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On β-expansions of unity for Perron power series

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Abstract

The aim of this paper is to prove that the stings of 0 *in the* β*-expansion of* 1 *exhibit a lacunarity bounded when* β *is a Perron power series over the finite field* $\mathbb{F}_q$ *with* $|\beta| > 1$.

Keywords: *formal power series,* β*-expansion, Perron series.*

1 Introduction

β-expansions of real numbers were introduced by A. Rényi [13]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors.
Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0, 1]$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\{0, 1, ..., [\beta]\}$ produced by the β-transformation $T_\beta : \ x \longrightarrow \ \beta x \pmod 1$ as follows :

$$\forall \ i \geq 1, \ x_i = [\beta T_\beta^{i-1}(x)], \ \textit{and thus } x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

An expansion is finite if $(x_i)_{i\geq 1}$ is eventually 0. A β-expansion is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic.

The β-expansion of 1 plays an important role in the study of the classification of algebraic numbers $\beta > 1$. Numbers β such that $d_\beta(1)$ is ultimately periodic are called Parry numbers and those such that $d_\beta(1)$ is finite are called simple Parry numbers. These families of numbers were introduced by Parry

[12] and its elements were initially called β-numbers and an easy argument implies that these elements are algebraic integer numbers. It is known that if β is a Pisot number (an algebraic integer whose conjugates have modulus <1), then β is a Parry number and it was proved in [5] that if β is a Salem number (an algebraic integer whose conjugates have modulus ≤ 1 and there exists at least one conjugate with modulus 1) of degree 4, then β is a Parry number. But, it is clair that there is not a full characterization of Parry numbers or simple Parry numbers.

In particular, Mahler [10], in an old result, has interested in the gaps between non-zero digits in $d_\beta(1)$ and he proved that if $\beta > 1$ is a algebraic number such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence in the following sense: There exists two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that :

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \cdots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \cdots$$

with $(s_n - m_n) \geq 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$, then

$$\limsup_{n\to+\infty}\left(\frac{s_n}{m_n}\right) < \infty.$$

Verger-Gaugry in [14] proved that the gaps in $d_\beta(1)$ are shown to exhibit a gappiness bounded using a version of Liouville's inequality which extends Mahler and Güting's approximation theorems and he obtained that if $\beta > 1$ is a algebraic number such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence, then

$$\limsup_{n\to+\infty}\left(\frac{s_n}{m_n}\right) \leq \frac{Log(M(\beta))}{Log(\beta)}. \qquad (1)$$

This result provides, in a natural way, a new classification of algebraic numbers $\beta > 1$. Allouche and Cosnard in [3] proved that there exists a smallest $q \in]1, 2[$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{+\infty} \delta_n q^{-n}$ with $\delta_n \in \{0, 1\}$. Furthermore, for this smallest q, the coefficient δ_n is equal to 0 (respectively, 1) if the sum of the binary digits of n is even (respectively, odd). This constant q is named Komornik-Loreti constant. Since the strings of zeros and 1's in the sequence δ_n are known and uniformly bounded, the constant q satisfies $\limsup_{n\to+\infty}(\frac{s_n}{m_n}) = 1$. However, authors in [3] have shown that q is a transcendental number.

A far reaching generalization of (1) with the same upper bound $\frac{Log(M(\beta))}{Log(\beta)}$ for a so-called Diophantine exponent of the sequence $d_\beta(1) = (a_i)_{i\geq 1}$ was obtained by Adamczewski and Bugeaud in [1]. Both in [1] and in the subsequent paper of Bugeaud [6] the main ingredient is Subspace Theorem.

Recently, Dubickas in [7] obtained an upper bound for two strings of consecutive zeros in $d_\beta(1)$ for rational β, unsing the theorem of Ridout, he proved that if $\beta = \frac{p}{q}$ satisfying $1 < \beta < 2$, then

$$\limsup_{n\to+\infty}(\frac{s_n}{m_n}) \leq \frac{Log(M(\beta))}{Log(\beta)} = \frac{Log(p)}{Log(\frac{p}{q})}.$$

In this paper, we consider an analogue of this concept in algebraic function over finite fields.

In section 2, We will define $\mathbb{F}_q((x^{-1}))$, the field of formal power series. Furthermore, we will provide the algebraic series as well as the analogues to Pisot and Salem numbers.

In section 3, we introduce the β-expansion algorithm for $\mathbb{F}_q((x^{-1}))$.

The last section is devoted to prove that if β is a Perron power series of algebraic degree $d \geq 2$ such that, $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence then the quotient of gaps in the string of 0 in the sequence $(a_i)_{i\geq 1}$ is bounded. This implies that if $d_\beta(1)$ has unbounded quotient of gaps so β has at last one conjugate $\widetilde{\beta}$ with $|\widetilde{\beta}| < |\beta|$.

2 Field of formal power series

Let p be a prime , q be a power of p, and let $\mathbb{F}_q$ be the finite field of q elements. By analogy with the real case , one can classically extend arithmetical results concerning the ring $\mathbf{Z}$ to the ring $\mathbb{F}_q[x]$ of polynomials with coefficient in $\mathbb{F}_q$ and the field $\mathbb{F}_q(x)$ to the field of rational functions.
Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series of the form :

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q$$

where

$$l = deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{for } f \neq 0; \\ -\infty & \text{for } f = 0. \end{cases}$$

Define the absolute value

$$|f| = \begin{cases} q^{deg\ f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases}$$

Since $|.|$ is not archimedean, $|.|$ fulfills the strict triangle inequality

$$\begin{aligned} |f+g| &\leq \max\,(|f|,|g|) && \text{and} \\ |f+g| &= \max\,(|f|,|g|) && \text{if } |f| \neq |g|. \end{aligned}$$

Let $f \in \mathbb{F}_q((x^{-1}))$, define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k x^k$ where the empty sum, as usual, is defined to be zero.
Therefore $[f] \in \mathbb{F}_q[x]$ and $(f - [f]) = \{f\}$ is in the unit disk $D(0,1)$ for all $f \in \mathbb{F}_q((x^{-1}))$.

Let $f \in \mathbb{F}_q((x^{-1}))$, we say that f is an algebraic series over $\mathbb{F}_q[x]$ if it is root of a polynomial $P \in \mathbb{F}_q[x][y]$ and it is called an integer algebraic series when P is unit irreducible polynomial. Note that if f is not an algebraic series, we say that f is a transcendental power series.

Theorem 2.1 [*11*] *Let $\mathbb{K}$ be complete with respect to $|.|$ and $\mathbb{L}/\mathbb{K}$ be an algebraic extension of degree n. Then $|.|$ has a unique extension to $\mathbb{L}$ defined by $|a| = \sqrt[n]{|\mathbf{N}_{\mathbb{L}/\mathbb{K}}(a)|}$, and $\mathbb{L}$ is complete with respect to this extension.*

We apply Proposition 2.1 to algebraic extensions of $\mathbb{F}_q((x^{-1}))$.
Therefore, since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, every algebraic element over $\mathbb{F}_q[x]$ can be valuated. However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed, such an element do not necessarily expressed as a power series. For a full characterization of the algebraic closure of $\mathbb{F}_q[x]$, we refer to K. S. Kedlaya [9].

3 β-expansions in $\mathbb{F}_q((x^{-1}))$

Similarly to the classical β-expansions for real numbers, we define the β-expansions for power series. For this, let β, $f \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$ and $|f| < 1$. A representation in base β (or β-representation) of f is an infinite sequence $(a_i)_{i\geq 1}$ in $\mathbb{F}_q[x]$, such

$$f = \sum_{i\geq 1} \frac{a_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f in base β, noted $d_\beta(f)$, which is obtained by using the β-transformation T_β in the unit disk which is given by $T_\beta(f) = \beta f - [\beta f]$. Then $d_\beta(f) = (a_i)_{i\geq 1}$ where

$$a_i = [\beta T_\beta^{i-1}(f)] \qquad (*)$$

The following theorem provides an analogue to the Parry condition in the real case.

Theorem 3.1 [*8*] *A β-representation $(a_i)_{i\geq 1}$ is the β-expansion of f in the unit disk $D(0,1)$ if and only if $|a_i| < |\beta|$ for all $i \geq 1$.*

Now let $f \in \mathbb{F}_q((x^{-1}))$ be an element with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}^*$ such that $|\beta|^{k-1} \leq |f| < |\beta|^k$, so $|\frac{f}{\beta^k}| < 1$ and we can represent f by shifting $d_\beta(\frac{f}{\beta^k})$ by k digits to the left. Therefore, if $d_\beta(f) = 0.a_1a_2a_3...$ then $d_\beta(\beta f) = a_1.a_2a_3...$
Note that the β-expansion is finite if $(a_i)_{i\geq 1}$ is eventually 0, it is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $a_k = a_{k+p}$ holds for all $k \geq m$; if $a_k = a_{k+p}$ holds for all $k \geq 1$, then it is purely periodic.

In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if f and $g \in \mathbb{F}_q((x^{-1}))$, we have $d_\beta(f+g) = d_\beta(f) + d_\beta(g)$ digitwise.

It is clair that the natural β-expansion of 1 is 1.0^∞. However, if we replace f by 1 in the previous algorithm $(*)$, we get a sequence $(a_i)_{i\geq 1}$ which plays an important role in the study of the classification of algebraic series β (in this case $|a_1| = |\beta|$). In the sequel, $d_\beta(1)$ mains the sequence obtained by $(*)$.

This following theorem characterizes the sequence of polynomial $(a_i)_{i\geq 1}$ able to be a certain β-expansion of 1.

Theorem 3.2 [8] *For all sequence of polynomial $(a_i)_{i\geq 1}$ such that $deg(a_1) > deg(a_i)$ for all $i \geq 2$, there exist $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$ such that $d_\beta(1) = (a_i)_{i\geq 1}$.*

Series β such that $d_\beta(1)$ is ultimately periodic are called β-series and those such that $d_\beta(1)$ is finite are called simple β-series and also an easy argument implies that these elements are algebraic integer series.
An element $\beta = \beta^{(1)} \in \mathbb{F}_q((x^{-1}))$ is said to be Pisot if it is an algebraic integer over $\mathbb{F}_q[x]$, $|\beta| > 1$ and $|\beta^{(j)}| < 1$ holds for all its conjugates $\beta^{(j)}$. An element $\beta = \beta^{(1)} \in \mathbb{F}_q((x^{-1}))$ is said to be Salem if it is an algebraic integer over $\mathbb{F}_q[x]$, $|\beta| > 1$ and $|\beta^{(j)}| \leq 1$ and there exists at least one conjugate $\beta^{(k)}$ such that $|\beta^{(k)}| = 1$. An element $\beta = \beta^{(1)} \in \mathbb{F}_q((x^{-1}))$ is said to be Perron if it is an algebraic integer over $\mathbb{F}_q[x]$, $|\beta| > 1$ and $|\beta^{(j)}| \leq |\beta|$.

P. Bateman and A. L. Duquette [4] characterized the Pisot and Salem power series:

Theorem 3.3 *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and*

$$P(y) = y^n - A_1y^{n-1} - \cdots - A_n, \quad A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then

(i) β is a Pisot series if and only if $|A_1| > \max_{2\leq i\leq n} |A_i|$.

(ii) β is a Salem series if and only if $|A_1| = \max_{2\leq i\leq n} |A_i|$.

Hbaib and Mkaouar [8] have proved that the simple β-series are the Pisot series and the β-series are the Salem series.

4 Results

In the previous section, we have seen that the β-expansion of 1 play a crucial role in the study of the algebraicity of power series β. This motivates the following problem first investigated in [8]. As a first result, we establish the following theorem:

Theorem 4.1 *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be a Perron series of algebraic degree $d \geq 2$ such that $d_\beta(1) = (a_i)_{i\geq 1}$ is an infinite and lacunary sequence in the following sense:*
There exists two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that:

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \cdots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \cdots$$

with $(s_n - m_n) \geq 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$.
Then,

$$\limsup_{n\to+\infty} (\frac{s_n}{m_n}) \leq d$$

In order to prove this result, we need these Lemmas:

Lemma 4.2 *Let $Q \in F_q[x][y]$ and $F(y^{(1)}, y^{(2)}, ..., y^{(d)}) = Q(y^{(1)})Q(y^{(2)})...Q(y^{(d)})$. Then, there exists a polynomial T with d variables and coefficients in $\mathbf{F}_q[x]$ such that*

$$F(y^{(1)}, y^{(2)}, ..., y^{(d)}) = T(\sigma_1, \sigma_2, ..., \sigma_d)$$

where:

$$\begin{cases} \sigma_1 = \sum_{i=1}^{d} y^{(i)} \\ \sigma_2 = \sum_{1\leq i<j\leq d} y^{(i)} y^{(j)} \\ \sigma_3 = \sum_{1\leq i<j<k\leq d} y^{(i)} y^{(j)} y^{(k)} \\ \vdots \\ \sigma_d = y^{(1)} y^{(2)} ... y^{(d)} \end{cases}$$

Note moreover that the total degree of T is lower or equal to the degree of Q.
Proof:
Let $\alpha_1 = \deg(Q)$.
Among terms containing $(y^{(1)})^{\alpha_1}$, we designate by α_2 the maximal exponent of $y^{(2)}$.
Among terms containing $(y^{(1)})^{\alpha_1}(y^{(2)})^{\alpha_2}$, we designate by α_3 the maximal exponent of $y^{(3)}$ and so on. We define ,thus, the dominant term of the forms $A(y^{(1)})^{\alpha_1}(y^{(2)})^{\alpha_2}...(y^{(d)})^{\alpha_d}$. Since F is symmetrical, we have $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$

(indeed, F contains all terms $A(y^{(\pi(1))})^{\alpha_1}(y^{(\pi(2))})^{\alpha_2}...(y^{(\pi(d))})^{\alpha_d}$ where π is a permutation of $\{1,2,...,d\}$).
We remark ,thus, that the dominant term of $\sigma_1^{\alpha_1-\alpha_2}...\sigma_{d-1}^{\alpha_{d-1}-\alpha_d}\sigma_d^{\alpha_1}$ is $(y^{(1)})^{\alpha_1}(y^{(2)})^{\alpha_2}...(y^{(d)})^{\alpha_d}$, hence by calculating $F(y^{(1)},y^{(2)},...,y^{(d)})-A\sigma_1^{\alpha_1-\alpha_2}...\sigma_{d-1}^{\alpha_{d-1}-\alpha_d}\sigma_d^{\alpha_d}$, we eliminate of F all terms of the forms $A(y^{(\pi(1))})^{\alpha_1}(y^{(\pi(2))})^{\alpha_2}...(y^{(\pi(d))})^{\alpha_d}$.
Finally we get the result by induction.

Lemma 4.3 *Let β be a Perron series with minimal polynomial $P_\beta(y) = y^d + A_{d-1}y^{d-1} + \cdots + A_0$, where $A_i \in \mathbb{F}_q[x]$ for all $0 \le i \le d$.*
Let $K(y) = B_m y^m + B_{m-1}y^{m-1} + \cdots + B_0$ where $B_i \in \mathbb{F}_q[x]$ for all $0 \le i \le m$ and $m \ge d$. Then,

$$|K(\beta)| \ge \frac{1}{H(K)^{d-1}|\beta|^{(d-1)m}},$$

where $H(K)$ is the height of K defined by $H(K) = \max_{0\le i\le m} |B_i|$.

Proof:
Let $K(y) = B_m y^m + B_{m-1}y^{m-1} + \cdots + B_0$ a polynomial of degree $m \ge d$.
Since $\beta^d = -A_{d-1}\beta^{d-1} - \cdots - A_0$, there exist $C_{(i,s)} \in \mathbb{F}_q[x]$ such that

$$\beta^{d+s-1} = C_{(d-1,s)}\beta^{d-1} + \cdots + C_{(0,s)} \text{ for all } s \ge 1.$$

Let $\beta = \beta^{(1)}$ and $\beta^{(2)}, \ldots, \beta^{(d)}$ be the conjugates of β. For $s = m-d+1$, there exist $D_i \in \mathbb{F}_q[x]$ such that

$$K(\beta^{(j)}) = D_{d-1}(\beta^{(j)})^{d-1} + \cdots + D_0 \text{ for all } 1 \le j \le d.$$

By Lemma 4.2, there exists a polynomial T with d variables and coefficients in $\mathbf{F}_q[x]$ such that

$$\prod_{j=1}^{d} K(\beta^{(j)}) = T(\sigma_1, \sigma_2, ..., \sigma_d)$$

with $|\sigma_i| = |\sum_{1\le j_1<j_2<...\le d} \beta^{(j_1)}\beta^{(j_2)}...\beta^{(j_i)}| = |A_{d-i}|$ for all $1 \le i \le d$, and the total degree of T is lower or equal to the degree of d which implies that

$$\prod_{j=1}^{d} K(\beta^{(j)}) \in \mathbb{F}_q[x].$$

We obtain

$$|K(\beta)| \ge \frac{1}{\prod_{j=2}^{d} |K(\beta^{(j)})|}.$$

However, for all $2 \le j \le d$, we have

$$\begin{array}{rcl} |K(\beta^{(j)})| & = & |B_m(\beta^{(j)})^m + B_{m-1}(\beta^{(j)})^{m-1} + \cdots + B_0| \\ & \le & H(K)|\beta^m| \end{array}$$

which implies that

$$|\prod_{j=2}^{d} K(\beta^{(j)})| \le H(K)^{d-1}|\beta|^{(d-1)m}$$

Finally, we have

$$|K(\beta)| \ge \frac{1}{H(K)^{d-1}|\beta|^{(d-1)m}}$$

Proof of Theorem 4.1.

We consider the polynomial

$$K_n(y) := -y^{m_n} + a_1 y^{m_n-1} + a_2 y^{m_n-2} + \cdots + a_{m_n}.$$

It is clair that $K_n(y)$ is a polynomial of degree m_n and $H(K_n) = |\beta|$.
Let now $P_\beta(y) = y^d + A_{d-1}y^{d-1} + \cdots + A_0$ be the minimal polynomial of β.
By lemma 4.3, we have

$$|K_n(\beta)| \ge \frac{1}{|\beta|^{(d-1)(m_n+1)}}. \qquad (1)$$

On the other hand, $K_n(\beta) = \beta^{m_n}(a_{s_n}\beta^{-s_n} + a_{s_{n+1}}\beta^{-s_{n+1}} + \cdots)$. Then,

$$|K_n(\beta)| \le \frac{|\beta|^{m_n}|\beta|}{|\beta|^{s_n}} = |\beta|^{m_n-s_n+1} \qquad (2)$$

Combining (1) and (2), we get

$$|\beta|^{s_n-m_n-1} < |\beta|^{2d-1}$$

which implies that

$$(s_n - d).deg(\beta) < (d-1)(m_n+1).deg(\beta).$$

So,

$$\frac{s_n}{m_n} - \frac{d}{m_n} < \frac{2d-1}{m_n}$$

Finally, we get that

$$\limsup_{n\to+\infty}(\frac{s_n}{m_n}) \le d$$

Corollary 4.4 *Let $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$ such that $d_\beta(1)$ is an infinite and lacunary sequence in the following sense: There exists two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that:*

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \cdots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \cdots$$

with $(s_n - m_n) \geq 2$, $a_{m_n} \neq 0$, $a_{s_n} \neq 0$ and $a_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$. If $\limsup\limits_{n\to+\infty}(\frac{s_n}{m_n}) = +\infty$ then β has at last one conjugate with $|\widetilde{\beta}| < |\beta|$.

5 Open Problem

Is it possible to generalize Theorem 4.1 for any algebraic series by introducing its Mahler measure in the analogue way of real case.

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