Int. J. Open Problems Comput. Math., Vol. 5, No. 3, September, 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

On The Hermite-Hadamard Type Integral Inequalities Involving Several Log-Convex Functions

Mehmet Zeki Sarikaya Hatice Yaldiz and Hakan Bozkurt

Department of Mathematics, Faculty of Science and Arts, University of Dzce, Turkey.
e-mail: sarikayamz@gmail.com
Department of Mathematics, Faculty of Science and Arts, University of Dzce, Turkey.
e-mail:yaldizhatice@gmail.com.
Department of Mathematics, Faculty of Science and Arts, University of Dzce, Turkey.
e-mail:insedi@yahoo .com.

Abstract

In this paper, new integral inequalities of Hermite-Hadamard type involving several differentiable log-convex functions are given.

Keywords: *Hermite-Hadamard's inequalities, log-convex functions, Logarithmic mean, Jensen's integral inequality.*

1 Introduction

The following inequality is well known in the literature as the Hermite-Hadamard inequality (see $[5, p.137]$):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

It is well known that the Hermite-Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, see ([1]-[8], [11], [12]) and the books [5],[9],[10] where further references are given.

In [6], Dragomir has established the following interesting refinements of Hadamard's inequalities for log-convex functions:

Let $f : I \to (0, \infty)$ be a differentiable log-convex function on the interval of real numbers I^0 (the interior of I) and $a, b \in I^0$ with $a < b$. Then the following inequalities hold:

$$\begin{aligned} & \frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} \\ \geq\ & L\left(\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\right) \\ \geq\ & 1, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{\frac{f(a)+f(b)}{2}}{\frac{1}{b-a}\int_a^b f(x)\,dx} & \geq 1 + \log\left[\frac{\int_a^b f(x)\,dx}{\int_a^b f(x)\exp\left[\frac{f'(x)}{f(x)}\left(\frac{a+b}{2} - x\right)\right]dx}\right] \\ & \geq 1 + \log\left[\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)}\right] \geq 1. \end{aligned} \tag{2}$$

Recently in [11], Pachpatte has proved the general versions of the inequalities (1) and (2) involving several differentiable log-convex functions.

In this paper , we prove another new integral inequalities Hermite-Hadamard type involving several differentiable log-convex functions. The method employed in our analysis is based on the basic properties of logarithms and the application of the well known Jensen's integral inequality.

2 Main Results

Now, we start with the following our main theorem.

Let $f, g : I \to (0, \infty)$ be differentiable log-convex functions on the interval of real numbers I^0 (the interior of I) and $a, b \in I^0$ with $a < b$. Then, the following inequalities holds:

$$
\begin{aligned}
& 2(b-a)\int_a^b f(x)\,g(x)\,dx \qquad (3)\\
\geq\; & \left(\int_a^b g(y)\,dy\right)\left(\int_a^b f(x)\exp\left[1+\frac{(x-b)\,g(b)-(x-a)\,g(a)}{\int_a^b g(y)\,dy}\right]dx\right)\\
& +\left(\int_a^b f(y)\,dy\right)\left(\int_a^b g(x)\exp\left[1+\frac{(x-b)\,f(b)-(x-a)\,f(a)}{\int_a^b f(y)\,dy}\right]dx\right).
\end{aligned}
$$

Let $f,\ g$ be diffrerentiable log-convex functions. Then

$$
\begin{aligned}
\log f(x)-\log f(y) &\geq \frac{d}{dy}(\log f(y))\,(x-y)\\
\log g(x)-\log g(y) &\geq \frac{d}{dy}(\log g(y))\,(x-y)
\end{aligned}
$$

for all $x, y \in I^0$, which implies that

$$
\log\frac{f(x)}{f(y)} \geq \frac{f'(y)}{f(y)}(x-y).
$$

That is

$$
f(x) \geq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] \qquad (4)
$$

$$
g(x) \geq g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right]. \qquad (5)
$$

Multiplying both sides of (4) and (5) by $g(x)$ and $f(x)$ respectively and adding the resultant, we obtain,

$$
\begin{aligned}
& 2f(x)\,g(x) \qquad (6)\\
\geq\; & g(x)\,f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right]+f(x)\,g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right].
\end{aligned}
$$

Integrating (6) the above inequality with respect to y on $[a,b]$.

$$
\begin{aligned}
& 2(b-a) f(x) g(x) && (7) \\
\geq\ & g(x) \int_a^b f(y) \exp\left[\frac{f'(y)}{f(y)}(x-y)\right] dy + f(x) \int_a^b g(y) \exp\left[\frac{g'(y)}{g(y)}(x-y)\right] dy
\end{aligned}
$$

Now, for integrals in right hand side of (7), using Jensen's integral inequality for $exp(.)$ functions, we have

$$
\begin{aligned}
& \int_a^b f(y) \exp\left[\frac{f'(y)}{f(y)}(x-y)\, dy\right] && (8) \\
\geq\ & \int_a^b f(y)\, dy \exp\left[\frac{\int_a^b f(y) \frac{f'(y)}{f(y)} (x-y)\, dy}{\int_a^b f(y)\, dy}\right] \\
\geq\ & \int_a^b f(y)\, dy \exp\left[1 + \frac{(x-b) f(b) - (x-a) f(a)}{\int_a^b f(y)\, dy}\right]
\end{aligned}
$$

and similarly we get,

$$
\begin{aligned}
& \int_a^b g(y) \exp\left[\frac{g'(y)}{g(y)}(x-y)\, dy\right] && (9) \\
\geq\ & \int_a^b g(y)\, dy \exp\left[1 + \frac{(x-b) g(b) - (x-a) g(a)}{\int_a^b g(y)\, dy}\right]
\end{aligned}
$$

Writing (8) and (9) in (7), it follows that

$$
\begin{aligned}
& 2(b-a) f(x) g(x) && (10) \\
\geq\ & f(x) \left(\int_a^b g(y)\, dy\right) \exp\left[1 + \frac{(x-b) g(b) - (x-a) g(a)}{\int_a^b g(y)\, dy}\right]
\end{aligned}
$$

$$+g(x)\int_a^b f(y)\,dy\exp\left[1+\frac{(x-b)f(b)-(x-a)f(a)}{\int_a^b f(y)\,dy}\right]$$

Integrating (10) the above inequality with respect to x on $[a,b]$, we get the required inequality in (3).

Under the asumptions of Theorem 2, we have

$$\begin{aligned}
&2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \qquad (11)\\
\geq\ & f\left(\frac{a+b}{2}\right)\left(\int_a^b g(y)\,dy\right)\exp\left[1-\frac{\frac{g(a)+g(b)}{2}(b-a)}{\int_a^b g(y)\,dy}\right]\\
&+g\left(\frac{a+b}{2}\right)\left(\int_a^b f(y)\,dy\right)\exp\left[1-\frac{\frac{f(a)+f(b)}{2}(b-a)}{\int_a^b f(y)\,dy}\right].
\end{aligned}$$

$$\exp\left[1-\frac{f(a)+f(b)}{2}(b-a)\right]+\exp\left[1-\frac{g(a)+g(b)}{2}(b-a)\right]\leq 2. \qquad (12)$$

If we take $x=\frac{a+b}{2}$ in Theorem 2, we get the required inequality in (11). By using inequality (1) in (11), then we obtain the required inequality in (12).

Under the asumptions of Theorem 2 and with $y=\frac{a+b}{2}$, we have

$$\begin{aligned}
&2\int_a^b f(x)g(x)\,dx \qquad (13)\\
\geq\ & f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\exp\left[1+\frac{(x-b)f(b)-(x-a)f(a)}{(b-a)f\left(\frac{a+b}{2}\right)}\right]dx\right)\\
&+g\left(\frac{a+b}{2}\right)\left(\int_a^b f(x)\exp\left[1+\frac{(x-b)g(b)-(x-a)g(a)}{(b-a)g\left(\frac{a+b}{2}\right)}\right]dx\right).
\end{aligned}$$

The proof is obvious by the above theorem 2.

3 Open Problem

It is well known that if f is a convex function on the interval $I = [a, b]$ with $a < b$, then the Hermite-Hadamard inequality holds for the convex functions. It has already been proved a lot of this type inequalities for several convex functions. So, there are two questions as follows:

1) How can be established the general versions of the inequalities (3), (11) and (13) involving several differentiable log-convex functions.

2) How to obtain similar results without using Jensen's inequality in the proof of theorem 2.

References

[1] Dragomir S.S. and Agarwal R.P.[1998], *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., 11(5), 91–95.

[2] Dragomir S. S.[1992] , *Two mappings in connection to Hadamard's inequalities*, J. Math. Anal. Appl. 167 , 49–56.

[3] Dragomir S. S. , Cho Y. J., and Kim S. S.[2000], *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl. 245 , 489–501.

[4] Dragomir S. S. and Mond B.[1998], *Integral inequalities of Hadamard type for Hadamard type for Log-convex functions*, Demonstratio 31,354-364.

[5] Dragomir S. S. and Pearce C. E. M.[2000], *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University.

[6] Dragomir S. S.[2000] , *Refinments of the Hermite-Hadamard integral inequality for Log-convex functions*, RGMIA Research Report Collection 3, 527-533.

[7] Kırmacı U.S.[2004], *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., 147 , 137-146.

[8] Kırmacı U.S.[2008] , *Improvement and further generalization of inequalities for differentiable mappings and applications*, Computers and Math. with Appl., 55 , 485-493.

[9] Mitrinovic D. S., Pecaric J. E. and Fink A. M.[1993], *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht.

[10] Pecaric J. E., Proschan F. and Tong Y. L.[1991], *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York.

[11] Pachpatte B.G.[2005], *A note Hadamard type integral inequalities involving several log-convex functions*, Tamkang Jour. of Math., 38(1), 43-47.

[12] Yang G. S. and Tseng K. L.[1999], *On certain integral inequalities related to Hermite–Hadamard inequalities*, J. Math. Anal. Appl. 239 , 180–187.