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Left Multipliers Satisfying Certain Algebraic Identities on Lie Ideals of Rings With Involution

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Abstract

In this note we investigate left multipliers satisfying certain algebraic identities on Lie ideals of rings with involution and discuss related results. Moreover we provide examples to show that the assumed restrictions cannot be relaxed.

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1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Recall that R is 2-torsion free if $2x = 0$ forces $x = 0$ and R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R admits an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. Indeed, if R is a prime ring with opposite R^o, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and therefore $*$-prime rings constitute a more general class of prime rings.
An additive map $F : R \longrightarrow R$ is called a left multiplier (resp. derivation) if $F(xy) = F(x)y$ (resp. $F(xy) = F(x)y + xF(y)$) for all x, y in R. A left

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multiplier F on R is said to be trivial if F is the identity map on R. An additive map $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Obviously, generalized derivation with $d = 0$ covers the concept of left multipliers.
There has been an ongoing interest concerning the relationship between the commutativity of a prime ring R and the behavior of a generalized derivation of R, with associated *nonzero* derivation. Many of obtained results extend other ones previously proven just for the action of the generalized derivation on the whole ring. In this direction, it seems natural to ask what we can say about the commutativity of R if the generalized derivation is replaced by a left multiplier. Our aim in this paper is to investigate the commutativity of a ring with involution $(R, *)$ satisfying certain identities involving a left multiplier acting on Lie ideals.

2 Left multipliers acting on Lie ideals

Throughout, $(R, *)$ will represent an associative ring with involution and $Sa_*(R) := \{r \in R / r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R. We shall need the following lemmas quoted from [4] and [5].

Lemma 2.1 ([4], Lemma 4) *If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of a 2-torsion free $*$-prime ring R and $a, b \in R$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([5], Lemma 2.3) *Let $0 \neq U$ be a $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U, U] = 0$, then $U \subseteq Z(R)$.*

We first fix the following facts which will be used in the sequel.

Fact 1. Let U be a noncentral $*$-Lie ideal of a $*$-prime ring R. If $aU = 0$ or $Ua = 0$, then $a = 0$. Indeed, if $aU = 0$ (resp. $Ua = 0$), then $aUa = 0 = aUa^*$ (resp. $aUa = 0 = a^*Ua$) and Lemma 2.1 yields $a = 0$.

Fact 2. Every $*$-prime ring is semiprime. Indeed, if $aRa = 0$ then $aRaRa^* = 0$ so that $a = 0$ or $aRa^* = 0$. But $aRa^* = 0$ together with $aRa = 0$ force $a = 0$.

Theorem 2.3 *Let R be a 2-torsion free $*$-prime ring and U be a $*$-Lie ideal of R. If R admits a left multiplier F such that $F(xy) - xy \in Z(R)$ for all $x, y \in U$, then either F is trivial or $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. From $F(uv) - uv \in Z(R)$ it follows that

$$F(u)v - uv \in Z(R) \text{ for all } u, v \in U. \tag{1}$$

Using (1) together with $F(v)u - vu \in Z(R)$ we get

$$F([u,v]) - [u,v] \in Z(R) \quad \text{for all} \;\; u,v \in U. \tag{2}$$

Replacing u by $[x,y]$ in (1), where $x,y \in U$, we obtain

$$(F([x,y]) - [x,y])v \in Z(R) \;\; \text{for all} \;\; v,x,y \in U. \tag{3}$$

Hence $[(F([x,y]) - [x,y])v, r] = 0$ for all $r \in R$ which, because of (2), yields

$$(F([x,y]) - [x,y])[v,r] = 0 \;\; \text{for all} \;\; v,x,y \in U, \;\; r \in R. \tag{4}$$

In particular, equation (4) yields

$$(F([x,y]) - [x,y])[U,U] = 0 \;\; \text{for all} \;\; x,y \in U. \tag{5}$$

Since $U \not\subseteq Z(R)$ and R is a 2-torsion free semiprime ring, then $[U,U]$ is a noncentral $*$-Lie ideal of R. Using Fact 1, equation (5) forces

$$F([x,y]) = [x,y] \;\; \text{for all} \;\; x,y \in U. \tag{6}$$

Since F is a left multiplier, then

$$F([[x,y],z]) = [x,y]z - F(z)[x,y] \;\; \text{for all} \;\; x,y,z \in U. \tag{7}$$

Employing $F([[x,y],z]) = [[x,y],z]$ by (6), then (7) reduces

$$(F(z) - z)[x,y] = 0 \;\; \text{for all} \;\; x,y,z \in U$$

in such a way that

$$(F(z) - z)[U,U] = 0 \;\; \text{for all} \; z \in U. \tag{8}$$

Once again using the fact that $[U,U]$ is a noncentral $*$-Lie ideal of R, in light of equation (8), Fact 1 forces to be

$$F(z) = z \quad \text{for all} \;\; z \in U.$$

From $F([u,r]) = [u,r]$, it follows that $(F(r) - r)u = 0$ and thus

$$(F(r) - r)U = 0 \quad \text{for all} \;\; r \in R. \tag{9}$$

Using Fact 1, equation (9) forces $F(r) = r$ for all $r \in R$ so that F is the identity map on R. ∎

Theorem 2.4 *Let R be a 2-torsion free $*$-prime ring and U be a $*$-Lie ideal of R. If R admits a left multiplier F such that $F(xy) + xy \in Z(R)$ for all $x, y \in U$, then either $-F$ is trivial or $U \subseteq Z(R)$.*

Proof. If F is a left multiplier satisfying the property $F(xy)+xy \in Z(R)$ for all $x, y \in U$, then the left multiplier $(-F)$ satisfies the condition $(-F)(xy)-xy \in Z(R)$ for all $x, y \in U$ and hence Theorem 2.3 forces $U \subseteq Z(R)$ or $-F = Id$. ∎

Remark 1. If we choose U a $*$-ideal instead of a $*$-Lie ideal, then Theorems 2.3 and 2.4 hold without the assumption on the characteristic of the ring.

Corollary 2.5 *Let R be a $*$-prime ring and I be a nonzero $*$-ideal of R. If R admits a left multiplier F such that neither F nor $(-F)$ is trivial and $F(xy) - xy \in Z(R)$ (or $F(xy) + xy \in Z(R)$) for all $x, y \in I$, then R is commutative.*

The following example proves that the $*$-primeness hypothesis in the above theorems is not superfluous.

Example 1.
Let (R, σ) be a noncommutative prime ring with involution and set $\mathcal{R} = R \times R$. Consider $U = R \times \{0\}$ and define an involution $*$ on $\mathcal{R}$ by $(x, y)^* = (\sigma(x), \sigma(y))$. It is straightforward to check that U is a $*$-Lie ideal of the semiprime ring $\mathcal{R}$. Moreover, if we set $F(x, y) = (x, 0)$ (resp. $F(x, y) = (-x, 0)$), the F is a left multiplier such that $F(uv) - uv \in Z(\mathcal{R})$ (resp. $F(uv) + uv \in Z(\mathcal{R})$) for all $u, v \in U$; but $U \nsubseteq Z(\mathcal{R})$. Hence Theorems 1 and 2 cannot be extended to semiprime rings.

As an application of Theorems 2.3 and 2.4, the following result improves Theorem 3.1 & 3.2 of [1], but only with further assumption that the ring R be 2-torsion free.

Theorem 2.6 *Let R be a 2-torsion free prime ring and U be a Lie ideal of R. If R admits a left multiplier F such that $F(xy) - xy \in Z(R)$ (resp. $F(xy) + xy \in Z(R)$) for all $x, y \in U$, then F (resp. $(-F)$) is trivial or $U \subseteq Z(R)$.*

Proof. Assume that $F(xy) - xy \in Z(R)$ for all $x, y \in U$. Let $\mathcal{F}$ be the left multiplier defined on the $*_{\text{ex}}$-prime ring $\mathcal{R} = R \times R^0$ by $\mathcal{F}(x, y) = (F(x), y)$. If we set $W = U \times U$, then W is a $*_{\text{ex}}$-Lie ideal of $\mathcal{R}$. Moreover, we have

$$\mathcal{F}((x, y)(u, v)) - (x, y)(u, v) = (F(xu) - xu, 0) \quad \text{for all } (x, y), (u, v) \in W,$$

which forces $\mathcal{F}((x, y)(u, v)) - (x, y)(u, v) \in Z(\mathcal{R})$. In view of Theorem 2.3, either $\mathcal{F}$ is trivial or $W \subseteq Z(\mathcal{R})$. Accordingly, F is trivial or $U \subseteq Z(R)$.
If $F(xy) + xy \in Z(R)$ for all $x, y \in U$, then $\mathcal{F}(x, y) = (F(x), -y)$ is a left multiplier on $\mathcal{R}$ such that $\mathcal{F}((x, y)(u, v)) + (x, y)(u, v) \in Z(\mathcal{R})$ for all

$(x, y), (u, v) \in \mathcal{R}$. The same above strategy combined with Theorem 2.4 leads to the result. ∎

Application of Theorem 2.6 together with Remark 1 yield the following result which improves ([2], Theorem 2.1).

Theorem 2.7 *Let R be a prime ring and F be a generalized derivation which is not the identity map on R. If $F(xy) - xy \in Z(R)$ for all x, y in a nonzero ideal I of R, then R is commutative.*

Proof. Assume that F is a generalized derivation associated with a derivation d. When the associated derivation $d = 0$, then using Theorem 2.6, we get the required result. On the other hand if $d \neq 0$, then the proof follows from Theorem 2.1 of [2]. ∎

Similarly, in view of Theorem 2.6, we obtain the following result which improves ([2], Theorem 2.2)

Theorem 2.8 *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(xy) + xy \in Z(R)$ for all $x, y \in I$, then R is commutative.*

3 Open Problem

We conclude our paper with following open questions:

(i) Does Theorem 2.3 remain valid without 2-torsion freeness hypothesis?

(ii) What can we say if the condition "$F(xy) - xy \in Z(R)$" is replaced by "$F(xy) - yx \in Z(R)$"?

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