putting $p = (1, 1, \cdots, 1)$ in Theorem 1.4, it's clear that we have equality in Theorem 1.4 if and only if that $a = (a_1, \cdots, a_n)$ is arithmetic sequence ( i. e., $a_{i+2} + a_i = 2a_{i+1}$ for all $i = 1, \cdots, n-2$) and (1.8) become

$$\sum_{i=1}^{n} a_i = \frac{n}{2} (a_1 + a_n),$$

which is the sum of $n$ terms of arithmetic sequence.

# 5 Open problem

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [10, Chapter X] established the following discrete version of Grüss inequality:

**Theorem D** *Let* $a = (a_1, a_2, \cdots, a_n), b = (b_1, b_2, \cdots, b_n)$ *be two* $n$*-tuples of real numbers such that* $r \leq a_i \leq R$ *and* $s \leq b_i \leq S$ *for* $i = 1, 2, \cdots, n$. *Then one has*

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n^2}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)\right| \leq$$

$$\frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s),$$

*where* $[x]$ *denotes the integer part of* $x$, $x \in \mathbb{R}$.

The following question arises: *Can we obtain an analogue result for convex* $n$*-tuples* $a = (a_1, a_2, \cdots, a_n)$ *and* $b = (b_1, b_2, \cdots, b_n)$?

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