Int. J. Open Problems Comput. Math., Vol. 5, No. 3, September, 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

Ultra Linear Continuous Functionals and Ultra Generalized Complex Numbers In The Ultra Generalized Function Spaces

Sabra Ramadan
Departement of Mathematics
Faculty of Science
Jazan University
e-mail:rmsabra@jazanu.edu.sa

Abstract

To study mathematical models in the Ultra Generalized Function Space $L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ constructed in [6]it is important to define some tools like Ultra Generalized Complex Numbers and Ultra Generalized Functionals . In this paper , the Ultra Generalized Complex Numbers $C^*_\alpha = K^*(C)/I^*(C)$, and the Ultra Generalized Linear continuous Functionals in the Space $L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ are defined. Their important properties are also proved.

Key words: New Generalized Function Space, Rome- Helfand- Shilov Spaces.

1 Introduction

In [6] the Ultra Generalized Function Space $L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ were defined in the following way: if α and β are nonnegative real numbers and $k, q \in \mathbb{N}$, define the following sets [2]:

$S^\beta_\alpha = \{f \in C^\infty(R) : \exists\ A > 0, \exists\ B > 0,\ \forall k\ \forall q\ \exists\ C > 0$
such that $|x^k f^{(q)}(x)| \le C A^k B^q K^{k\alpha} q^{q\beta}\}$

If $\alpha > 0$ and $\beta = \alpha$, then the Space $S^\alpha_\alpha(\mathbb{R})$ is said to be Rome-Helfand-Shilov Space.

The Topology in $S^\alpha_\alpha(\mathbb{R})$ is defined by the system of semi norms in the following way :

$$p_{n,l} = \sup_{\substack{k \le n \\ m \le l}} q_{k,m}$$

where

$$q_{k,m} = \sup_{x \in \mathbb{R}} \frac{x^k f^{(m)}(x)}{A^k B^m k^{\alpha k} m^{\alpha m}}$$

The following theorem is true see [3,6]

Theorem 1.1 If $f, g \in S_\alpha^\alpha(\mathbb{R})$, then for each n, l there is a constant $C_{n,l} > 0$ such that $p_{n,l}(fg) \le C_{n,l}\ p_{n,l}(f)\ p_{n,l}(g), \forall f, g \in S_\alpha^\alpha(\mathbb{R})$

Now , let X be separated complete locally convex algebra [1] with topology defined by the family of semi norms $P_{i\epsilon I}$ such that for each $i \in I$ there is $j \in I$ and a constant $C_i > 0$ for which

$$p_i(xy) \le C_i\ p_j(x)\ p_j(y)\ \ \forall\ x, y \in X \quad (*)$$

if we denote by $G(X)$ the set of all possible sequences (x_k) in X , then $G(X)$ is an algebra with operations of coordinate wise multiplication.
Let $\alpha > 1$ be positive real number , define the following sets [6]:

$$G_\alpha(X) = \{x = (x_k) \in G(X) : \exists\ m,\ \forall\ i \in I,\ \exists C_i > 0,\ p_i(x_k) \le C_i\ exp(mk^{\frac{1}{\alpha}}),\ \forall k\}$$

$$N_\alpha(X) = \{x = (x_k) \in G(X) : \exists\ m,\ \forall\ i \in I,\ \exists C_i > 0,\ p_i(x_k) \le C_i\ exp(-mk^{\frac{1}{\alpha}}),\ \forall k\}$$

Theorem 1.2 The space $G_\alpha(X)$ is a subalgebra of the Algebra $G(X)$, and $N_\alpha(X)$ is an ideal of $G_\alpha(X)$.

Define the Ultra space $L_\alpha(X)$ as a factor space $L_\alpha(X) = G_\alpha(X)/N_\alpha(X)$. Since, $S_\alpha^\alpha(\mathbb{R})$ is complete separated locally convex algebra and $p_{n,l}$ satisfy $(*)$, then the Ultra Generalized Functions Space is defined in the following way:

$$L_\alpha(S_\alpha^\alpha(\mathbb{R})) = G_\alpha(S_\alpha^\alpha(\mathbb{R}))/N_\alpha(S_\alpha^\alpha(\mathbb{R}))$$

The embedding of the spaces $S_\alpha^\alpha(\mathbb{R})$ and $[S_\alpha^\alpha(\mathbb{R})]'$ in to the Algebra $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ have been defined [6].Therefore , we can write $S_\alpha^\alpha(\mathbb{R}) \subset L_\alpha(S_\alpha^\alpha(\mathbb{R})$, $[S_\alpha^\alpha(\mathbb{R})]' \subset L_\alpha(S_\alpha^\alpha(\mathbb{R})$.

2 Ultra Generalized Complex Numbers

Now our aim is to construct tools in the space $L_\alpha(S_\alpha^\alpha(\mathbb{R})$. For example we need Ultra Generalized Numbers C^* to study mathematical models as Cauchy's

$$\begin{cases} Df = fg & f(0) = z^* \\ f(0) = z^* & u, v \in L_\alpha(S_\alpha^\alpha(\mathbb{R})) \\ u, v \in L_\alpha(S_\alpha^\alpha(\mathbb{R})) \end{cases}$$

or

$$\begin{cases} Df = \delta^n f & f(a) = b \\ f(a) = b & a, b \in C^* \\ f \in L_\alpha(S_\alpha^\alpha(\mathbb{R})) \end{cases}$$

We define the Ultra Generalized Complex Numbers corresponding to the Space of the Ultra Generalized Functions $L_\alpha(S_\alpha^\alpha(\mathbb{R})$ in the following way:

let $K(C)$ be the set of all sequences of complex numbers . Define $K^*(C)$ as the set of all sequences $(z_k) \in K(C)$ such that there is a natural numbers $m \in N$ and a constant $C > 0$, such that $| z_k | \leq C.exp(mk^{\frac{1}{\alpha}})$, for each k. Define $I^*(C)$ as the set of all sequences $(\eta_k) \in K(C)$ such that for each $m \in N$ and for each k there is a constant $d > 0$ such that $| \eta_k | \leq d.exp(-mk^{\frac{1}{\alpha}})$, for each k in the domain of the sequence (η_k).

Theorem 2.1

(a) The set $K^*(C)$ is an algebra

(b) The set $I^*(C)$ be an ideal in $K^*(C)$.

Proof. a) Suppose $z_1 = (z_k), z_2 = (z_k)'$ are elements in $K^*(C)$, then there are natural numbers m_1, m_2 and the constants $C_1 > 0, C_2 > 0$ such that $| z_k | \leq C_1.exp(m_1 k^{\frac{1}{\alpha}})$ and $| z'_k | \leq C_2.exp(m_2 k^{\frac{1}{\alpha}})$. Then, $| z_k z'_k | \leq C_1 C_2.exp((m_1 + m_2)k^{\frac{1}{\alpha}})$ and hence $z_1.z_2 \in K^*(C)$.

b) Now suppose that $z = (z_k) \in K^*(C)$, then there is a natural number m_1 and a constant $C > 0$ such that $| z_k | \leq C.exp(m_1 k^{\frac{1}{\alpha}})$, for each k. Now if $\eta = (\eta_k) \in I^*(C)$ then for each $m \in N$ there is a constant $d > 0$ such that $| \eta |=| \eta_k | \leq d.exp(-mk^{\frac{1}{\alpha}})$ for each k. Now consider $| \eta z |=| \eta_k z_k | \leq Cd.exp((m_1 - m)k^{\frac{1}{\alpha}})$, that is $z\eta \in I^*(C)$.

Theorem 2.2

(a) If $h = (h_k) \in G_\alpha(S^\alpha_\alpha(R))$ and $\mu_0 \in R$, then $h(\mu_0) = (h_k(\mu_0)) \in K^*(C)$

(b) If $h = (h_k) \in N_\alpha(S^\alpha_\alpha(R))$ and $\mu_0 \in R$, then $\eta(\mu_0) = (\eta_k(\mu_0)) \in I^*(C)$

Proof. The theorem is proved by using definitions of $G_\alpha(S^\alpha_\alpha(R)$, $N_\alpha(S^\alpha_\alpha(R))$, $K^*(C)$, and $I^*(C)$

Definition 2.1 The Space of Ultra Generalized Complex Numbers is defined as a factor Algebras

$$C^*_\alpha = K^*(C)/I^*(C).$$

The definition of C^*_α and theorem 2.2 play important role when we study the models like Cauchy's in the Space of Ultra Generalized Functions $L_\alpha(S^\alpha_\alpha(\mathbb{R}))$.

Moreover , we define embedding of the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{R}$ into the Space of Ultra Generalized Complex Numbers C^*_α by the following

$$\begin{aligned} j_1 &: x \in R \to (x_k + 0i) \in C^*_\alpha, \text{where } x_k = x \ \forall k \\ j_2 &: z \in C \to (z_k) \in C^*_\alpha, \text{where } z_k = z \ \forall k \end{aligned}$$

3 Ultra Linear Continuous Functionals in $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$

Let $A : S_\alpha^\alpha(\mathbb{R}) \to S_\alpha^\alpha(\mathbb{R})$ be a linear continuous operator , then [1] for each $i \in I$ there exists j and a constant $C_i > 0$ such that $p_i(A(\varphi(x))) \leq C_i p_j(\varphi(x)), \forall \varphi \in S_\alpha^\alpha(R)$. The operator A is lifted coordinate wise to a map which we denote by $A^* : G(S_\alpha^\alpha(\mathbb{R})) \to G(S_\alpha^\alpha(\mathbb{R}))$.

Theorem 3.1

(a) $A^*[G_\alpha(S_\alpha^\alpha(\mathbb{R}))] \subset G_\alpha(S_\alpha^\alpha(\mathbb{R}))$

(b) $A^*[N_\alpha(S_\alpha^\alpha(\mathbb{R}))] \subset N_\alpha(S_\alpha^\alpha(\mathbb{R}))$.

Proof. The proof of this theorem follows from the definition of sets $G_\alpha(S_\alpha^\alpha(\mathbb{R}))$, $N_\alpha(S_\alpha^\alpha(\mathbb{R}))$, and by using the continuity of the operator A.

Now the operator A can be lifted to a map which we will denote by

$$A_\alpha^* : L_\alpha(S_\alpha^\alpha(\mathbb{R})) \to L_\alpha(S_\alpha^\alpha(\mathbb{R}))$$

Theorem 3.2

(a) The Operator A_α^* is independent on a representative

(b) if $A : S_\alpha^\alpha(\mathbb{R}) \to S_\alpha^\alpha(\mathbb{R})$ is an isomorphism of $S_\alpha^\alpha(\mathbb{R})$, then $A_\alpha^* : L_\alpha(S_\alpha^\alpha(\mathbb{R})) \to L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ is an isomorphism of $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$

Proof. a) Let $f \in L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ and let (f_k) and (g_k) are two representatives of f, then

$$(f_k - g_k) \in N_\alpha(S_\alpha^\alpha(\mathbb{R})).$$

Since A^* is continuous, then

$$p_i[A_\alpha^*(f_k) - A_\alpha^*(g_k)] = p_i(A_\alpha^*(f_k - g_k)) \leq C_j p_j(f_k - g_k) \leq C_i C_j exp(-mk^{\frac{1}{\alpha}}), \forall k,$$

that is $[A^*_\alpha(f_k) - A^*_\alpha(g_k)] \in N_\alpha(S^\alpha_\alpha(\mathbb{R}))$.

b) The proof follows immediately from the definition of A^*_α and by the fact that $A : S^\alpha_\alpha(\mathbb{R}) \to S^\alpha_\alpha(\mathbb{R})$ is an isomorphism of $S^\alpha_\alpha(\mathbb{R})$

Now , if $h : S^\alpha_\alpha(\mathbb{R}) \to C$ be linear continuous functional , then it is lifted coordinate wise to $h : G(S^\alpha_\alpha(\mathbb{R})) \to C$ and $h^*_\alpha : L_\alpha(S^\alpha_\alpha(\mathbb{R})) \to C$ and the functional h^*_α is well defined by virtue of the following results.

Corollary 3.3

(a) $h^*_\alpha[G^*_\alpha(S^\alpha_\alpha(\mathbb{R}))] \subset K^*(C)$

(b) $h^*_\alpha[N^*_\alpha(S^\alpha_\alpha(\mathbb{R}))] \subset I^*(C)$,

(c) The functional h^*_α is independent on a representative.

Now we can define the Lebege's Integral on $L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ as a linear continuous functional by the following way: Let $K \subset \mathbb{R}$ be any compact set and $\varphi \in L_\alpha(S^\alpha_\alpha(\mathbb{R}))$, then define $\int_K \varphi(x)d\mu = \int_K \varphi_k(x)d\mu$, where, $\varphi_k(x)$ be any representative of φ.

Definition 3.1 The ultra generalized complex number $z^* = \int_K \varphi(x)d\mu = \int_K \varphi_k(x)d\mu$ is called the generalized integral of ultra generalized function $\varphi \in L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ over the compact K .

The generalized integral defined above preserve many properties of usual Lebege's integral defined in $S^\alpha_\alpha(\mathbb{R})$, for example, the following properties are preserved :

(1) $\int_K[\lambda(x) \pm \eta(x)]d\mu = \int_K \lambda(x)d\mu \pm \int_K \eta(x)d\mu$

(2) $\int_K a\lambda(x)d\mu = a\int_K \lambda(x)d\mu, \forall a \in C^*_\alpha$

4 Open Problem

How to define the Extended Laplace Transform in the Ultra Generalized Function Space $S^\alpha_\alpha(R)$

References

[1] A.B.Antonevich , Ya.V.Radyno , Functional Analysis and Integral Equations ,Minsk, 1984.

[2] Helfand Y.M.,G.E.Shilov,The Space of test and Generalized Functions,Moscow,1985.

[3] Radyno Ya.V., Ngo Fou Thchan, and R.Sabra , Fourier Transform in the Algebra of New Generalized Functions, Dokl.RAN,327(1992)20-24.

[4] Sabra Ramadan ,Linear and Bilinear Operations in the $L(E)$ Space,Comp.Sc.and Math.er.,8(1997)81-86.

[5] Sabra Ramadan,The New Generalized Functions Spaces $L(D(\mathbb{R}))$ and $L(Z(\mathbb{R}))$,Dirasat,25(1998)20-24.

[6] Sabra R.M., On Algebra of New Generalized Functions Space $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$,Dirasat,27(1999)10-14.

[7] Sabra Ramadan, One Algebra of New Generalized Functions ,J.Math. and Stat.,3(2007)15-16.

[8] Sabra Ramadan,Generalized Cauchy's Models and Generalized Integrals,J.Math. and Stat.,3(2007)21-23.

[9] Sabra Ramadan,The Extended Laplace Transform ,J.Math. and Stat.,3(2007)168–171.