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On The Second Change of Rings Theorems for the Gorenstein Homological Dimensions

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Abstract

This paper is concerned with the second change of rings theorems for the Gorenstein homological dimensions. More precisely, we give the ultimate versions of these theorems relatively to these new invariants.

Keywords: *Gorenstein projective dimension, Gorenstein injective dimension, Gorenstein flat dimension, exact sequence, functor, resolution.*

1 Introduction

Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R-modules. If x is a central element of R, when no confusion is likely, R^* denotes the factor ring $\dfrac{R}{xR}$ and, for any R-module A, $Z(A)$ denotes the set of all zero-divisors of A.

The Gorenstein homological algebra has reached an advanced level since the pioneering works of M. Auslander and M. Bridger in [1, 2]. One of the central points of the theory is its ability to recognize Gorenstein rings. A Noetherian local commutative ring R is called Gorenstein if it is Cohen-Macaulay and has an irreducible parameter ideal. It is worth noting that classical homological algebra might be viewed as being based on projective modules. Whereas, in Gorenstein homological algebra, one replaces the projective modules with the class of Gorenstein projective modules.

Recall that an R-module M is said to be Gorenstein projective (G-projective for short), if there exists an exact sequence $\mathbf{P}$ of projective modules, called a complete projective resolution, with

$$\mathbf{P} = \ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \ldots$$

such that the complex $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for each projective module Q (see [6, 12]), and $M = \mathrm{Im}(P_0 \to P_{-1})$. Also, the Gorenstein projective dimension of an R-module M ($\mathrm{Gpd}_R(M)$ for short) is the least positive integer n such that there exists an exact sequence $0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ with the G_i are Gorenstein projective modules, $+\infty$ otherwise. The Gorenstein projective dimension is a refinement of the classical projective dimension of a module M, in the sense that $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ with equality when $\mathrm{pd}_R(M)$ is finite. It was introduced by Enochs and Jenda in [7] to extend the notion of the G-dimension defined by Auslander and Bridger. The Gorenstein injective dimension is defined dually. Moreover, an R-module M is said to be Gorenstein flat (G-flat for short), if there exists an exact sequence $\mathbf{F}$ of flat modules, called a complete flat resolution, with

$$\mathbf{F} = \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \ldots$$

such that the complex $I \otimes_R \mathbf{F}$ is exact for each injective right R-module I, and $M = \mathrm{Im}(F_0 \to F_{-1})$. The reader is kindly referred to [5, 6, 7, 8, 9, 10, 12] for basics and in-depth investigations on Gorenstein homological theory.

On the other hand, a recent paper of Bennis and Mahdou, [4], established the following second change of rings theorems for the Gorenstein projective dimension and the Gorenstein injective dimension:

Theorem A [4, Theorem 3.1]. *Let x be a central non zero-divisor in R. If M is an R-module such that x is a non zero-divisor on M, then*

$$\mathrm{Gpd}_{R^*}\Big(\frac{M}{xM}\Big) \leq \mathrm{Gpd}_R(M).$$

Theorem B [4, Theorem 3.4]. *Let x be a central non zero-divisor in R. If M is an R-module such that x is a non zero-divisor on M, then*

$$1 + \mathrm{Gid}_{R^*}\Big(\frac{M}{xM}\Big) \leq \mathrm{Gid}_R(M)$$

except when M is a Gorenstein injective R-module (in which case $M = xM$).

The aim of this paper is, on the one hand, to generalize Theorem A and Theorem B dropping the hypothesis "$x \notin Z(M)$" and involving instead the submodule ${}_xM := \{z \in M : xz = 0\}$ of M annihilated by x, and to give the second change of rings theorem for the Gorenstein flat dimension as well as its ultimate version in the setting of a right coherent ring, on the other. Our approach allows to assign the dual result of Theorem A and Theorem B and to prove that each version of second change of rings theorem for the Gorenstein projective dimension has a counterpart in the case of the Gorenstein injective dimension. Our ultimate second change of rings theorem, Theorem 2.13, generalizes Theorem B by stating the following: *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not a Gorenstein injective R-module. Then*

$$1 + \mathrm{Gid}_{R^*}\Big(\frac{M}{xM}\Big) \leq \mathrm{Gid}_R(M) \text{ if and only if } \mathrm{Gid}_{R^*}({}_xM) - 1 \leq \mathrm{Gid}_R(M).$$

As to the Gorenstein projective dimension, we transfer to the Gorenstein case and give its ultimate form a version of the second change of rings theorem for the projective dimension, that is, [13, Exercise 1, p. 155]. In fact, we prove the following:

Theorem C. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module such that $M = xM$, that is, $R^* \otimes_R M = 0$.* Then

$$1 + \mathrm{Gpd}_{R^*}({}_xM) \leq \mathrm{Gpd}_R(M)$$

except when M is Gorenstein projective over R in which case ${}_xM = 0$.

Further, through Theorem 2.15, we give the ultimate version of Theorem C by proving the following: *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not Gorenstein projective over R. Then*

$$1 + \mathrm{Gpd}_{R^*}({}_xM) \leq \ \mathrm{Gpd}_R(M) \textit{ if and only if } \mathrm{Gpd}_{R^*}\Big(\frac{M}{xM}\Big) - 1 \leq \ \mathrm{Gpd}_R(M).$$

Section 3 deals with the Gorenstein flat dimension. Actually, we provide the second change of rings theorem for the Gorenstein flat dimension as well as the ultimate form of the Gorenstein flat version of Theorem 2.15 in the setting of a right coherent ring R.

2 Case of the Gorenstein projective dimension and Gorenstein injective dimension

This section aims at giving a general version of the second change of rings theorems for the Gorenstein projective dimension and the Gorenstein injective one, namely, Theorem A and Theorem B.

First, for the convenience of the reader, we catalog the following results from [12] and [4] which will be useful to prove our theorems.

Proposition 2.1 [12, Theorem 2.20]. *Let M be an R-module with finite Gorenstein projective dimension and let $n \geq 0$ be an integer. Then the following assertions are equivalent:*
1) $\mathrm{Gpd}_R(M) \leq n$;
2) $\mathrm{Ext}^i_R(M,P) = 0$ *for all $i > n$ and all projective R-modules P.*

Proposition 2.2 [12, Theorem 2.22]. *Let M be an R-module with finite Gorenstein injective dimension and let $n \geq 0$ be an integer. Then the following assertions are equivalent:*
1) $\mathrm{Gid}_R(M) \leq n$;
2) $\mathrm{Ext}^i_R(I,M) = 0$ *for all $i > n$ and all injective R-modules I.*

Proposition 2.3 [12, Theorem 2.24]. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. If any two of the modules M, M', M'' have finite Gorenstein projective dimension, then so has the third.*

Proposition 2.4 [12, Theorem 2.25]. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. If any two of the modules M, M', M'' have finite Gorenstein injective dimension, then so has the third.*

Lemma 2.5 [4, Lemma 2.4]. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. Then*

1) $\mathrm{Gpd}_R(A) \leq \max\{\mathrm{Gpd}_R(B), \mathrm{Gpd}_R(C)-1\}$ *with equality when* $\mathrm{Gpd}_R(B) \neq \mathrm{Gpd}_R(C)$.

2) $\mathrm{Gpd}_R(B) \leq \max\{\mathrm{Gpd}_R(A), \mathrm{Gpd}_R(C)\}$ *with equality when* $\mathrm{Gpd}_R(C) \neq \mathrm{Gpd}_R(A) + 1$.

3) $\mathrm{Gpd}_R(C) \leq \max\{\mathrm{Gpd}_R(B), \mathrm{Gpd}_R(A)+1\}$ *with equality when* $\mathrm{Gpd}_R(B) \neq \mathrm{Gpd}_R(A)$.

Lemma 2.6 [4, Lemma 2.5]. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. Then*

1) $\mathrm{Gid}_R(A) \leq \max\{\mathrm{Gid}_R(B), \mathrm{Gid}_R(C)+1\}$ *with equality when* $\mathrm{Gid}_R(B) \neq \mathrm{Gid}_R(C)$.

2) $\mathrm{Gid}_R(B) \leq \max\{\mathrm{Gid}_R(A), \mathrm{Gid}_R(C)\}$ *with equality when* $\mathrm{Gid}_R(A) \neq \mathrm{Gid}_R(C) + 1$.

3) $\mathrm{Gid}_R(C) \leq \max\{\mathrm{Gid}_R(B), \mathrm{Gid}_R(A)-1\}$ *with equality when* $\mathrm{Gid}_R(B) \neq \mathrm{Gid}_R(A)$.

Theorem 2.7 [4, Theorem 4.1 and Theorem 4.2]. *Let x be a central* *element of R such that $x \notin Z(R)$. Let M be an R-module such that $xM = 0$. Then*

$$\begin{cases} \mathrm{Gpd}_R(M) & = & 1 + \mathrm{Gpd}_{R^*}(M) \\ \\ \mathrm{Gid}_R(M) & = & 1 + \mathrm{Gid}_{R^*}(M) \end{cases}$$

In particular, $\mathrm{Gpd}_R(M)$ *and* $\mathrm{Gpd}_{R^*}(M)$ *(resp.,* $\mathrm{Gid}_R(M)$ *and* $\mathrm{Gid}_{R^*}(M)$*) are simultaneously finite.*

Next, we establish the following results which will be useful in the sequel.

Lemma 2.8. *Let M be an R-module and x a central element of R such that $x \notin Z(R)$. Then* $\mathrm{Hom}_R(R^*, M) \cong {}_xM$, $\mathrm{Ext}^1_R(R^*, M) \cong \dfrac{M}{xM}$ *and* $\mathrm{Tor}^R_1(R^*, M) \cong {}_xM$.

Proof. As $x \notin Z(R)$, the following sequence is exact $0 \longrightarrow R \xrightarrow{x} R \longrightarrow R^* \longrightarrow 0$. Applying the functor $\mathrm{Hom}_R(-, M)$, we get the next exact sequence

$$0 \to \mathrm{Hom}_R(R^*, M) \to \mathrm{Hom}_R(R, M) \xrightarrow{x} \mathrm{Hom}_R(R, M) \to \mathrm{Ext}^1_R(R^*, M) \to 0.$$

Since $\mathrm{Hom}_R(R, M) \cong M$, this latter sequence turns out to be the following exact one

$$0 \to \mathrm{Hom}_R(R^*, M) \to M \xrightarrow{x} M \to \mathrm{Ext}^1_R(R^*, M) \to 0.$$

Then the first two isomorphisms easily follows. Applying the functor $\otimes_R M$ instead of $\mathrm{Hom}_R(-, M)$ to the initial exact sequence yields the last isomorphism. $\Box$

Lemma 2.9. *Let x be a central element of R such that $x \notin Z(R)$. Let $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{\alpha} M \longrightarrow 0$ be an exact sequence of R-modules with $x \notin Z(E)$. Then the natural sequence of R^*-modules*

$$0 \longrightarrow {}_xM \longrightarrow \frac{N}{xN} \xrightarrow{\bar{i}} \frac{E}{xE} \xrightarrow{\bar{\alpha}} \frac{M}{xM} \longrightarrow 0$$

is exact.

Proof. Tensoring with R^* the sequence $0 \longrightarrow N \xrightarrow{i} E \xrightarrow{\alpha} M \longrightarrow 0$ yields, by Lemma 2.8, the exact sequence of R^*-modules

$$0 = {}_xE = \mathrm{Tor}^R_1(R^*, E) \to {}_xM = \mathrm{Tor}^R_1(R^*, M) \to \frac{N}{xM} \xrightarrow{\bar{i}} \frac{E}{xE} \xrightarrow{\bar{\alpha}} \frac{M}{xM} \to 0$$

yielding the desired exact sequence. $\Box$

Lemma 2.10. *Let x be a central element of R such that $x \notin Z(R)$. Let $0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\alpha} N \longrightarrow 0$ be an exact sequence of R-modules such that $E = xE$. Then the natural sequence of R^*-modules*

$$0 \longrightarrow {}_xM \xrightarrow{xi} {}_xE \xrightarrow{x\alpha} {}_xN \longrightarrow \frac{M}{xM} \longrightarrow 0$$

is exact.

Proof. Applying the functor $\mathrm{Hom}_R(R^*, -)$ to the sequence $0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\alpha} N \longrightarrow 0$, we get the exact sequence

$$0 \longrightarrow {}_xM \xrightarrow{xi} {}_xE \xrightarrow{x\alpha} {}_xN \longrightarrow \mathrm{Ext}^1_R(R^*, M) \longrightarrow \mathrm{Ext}^1_R(R^*, E).$$

As, by Lemma 2.8, $\mathrm{Ext}^1_R(R^*, M) \cong \dfrac{M}{xM}$ and $\mathrm{Ext}^1_R(R^*, E) \cong \dfrac{E}{xE} = 0$, we obtain the desired exact sequence. □

In light of Lemma 2.8, a new interpretation of the hypotheses of Theorem A, that is, $x \notin Z(M)$, arises as $({}_xM :=)\ \mathrm{Hom}_R(R^*, M) = 0$ allowing to state Theorem A in the following way: *Let x be a central non zero-divisor in R. If M is an R-module such that* $\mathrm{Hom}_R(R^*, M) = 0$, *then*

$$(\ \mathrm{Gpd}_{R^*}(R^* \otimes_R M) =)\ \mathrm{Gpd}_{R^*}\Big(\frac{M}{xM}\Big) \leq\ \mathrm{Gpd}_R(M).$$

In view of this, it becomes clear that the dual result of Theorem A is the following theorem:

Theorem 2.11 [4, Lemma 3.3]. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module such that $M = xM$, that is, $R^* \otimes_R M = 0$. Then*

$$\Big(\mathrm{Gid}_{R^*}(\mathrm{Hom}_R(R^*, M)) = \Big)\ \mathrm{Gid}_{R^*}({}_xM) \leq\ \mathrm{Gid}_R(M).$$

Notice that Theorem 2.11 is the Gorenstein version of [13, Theorem 204] and that Theorem C stands as the dual of Theorem B.

Next, through Theorem 2.12 and Theorem 2.13, we generalize the second change of rings theorem for the Gorenstein injective dimension, that is Theorem B. First, notice that if M is a Gorenstein injective R-module and x is a central element of R such that $x \notin Z(R)$, then $M = xM$.

Theorem 2.12. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not Gorenstein injective over R. Then*

1) $1 + \mathrm{Gid}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \max\{\mathrm{Gid}_{R^*}({}_xM) - 1,\ \mathrm{Gid}_R(M)\}$.

2) *If* $\mathrm{Gid}_R(M) < +\infty$, *then* $\mathrm{Gid}_{R^*}\Big(\dfrac{M}{xM}\Big)$ *and* $\mathrm{Gid}_{R^*}({}_xM)$ *are simultaneously finite.*
3) *Assume that* $\mathrm{Gid}_{R^*}({}_xM) > \mathrm{Gid}_R(M)$. *Then* $\mathrm{Gid}_{R^*}({}_xM) = 2+\mathrm{Gid}_{R^*}\Big(\dfrac{M}{xM}\Big)$.

Proof. 1) (and (2)) If $\mathrm{Gid}_R(M) = +\infty$, then we are done. Next, assume that $1 \leq \mathrm{Gid}_R(M) < +\infty$. In view of Theorem 2.7, it suffices to prove that

$$\mathrm{Gid}_R\Big(\frac{M}{xM}\Big) \leq \max\{\mathrm{Gid}_R({}_xM) - 2,\ \mathrm{Gid}_R(M)\}.$$

Consider the following two exact sequences of R-modules

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow M \xrightarrow{x} xM \longrightarrow 0 \ (*) \\ \\ 0 \longrightarrow xM \longrightarrow M \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (**). \end{cases}$$

First, by Proposition 2.4, $\mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big)$ is finite if and only $\mathrm{Gid}_R(xM)$ is finite if and only if $\mathrm{Gid}_R({}_xM)$ is finite. Then, by Theorem 2.7, (2) holds. Now, by virtue of Lemma 2.6, via the sequence($**$), $\mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big) \leq \max\{\mathrm{Gid}_R(M),\ \mathrm{Gid}_R(xM) - 1\}$, and via ($*$), $\mathrm{Gid}_R(xM) \leq \max\{\mathrm{Gid}_R(M),\ \mathrm{Gid}_R({}_xM) - 1\}$. Hence $\mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big) \leq \max\{\mathrm{Gid}_R(M),\ \mathrm{Gid}_R({}_xM) - 2\}$ establishing (1).
3) Assume that $\mathrm{Gid}_{R^*}({}_xM) > \mathrm{Gid}_R(M)$. Then, by Theorem 2.7, $\mathrm{Gid}_R({}_xM) > \mathrm{Gid}_R(M) + 1$. Now, by (1), $\mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big) \leq \mathrm{Gid}_R({}_xM) - 2$. Conversely, by Lemma 2.6, via the sequence ($*$), $\mathrm{Gid}_R({}_xM) \leq \max\{\mathrm{Gid}_R(M), \mathrm{Gid}_R(xM)+1\}$ and via the sequence ($**$), $\mathrm{Gid}_R(xM) \leq \max\Big\{\mathrm{Gid}_R(M),\ \mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big) + 1\Big\}$. Then $\mathrm{Gid}_R({}_xM) - 1 \leq \max\Big\{\mathrm{Gid}_R(M),\ \mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big) + 1\Big\}$. As by hypotheses, $\mathrm{Gid}_R({}_xM) > \mathrm{Gid}_R(M) + 1$, we get $\mathrm{Gid}_R({}_xM) - 1 \leq \mathrm{Gid}_R\Big(\dfrac{M}{xM}\Big) + 1$ yielding the desired equality and completing the proof. □

Our next result represents the ultimate version of the second change of rings theorem for the Gorenstein injective dimension, that is, Theorem B from the introduction.

Theorem 2.13. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not a Gorenstein injective R-module. Then*

$$1 + \mathrm{Gid}_{R^*}\Big(\frac{M}{xM}\Big) \leq \mathrm{Gid}_R(M) \text{ if and only if } \mathrm{Gid}_{R^*}({}_xM) - 1 \leq \mathrm{Gid}_R(M).$$

Proof. If $\mathrm{Gid}_{R^*}({}_xM) - 1 \leq \mathrm{Gid}_R(M)$, then, by Theorem 2.12(1), $1+\mathrm{Gid}_{R^*}\left(\dfrac{M}{xM}\right) \leq \mathrm{Gid}_R(M)$. Conversely, assume that $1+\mathrm{Gid}_{R^*}\left(\dfrac{M}{xM}\right) \leq \mathrm{Gid}_R(M)$, that is, $\mathrm{Gid}_R\left(\dfrac{M}{xM}\right) \leq \mathrm{Gid}_R(M)$. A similar statement of the assertion (1) of Theorem 2.12 is established for the R^*-module ${}_xM$ in the proof of Theorem 2.12(3), that is, $\mathrm{Gid}_R({}_xM)-1 \leq \max\left\{\mathrm{Gid}_R(M), \mathrm{Gid}_R\left(\dfrac{M}{xM}\right)+1\right\}$. Hence $\mathrm{Gid}_R({}_xM) - 1 \leq \mathrm{Gid}_R(M) + 1$, so that, by Theorem 2.7, we get $\mathrm{Gid}_{R^*}({}_xM) - 1 \leq \mathrm{Gid}_R(M)$, as desired. □

Next, through Theorem 2.14 and Theorem 2.15, we generalize the second change of rings theorem for the Gorenstein projective dimension, that is Theorem A. Note that if M is a Gorenstein projective R-module and x is a central element of R such that $x \notin Z(R)$, then $\dfrac{M}{xM}$ is Gorenstein projective over R^* and ${}_xM = 0$. The proofs of Theorem 2.14 and Theorem 2.15 are similar to those of Theorem 2.12 and Theorem 2.13, respectively, using Lemma 2.5 and Theorem 2.7.

Theorem 2.14. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module. Then*
1) $\mathrm{Gpd}_{R^*}\left(\dfrac{M}{xM}\right) \leq \max\{2 + \mathrm{Gpd}_{R^*}({}_xM),\ \mathrm{Gpd}_R(M)\}$.
2) *If* $\mathrm{Gpd}_R(M) < +\infty$, *then* $\mathrm{Gpd}_{R^*}\left(\dfrac{M}{xM}\right)$ *and* $\mathrm{Gpd}_{R^*}({}_xM)$ *are simultaneously finite.*
3) *If* $1+\mathrm{Gpd}_{R^*}({}_xM) > \mathrm{Gpd}_R(M)$ and M is not Gorenstein projective over R, *then*

$$\mathrm{Gpd}_{R^*}\left(\frac{M}{xM}\right) = 2 + \mathrm{Gpd}_{R^*}({}_xM).$$

The following result is the dual of Theorem 2.13 and it represents the ultimate version of Theorem C from the introduction. One can easily notice the duality between the corresponding "if" statements of Theorem 2.13 and Theorem 2.15 as well as their "only if" statements.

Theorem 2.15. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not Gorenstein projective over R. Then*

$$1 + \mathrm{Gpd}_{R^*}({}_xM) \leq \mathrm{Gpd}_R(M) \textit{ if and only if } \mathrm{Gpd}_{R^*}\left(\frac{M}{xM}\right) - 1 \leq \mathrm{Gpd}_R(M).$$

3 Case of the Gorenstein flat dimension

The goal of this section is to establish the second change of rings theorem for the Gorenstein flat dimension and its ultimate version in the setting of a right coherent ring R.

Next, we collect useful results concerning basic properties of the Gorenstein flat dimension.

Proposition 3.1 [12, Theorem 3.14]. *Let R be a right coherent ring. Let M be an R-module with finite Gorenstein flat dimension and let $n \geq 0$ be an integer. Then the following assertions are equivalent:*
1) $\mathrm{Gfd}_R(M) \leq n$;
2) $\mathrm{Tor}_i^R(I, M) = 0$ *for all* $i > n$ *and all injective R-modules I*;
3) $\mathrm{Tor}_i^R(Q, M) = 0$ *for all* $i > n$ *and all R-modules Q with finite injective dimension.*

Proposition 3.2 [12, Theorem 3.15]. *Let R be a right coherent ring. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. If any two of the modules M, M', M'' have finite Gorenstein flat dimension, then so has the third.*

Lemma 3.3. *Let R be a right coherent ring. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. Then*
1) $\mathrm{Gfd}_R(A) \leq \max\{\mathrm{Gfd}_R(B), \mathrm{Gfd}_R(C) - 1\}$ *with equality when* $\mathrm{Gfd}_R(B) \neq \mathrm{Gfd}_R(C)$.
2) $\mathrm{Gfd}_R(B) \leq \max\{\mathrm{Gfd}_R(A), \mathrm{Gfd}_R(C)\}$ *with equality when* $\mathrm{Gfd}_R(C) \neq \mathrm{Gfd}_R(A) + 1$.
3) $\mathrm{Gfd}_R(C) \leq \max\{\mathrm{Gfd}_R(B), \mathrm{Gfd}_R(A) + 1\}$ *with equality when* $\mathrm{Gfd}_R(B) \neq \mathrm{Gfd}_R(A)$.

Proof. It is straightforward via Proposition 3.1 and Proposition 3.2. For a proof in the general setting of a GF-closed ring R, we refer to [3, Theorem 2.11]. □

The following result presents an additional argument that flat modules are almost projective modules.

Lemma 3.4. *1) Let x be a non zero-divisor element of R and let A be a direct limit of a direct system of nonzero projective R-modules. Then x is a non zero-divisor of M.*
2) Let x be a non zero-divisor element of R and let M be a flat R-module.

Then x is a non zero-divisor of M.

Proof. 1) Let $\{P_i, \varphi_j^i\}$ be a direct system of nonzero projective modules with $\varphi_j^i : P_i \longrightarrow P_j, \forall i \leq j$. Let $\lambda_i : P_i \longrightarrow \coprod_i P_i$ be the ith injection into the direct sum of the P_i. Let $H = \varinjlim P_i = \dfrac{\coprod P_i}{S}$ with S is the submodule of $\coprod P_i$ generated by all elements of the form $\lambda_j(\varphi_j^i(a_i)) - \lambda_i(a_i)$. Let $z \in H$ such that $xz = 0$. Then, by [15, Theorem 2.17(i)], there exists an index i and an element $a_i \in P_i$ such that $z = \overline{\lambda_i(a_i)}$. Then $xz = \overline{\lambda_i(xa_i)} = 0$. Now, applying [15, Theorem 2.17(ii)], there exists $t \geq i$ such that $x\varphi_t^i(a_i) = \varphi_t^i(xa_i) = 0$ with $\varphi_t^i(a_i) \in P_t$. As $x \notin Z(P_t)$, we get $\varphi_t^i(a_i) = 0$. A second application of [15, Theorem 2.17(ii)] yields $\overline{\lambda_i(a_i)} = \overline{0} = z$, as desired.

2) It follows easily from (1) as, by [14, Theorem 1.2], F is a direct limit of a direct system of finitely generated free R-modules. This completes the proof. $\square$

We next establish a version of Rees's theorem [15, Theorem 9.37] for the torsion functor.

Theorem 3.5. *Let x be a central element of R such that x is not a unit of R and $x \notin Z(R)$. Let A be a right R^*-module and B be a left R-module such that $x \notin Z(B)$. Then*

$$\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^{R^*}\Big(A, \frac{B}{xB}\Big).$$

Proof. 1) Let $... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ be a projective resolution of R-modules of B. Tensoring with R^* and as $x \notin Z(B) \cup Z(P_i)$ for each integer $i \geq 0$, by successive applications of Lemma 2.9, we get the following exact sequence of right R^*-modules

$$... \longrightarrow \frac{P_1}{xP_1} \longrightarrow \frac{P_0}{xP_0} \longrightarrow \frac{B}{xB} \longrightarrow 0$$

which is a projective resolution of $\dfrac{B}{xB}$ over R^*. Now, applying the functor $A\otimes_{R^*}$ to this last sequence yields the next complex

$$C\Big(A, \frac{B}{xB}\Big) = ... \longrightarrow A \otimes_{R^*} \frac{P_1}{xP_1} \longrightarrow A \otimes_{R^*} \frac{P_0}{xP_0} \longrightarrow A \otimes_{R^*} \frac{B}{xB} \longrightarrow 0.$$

The homology groups of this complex are the $H_n\Big(C\Big(A, \dfrac{B}{xB}\Big)\Big) = \mathrm{Tor}_n^{R^*}\Big(A, \dfrac{B}{xB}\Big)$. Note that $A\otimes_{R^*} \dfrac{P_i}{xP_i} \overset{\varphi_i}{\cong} A \otimes_R P_i$ for each integer $i \geq 0$ and $A\otimes_{R^*} \dfrac{B}{xB} \overset{\varphi}{\cong} A\otimes_R B$.

Therefore, we get the following commutative diagram

$$\begin{array}{ccccccccc}
C\Big(A,\dfrac{B}{xB}\Big) = \ldots \longrightarrow & A\otimes_{R^*}\dfrac{P_1}{xP_1} & \longrightarrow & A\otimes_{R^*}\dfrac{P_0}{xP_0} & \longrightarrow & A\otimes_{R^*}\dfrac{B}{xB} & \longrightarrow 0\\
& \varphi_1\downarrow & & \varphi_0\downarrow & & \varphi\downarrow & \\
C(A,B) = \ldots \longrightarrow & A\otimes_R P_1 & \longrightarrow & A\otimes_R P_0 & \longrightarrow & A\otimes_R B & \longrightarrow 0
\end{array}$$

with $H_n(C(A,B)) = \operatorname{Tor}_n^R(A,B)$ are the homology groups of the complex $C(A,B)$. Consequently,

$$H_n\Big(C\Big(A,\frac{B}{xB}\Big)\Big) \cong H_n(C(A,B))$$

for each positive integer n establishing the desired isomorphism. □

Theorem 3.5 allows easily to establish the second change of rings theorem for the flat dimension.

Theorem 3.6. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module such that $x \notin Z(M)$. Then*

$$\operatorname{fd}_{R^*}\Big(\frac{M}{xM}\Big) \leq \operatorname{fd}_R(M).$$

Proof. It is a direct consequence of Theorem 3.5.

We next present the second change of rings theorem for the Gorenstein flat dimension.

Theorem 3.7. *Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module such that $x \notin Z(M)$. Then*

$$\operatorname{Gfd}_{R^*}\Big(\frac{M}{xM}\Big) \leq \operatorname{Gfd}_R(M).$$

Proof. First, let us prove that if G is a Gorenstein flat R-module, then $\dfrac{G}{xG}$ is a Gorenstein flat R^*-module. In fact, assume that G is Gorenstein flat over R and let $\mathbf{E} = \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \ldots$ be a complete flat resolution of R-modules such that $G := \operatorname{Im}(F_0 \longrightarrow F_{-1})$. Let $G_i := \operatorname{Im}(F_i \to F_{i-1})$ for each integer i with $G = G_0$. Applying Lemma 2.9 to each short exact sequence of R-modules $0 \longrightarrow G_{i+1} \longrightarrow F_i \longrightarrow G_i \longrightarrow 0$ yields the following short exact sequence of R^*-modules $0 \longrightarrow \dfrac{G_{i+1}}{xG_{i+1}} \longrightarrow \dfrac{F_i}{xF_i} \longrightarrow \dfrac{G_i}{xG_i} \longrightarrow 0$ as, by Lemma 3.4, G_i is a submodule of a flat module and thus ${}_xG_i = 0$. Then, we get the exact sequence of R^*-modules $\dfrac{\mathbf{E}}{\mathbf{xE}} = ...\longrightarrow \dfrac{F_1}{xF_1} \longrightarrow \dfrac{F_0}{xF_0} \longrightarrow \dfrac{F_{-1}}{xF_{-1}} \longrightarrow ...$ with $\mathrm{Im}\Big(\dfrac{F_i}{xF_i} \to \dfrac{F_{i-1}}{xF_{i-1}}\Big) = \dfrac{G_i}{xG_i}$ for each integer i and each $\dfrac{F_i}{xF_i}$ is a flat R^*-module, by Theorem 3.6. Fix an integer i and let Q be an injective module over R^*. Then, by Theorem 3.5, $\mathrm{Tor}_1^{R^*}\Big(Q, \dfrac{G_i}{xG_i}\Big) \cong \mathrm{Tor}_1^R(Q, G_i)$. Since, by [13, Theorem 202], $\mathrm{id}_R(Q) = 1$, we get $\mathrm{Tor}_1^R(Q, G_i) = 0$, by Proposition 3.1. Hence $\mathrm{Tor}_1^{R^*}\Big(Q, \dfrac{G_i}{xG_i}\Big) = 0$ for each injective R^*-module Q. It follows that $\dfrac{\mathbf{E}}{\mathbf{xE}}$ is a complete flat resolution over R^* and thus $\dfrac{G}{xG}$ is a Gorenstein flat R^*-module, as claimed.

If $\mathrm{Gfd}_R(M) = +\infty$, then we are done. Assume that $\mathrm{Gfd}_R(M) < +\infty$ and let $0 \longrightarrow G_0 \longrightarrow G_1 \longrightarrow ... \longrightarrow G_n \longrightarrow M \longrightarrow 0$ be an exact sequence of Gorenstein flat R-modules with $n \geq 0$ is an integer. Now, tensoring this sequence with R^* yields, by Lemma 2.9, as $x \notin Z(M)$ and $x \notin Z(G_i)$ (see Lemma 3.4) for each integer $i = 0, ..., n$, the exact sequence of R^*-modules

$$0 \longrightarrow \frac{G_0}{xG_0} \longrightarrow \frac{G_1}{xG_1} \longrightarrow ...\frac{G_n}{xG_n} \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

By the first step, each $\dfrac{G_i}{xG_i}$ is a Gorenstein flat R^*-module. It follows that $\mathrm{Gfd}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq n$ yielding the desired inequality. $\square$

An analog version of Theorem C from the introduction for the Gorenstein flat dimension is the following in the setting of a right coherent ring.

Theorem 3.8. *Let R be a right coherent ring. Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module such that $M = xM$, that is, $R^* \otimes_R M = 0$.* Then

$$1 + \mathrm{Gfd}_{R^*}({}_xM) \leq \mathrm{Gfd}_R(M)$$

except when M is Gorenstein flat over R in which case ${}_xM = 0$.

We need the following result due to M. Harris.

Lemma 3.9 [11, Theorem 2]. *Let R be a right coherent ring and x be a central element of R. Then $R^* := \dfrac{R}{xR}$ is a right coherent ring.*

Proof of Theorem 3.8. Let $0 \longrightarrow N \longrightarrow G \longrightarrow M \longrightarrow 0$ be a short exact sequence of R-modules such that G is Gorenstein flat over R and $\mathrm{Gfd}_R(M) = 1+\mathrm{Gfd}_R(N)$. Then, applying the functor $R^* \otimes_R -$ and Lemma 2.9, we get the following exact sequence of R^*-modules

$$_xG = 0 \longrightarrow {}_xM \longrightarrow \frac{N}{xN} \longrightarrow \frac{G}{xG} \longrightarrow \frac{M}{xM} = 0.$$

Now, by Theorem 3.7, $\dfrac{G}{xG}$ is a Gorenstein flat R-module. Then, as, by Lemma 3.9, R^* is right coherent, we get, by Lemma 3.3, $\mathrm{Gfd}_{R^*}\Big(\dfrac{N}{xN}\Big) = \mathrm{Gfd}_{R^*}({}_xM)$. Therefore, another application of Theorem 3.7 yields $\mathrm{Gfd}_{R^*}({}_xM) \leq \mathrm{Gfd}_R(N) = \mathrm{Gfd}_R(M) - 1$, as desired. $\Box$

Through Theorem 3.10 and Theorem 3.11, we generalize Theorem 3.8 and give its ultimate version.

Theorem 3.10. *Let R be a right coherent ring. Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module. Then*
1) $\mathrm{Gfd}_{R^*}\Big(\dfrac{M}{xM}\Big) \leq \max\{2 + \mathrm{Gfd}_{R^*}({}_xM),\ \mathrm{Gfd}_R(M)\}$.
2) *If* $\mathrm{Gfd}_R(M) < +\infty$, *then* $\mathrm{Gfd}_{R^*}\Big(\dfrac{M}{xM}\Big)$ *and* $\mathrm{Gfd}_{R^*}({}_xM)$ *are simultaneously finite.*
3) *If* $1+\mathrm{Gfd}_{R^*}({}_xM) > \mathrm{Gfd}_R(M)$ and M is not Gorenstein flat over R, *then* $\mathrm{Gfd}_{R^*}\Big(\dfrac{M}{xM}\Big) = 2+\mathrm{Gfd}_{R^*}({}_xM)$.

Proof. 1) (and (2)) If $\mathrm{Gfd}_R(M) = +\infty$, then we are done. Also, if M is Gorenstein flat over R, then, by Theorem 3.7, $\dfrac{M}{xM}$ is Gorenstein flat over R^* and, by Lemma 3.4, ${}_xM = 0$. Then, assume that $1 \leq \mathrm{Gfd}_R(M) = n < +\infty$. Let $0 \longrightarrow A \longrightarrow G \longrightarrow M \longrightarrow 0$ $(*)$ be an exact sequence of R-modules such that G is Gorenstein flat over R. Note that as A is a submodule of a Gorenstein flat module and $x \notin Z(R)$, $x \notin Z(A)$, that is, ${}_xA = 0$. So, by Theorem 3.7, $\mathrm{Gfd}_{R^*}\Big(\dfrac{A}{xA}\Big) \leq \mathrm{Gfd}_R(A) \leq n-1$. On the other hand, tensoring the sequence $(*)$ with R^* yields, by Lemma 2.9, the exact sequence of R^*-modules

$$0 \longrightarrow {}_xM \longrightarrow \frac{A}{xA} \longrightarrow \frac{G}{xG} \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

Now, let $H = \mathrm{Im}\Big(\dfrac{A}{xA} \to \dfrac{G}{xG}\Big)$. We have the next two exact sequences of

R^*-modules

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow \dfrac{A}{xA} \longrightarrow H \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow H \longrightarrow \dfrac{G}{xG} \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***). \end{cases}$$

As, by Lemma 3.9, R^* is right coherent, it is then clear, by Proposition 3.2, that $\mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) < +\infty$ if and only if $\mathrm{Gfd}_{R^*}(H) < +\infty$ if and only if $\mathrm{Gfd}_{R^*}({}_xM) < +\infty$ establishing (2). Also, as R^* is right coherent, via the sequence $(**)$, we get, by Lemma 3.3, $\mathrm{Gfd}_{R^*}(H) \leq \max\left\{1+\mathrm{Gfd}_{R^*}({}_xM),\ \mathrm{Gfd}_{R^*}\left(\dfrac{A}{xA}\right)\right\}$. Moreover, by the sequence $(***)$, we have $\mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) \leq 1+\mathrm{Gfd}_{R^*}(H)$. It follows that

$$\begin{aligned} \mathrm{Gfd}_{R^*}\left(\frac{M}{xM}\right) &\leq 1 + \max\{1 + \mathrm{Gfd}_{R^*}({}_xM), n-1\} \\ &= \max\{2 + \mathrm{Gfd}_{R^*}({}_xM),\ \mathrm{Gfd}_R(M)\}, \text{ as desired.} \end{aligned}$$

3) Assume that $1+\mathrm{Gfd}_{R^*}({}_xM) > \mathrm{Gfd}_R(M) \geq 1$. Then, by (1), $\mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) \leq 2+\mathrm{Gfd}_{R^*}({}_xM)$. Conversely, proceeding as in (1), consider the above-mentioned two exact sequences

$$\begin{cases} 0 \longrightarrow {}_xM \longrightarrow \dfrac{A}{xA} \longrightarrow H \longrightarrow 0 \ (**) \\ \\ 0 \longrightarrow H \longrightarrow \dfrac{G}{xG} \longrightarrow \dfrac{M}{xM} \longrightarrow 0 \ (***). \end{cases}$$

If $\dfrac{M}{xM}$ is Gorenstein flat over R^*, then, as R^* is right coherent, H is Gorenstein flat over R^*, and thus $\mathrm{Gfd}_{R^*}({}_xM) = \mathrm{Gfd}_{R^*}\left(\dfrac{A}{xA}\right) \leq \mathrm{Gfd}_R(M) - 1$ which is contradictory to our initial assumption. Then $\mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) \geq 1$, so that $\mathrm{Gfd}_{R^*}(H) = \mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) - 1$. Also, via the sequence $(**)$ and by Lemma 3.3, as R^* is right coherent,

$$\begin{aligned} \mathrm{Gfd}_{R^*}({}_xM) &\leq \max\left\{\mathrm{Gfd}_{R^*}\left(\frac{A}{xA}\right), \mathrm{Gfd}_{R^*}(H) - 1\right\} \\ &\leq \max\left\{\mathrm{Gfd}_R(M) - 1,\ \mathrm{Gfd}_{R^*}\left(\frac{M}{xM}\right) - 2\right\}. \end{aligned}$$

This ensures, as $1+\mathrm{Gfd}_{R^*}({}_xM) > \mathrm{Gfd}_R(M)$, that $\mathrm{Gfd}_{R^*}({}_xM) \leq \mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) - 2$. It follows that $\mathrm{Gfd}_{R^*}\left(\dfrac{M}{xM}\right) = 2+\mathrm{Gfd}_{R^*}({}_xM)$ establishing (3) and completing the proof. □

The following result represents the ultimate version of Theorem 3.8.

Theorem 3.11. *Let R be a right coherent ring. Let x be a central element of R such that $x \notin Z(R)$. Let M be an R-module which is not Gorenstein flat over R. Then*

$$1+\mathrm{Gfd}_{R^*}({}_xM) \leq \mathrm{Gfd}_R(M) \text{ if and only if } \mathrm{Gfd}_{R^*}\left(\frac{M}{xM}\right)-1 \leq \mathrm{Gfd}_R(M).$$

Proof. Let $\mathrm{Gfd}_{R^*}\left(\frac{M}{xM}\right)-1 \leq \mathrm{Gfd}_R(M)$. If $1+\mathrm{Gfd}_{R^*}({}_xM) > \mathrm{Gfd}_R(M)$, then, by Theorem 3.10(3),

$$\begin{aligned}\mathrm{Gfd}_{R^*}\left(\frac{M}{xM}\right) &= 2+\mathrm{Gfd}_{R^*}({}_xM)\\ &> 1+\mathrm{Gfd}_R(M) \text{ which is absurd.}\end{aligned}$$

It follows that $1+\mathrm{Gfd}_{R^*}({}_xM) \leq \mathrm{Gfd}_R(M)$. Conversely, assume that $1+\mathrm{Gfd}_{R^*}({}_xM) \leq \mathrm{Gfd}_R(M)$. Then, using Theorem 3.10(1),

$$\begin{aligned}\mathrm{Gfd}_{R^*}\left(\frac{M}{xM}\right) &\leq 1+\max\{1+\mathrm{Gfd}_{R^*}({}_xM),\ \mathrm{Gfd}_R(M)\}\\ &\leq 1+\mathrm{Gfd}_R(M), \text{ as desired. } \square\end{aligned}$$

4 An open problem

We end this paper by the following problem concerning the change of rings theorem related to the Gorenstein projective and injective dimensions. Recall that it is shown in Theorem 2.13 and Theorem 2.15 that: If x is a central element of R such that $x \notin Z(R)$ and M is an R-module which is not Gorenstein injective (resp., M is not Gorenstein projective), then

$$1+\mathrm{Gid}_{R^*}\left(\frac{M}{xM}\right) \leq \mathrm{Gid}_R(M) \text{ if and only if } \mathrm{Gid}_{R^*}({}_xM)-1 \leq \mathrm{Gid}_R(M)$$

$$\left(\text{resp., } 1+\mathrm{Gpd}_{R^*}\left(\frac{M}{xM}\right) \leq \mathrm{Gpd}_R(M) \text{ if and only if } \mathrm{Gpd}_{R^*}({}_xM)-1 \leq \mathrm{Gpd}_R(M)\right).$$

From these two theorems arise the following natural question:

Problem. Let x be a central element of R such that $x \notin Z(R)$ and M be an R-module which is not Gorenstein injective (resp., M is not Gorenstein projective). Then, have we

$$1 + \mathrm{Gid}_{R^*}\Big(\frac{M}{xM}\Big) = \mathrm{Gid}_R(M) \text{ if and only if } \mathrm{Gid}_{R^*}({}_xM) - 1 = \mathrm{Gid}_R(M)?$$

$$\Big(\text{resp., } 1+\mathrm{Gpd}_{R^*}\Big(\frac{M}{xM}\Big) = \mathrm{Gpd}_R(M) \text{ if and only if } \mathrm{Gpd}_{R^*}({}_xM)-1 = \mathrm{Gpd}_R(M)?\Big)$$

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