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Cubic Spline Solutions For Two-Point Boundary Value Problems Using Quarter-Sweep SOR Method

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Abstract

In this paper, iterative methods particularly a family of Successive Over Relaxation (SOR) methods are used to solve system of linear equations generated from discretization of second-order two-point boundary value problems through cubic spline approaches. For the proposed problems, family of SOR methods namely Full-Sweep SOR (FSSOR), Half-Sweep SOR (HSSOR) and Quarter-Sweep SOR (QSSOR) has been considered to be the generated linear solver. In addition, the formulation and implementation of these three proposed methods were also presented. Comparison among all tested SOR methods were carried out to show their performance.

Keywords: *Quarter-sweep iteration, SOR, Cubic Spline Scheme*

1 Introduction

Presently, boundary value problems of ordinary differential equations play an essential role in many fields. This is due to many applications of sciences and engineering including modelling of chemical reactions and heat transfer governed by these problems. In addition to that, obtaining accurate and fast numerical solution of two-point BVPs is of great importance due to its wide application in scientific and engineering researches. Therefore, many numerical methods intensively have been proposed to solve two-point boundary value problems such as finite difference, finite element and finite volume methods [9], extended ADM (EADM) [14], precise time integration (PTI) method [7], shooting method [34], using Legendre polynomials and function approximation [16], Galerkin and Collocation methods [15] and using interpolation [28]. In this paper, however, discretization schemes based on spline scheme were used to discretize the proposed problems. Precisely, the cubic spline scheme will be taken into account in constructing a cubic spline approximation equation towards two-point boundary value problems.

Actually based on previous studies of spline solutions, the development and analysis of the methods towards two-point boundary value problems have also been discussed by many researchers [4,6,23,24]. These spline schemes are used to discretize two-point boundary value problems and then derive their corresponding spline approximation equations. Then each of these approximation equations yields a large and sparse linear system. Since the attributes of linear systems are large and sparse, iterative methods are the natural options for efficient solutions. As a matter of fact, various iterative methods also have been studied to yield fast numerical solution of linear systems (see Young [36]; Hackbusch [10]; Saad [25]). Apart from those iterative methods, the discovery on the concept of the half-sweep iterative method has been inspired by Abdullah [1] via Explicit Decoupled Group (EDG) iterative method in solving two-dimensional Poisson equations. The main characteristic of this concept is that the half-sweep iterative method includes the reduction technique in order to reduce the computational complexity of linear systems generated from corresponding approximation equations. Following to this concept, further investigations have been extensively conducted in [2,5,13,37] for demonstrating the capability of the half-sweep iteration. Apart from these one-stage iteration concepts, combinations between half-sweep iteration concept with two-stage iterative methods namely HSIADE [29], HSAM [18] and HSGM [20] have also been constructed and implemented for solving linear systems. They pointed out that these their proposed two-stage iterative methods are one of most efficient iterative methods in solving any system of linear equations. Due to the low computational complexity, this concept has been used to derive a family of multigrid methods [21,31]. Consequently, Hassan et al. [11,12] have established a family of FDTD methods using this concept in solving wave propagation problems. Meanwhile Saudi and Sulaiman [26,27] applied to solve the robotic path planning.

Differently from the half-sweep iteration approach, Othman and Abdullah [22] have expanded this approach to initiate the Modified Explicit Group (MEG) method based on the quarter-sweep approach. It is proved that this method is one of most efficient block iterative methods in solving any linear systems as compared with EG and EDG iterative methods. Later, many studies have been conducted to demonstrate the capability of the quarter-sweep iteration [17,19,30,32,33,38].

Due to the advantage of quarter-sweep approach, the main purpose of this paper is to examine the efficiency of family of SOR iterative methods such as FSSOR, HSSOR and QSSOR for solving two-point boundary value problems by using spline approximation equation based on cubic spline scheme.

Consider linear two-point boundary value problem be defined in general form as

$$y'' + l(x)y' + f(x)y = g(x), \ \ x \in [a,b] \tag{1}$$

subject to the boundary conditions

$$y(a) = A_1, \qquad y(b) = A_2$$

where $l(x)$, $f(x)$ and $g(x)$ are continuous on the interval $[a,b]$, through A_i, $i = 1,2$ are finite real constants and $l(x)$, $f(x)$ and $g(x)$ are known functions.

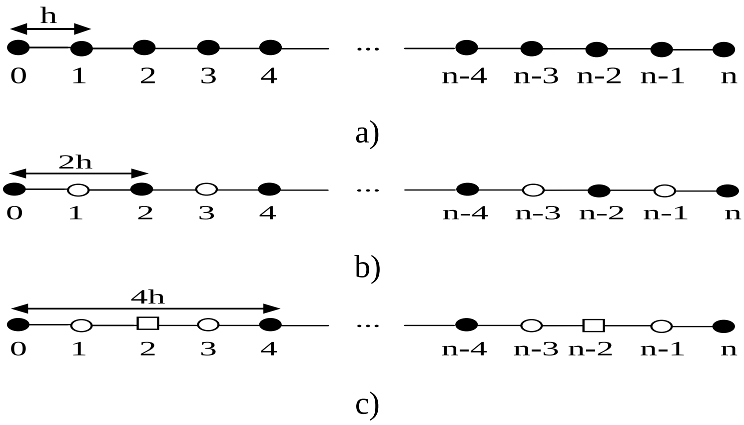

Figure 1 a), b) and c) show distribution of uniform solid node points for the full-, half- and quarter-sweep cases respectively

Based on Figure 1, the implementations of full- and half-sweep iterative methods will be performed to obtain approximate values onto node points of type ● only until convergence test will be figured out. Meanwhile, the approximation solutions for the remaining points can be computed by using direct method [1,13,29,30].

2 Quarter-sweep cubic spline approximation equations

By using full-, half-, and quarter-sweep spline approximation equations for solving problem (1), a finite set of grid points $x_i, i = 0, 1, 2, \cdots, n-1, n$ is established by partitioning the interval $[a,b]$ into $(n+1)$ uniformly subinterval

$$x_i = a + ih, \quad x_0 = a, \quad x_n = b, \quad h = \frac{(b-a)}{n+1}$$

Let $y(x)$ be the exact solution of problem (1) and S_i be an approximation to $y_i = y(x_i)$ determined by the segments of $Q_i(x)$ that are passing through to the points (x_i, S_i) and (x_{i+1}, S_{i+1}). The spline approximation in general form can be expressed as

$$S_k(x) = \sum_{m=0}^{n} C_m (x - x_k)^m \qquad n = 1,2,\ldots,2n-1. \tag{2}$$

where C_m are the coefficient which must be determined in each interval, while suppose that n refer to the order of spline. Then let the cubic polynomial spline from Eq. (2) be defined as $Q_i(x)$ in general form as

$$Q_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \tag{3}$$

for $i = 0p, 1p, 2p, ..., n-p, n$ with a_i, b_i and c_i are constant coefficients. The value of p, which equals to 1, 2 and 4, represents for the full-, half- and quarter-sweep cases respectively. In order to get the expression of three coefficients, a_i, b_i and c_i, the cubic spline segments $Q_i(x)$ in terms of S_{i+p} and M_{i+p} can be manipulated to derive by the following conditions:

$$\left.\begin{array}{ll} \text{i.} & Q_i\left(x_{i+p}\right) = S_{i+p}, \\ \text{ii.} & Q_i\left(x_i\right) = S_{i-p}, \\ \text{iii.} & Q_i''\left(x_{i+p}\right) = M_{i-p}. \end{array}\right\} \tag{4}$$

Then the following expressions can be obtained by the straightforward substitution

$$a_i = \frac{M_i - M_{i-p}}{6hp}, \quad b_i = \frac{M_{i-p}}{2}, \quad c_i = \left[\frac{S_i - S_{i-p}}{ph}\right] - \frac{ph^2}{6}\left[M_i + 2M_{i+p}\right], \tag{5}$$
$$d_i = S_{i-p}$$

with $i = 0p, 1p, 2p, ..., n-p, n$. Now using the expressions in (5) and the continuity conditions of $Q_{i-p}^{(k)}(x_i) = Q_i^{(k)}(x_i)$, where $k = 1, 2$ with respect to $M_i = -l_i y_i' - f_i y_i + g_i$ where $l_i = l(x_i)$, $f_i = f(x_i)$ and $g_i = g(x_i)$, the general cubic spline approximation equations for full-, half-, and quarter-sweep cases can be stated as

$$\alpha_{ip} S_{i-p} + \beta_{ip} S_{ip} + \sigma_{ip} S_{i+p} = F_{ip} \tag{6}$$

where

$$\alpha_{ip} = 6 - 3\frac{(ph)}{2} l_{i-p} + \frac{(ph)}{2} l_{i+p} - 2(ph) l_{i+p} + (ph)^2 f_{i-p},$$
$$\beta_{ip} = 4(ph)^2 f_i - 2(ph) l_{i+p} + 2(ph) l_{i-p} - 12,$$
$$\sigma_{ip} = 6 - \frac{(ph)}{2}\left(l_{i-p} - 2l_i - 3l_{i+p}\right) + (ph)^2 f_{i+p},$$
$$F_{ip} = (ph)^2\left(g_{i-p} + 4g_i + g_{i+p}\right),$$

with $i = 1p, 2p, ..., n-p$. Furthermore, the following linear system formed using cubic spline approximation equations in (6) can be easily shown in matrix form as follows

$$\mathrm{A}\underset{\sim}{S} = \underset{\sim}{F} \tag{7}$$

where,

$$A=\begin{bmatrix} \beta_{1p} & \sigma_{1p} & & & & \\ \alpha_{2p} & \beta_{2p} & \sigma_{2p} & & & \\ & \alpha_{3p} & \beta_{3p} & \sigma_{3p} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{n-2p} & \beta_{n-2p} & \sigma_{n-2p} \\ & & & & \alpha_{n-p} & \beta_{n-p} \end{bmatrix}_{\left(\frac{n}{p}-1\right)\times\left(\frac{n}{p}-1\right)},$$

$$\underset{\sim}{S}=\left[S_{1P},\quad S_{2P},\quad S_{3P},\quad \cdots,\quad S_{n-P}\right]^T,$$

$$\underset{\sim}{F}=\left[F_{1P}-\alpha_{1p},\quad F_{2P},\quad F_{3P},\quad \cdots,\quad F_{n-P}-\beta_{n-p}\right]^T.$$

3 Family of successive over relaxation iterative methods

As above-mentioned in the second section, the coefficient matrix, A of linear systems in Eq. (7) is sparse and large. Consequently, iterative methods are proposed being as the natural options for efficient solutions of sparse linear system. In this section, we show on how to construct FSSOR, HSSOR and QSSOR iterative methods being applied to solve linear systems (7).

To derive the formulation for FSSOR, HSSOR, and QSSOR iterative methods, let the coefficient matrix, *A* in Eq. (7) be decomposed as

$$A=D+L+U \tag{8}$$

where *L*, *D* and *U* are lower triangular, diagonal and upper triangular matrices respectively. By using the decomposition in Eq. (8), therefore, the general scheme for SOR method can be stated as [36]

$$\underset{\sim}{S}^{(k+1)}=(1-\omega)\underset{\sim}{S}^{(k)}+\omega(D+L)^{-1}\left(-U\underset{\sim}{S}^{(k)}+\underset{\sim}{F}\right) \tag{9}$$

where ω and $\underset{\sim}{S}^{(k)}$ represent as a relaxation factor and an unknown vector at the k^th iteration respectively. Therefore by determining values of matrices *D*, *L* and *U* as stated in Eq. (7), the general algorithm of family of SOR iterative methods would be generally described in Algorithm 1.

Algorithm 1: FSSOR, HSSOR and QSSOR schemes

i. Initialize

$$S_i^{(0)}\leftarrow 0,\quad l_i\leftarrow 0,\quad \varepsilon\leftarrow 10^{-10}$$

ii. For $i=1p,2p,3p,\cdots,n-p$, calculate

$$\alpha_{ip} = 6 - 3\frac{(ph)}{2} l_{i-p} + \frac{(ph)}{2} l_{i+p} - 2(ph) l_{i+p} + (ph)^2 f_{i-p},$$
$$\beta_{ip} = 4(ph)^2 f_i - 2(ph) l_{i+p} + 2(ph) l_{i-p} - 12,$$
$$\sigma_{ip} = 6 - \frac{(ph)}{2}\left(l_{i-p} - 2l_i - 3l_{i+p}\right) + (ph)^2 f_{i+p},$$
$$F_{ip} = (ph)^2\left(g_{i-p} + 4g_i + g_{i+p}\right)$$

iii. For $i = 1p, 2p, \cdots, n-p$, calculate

$$S_i^{(k+1)} \leftarrow (1-\omega)S_i^{(k)} + \frac{\omega}{\beta_{ip}}\left(F_i - \alpha_{ip} S_{i-p}^{(k+1)} - \sigma_i S_{i+p}^{(k)}\right)$$

iv. Check the convergence test, $\left|S_i^{(k+1)} - S_i^{(k)}\right| \le \varepsilon$. If yes, go to step (v). Otherwise go back to step (iii)

v. Display approximate solutions.

4 Numerical experiments

In this section, three examples of problems have been proposed to examine their effectiveness of FSSOR, HSSOR and QSSOR methods based on cubic spline approach. For comparison purpose, there are three parameters considered in numerical comparison namely number of iterations, execution time and maximum absolute error. Throughout the numerical simulations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$ and carried out on several different values of *n*.

Example 1 [15]

$$y'' - 4y = 4\cosh(1), \quad x \in [0,1] \tag{11}$$

The exact solution for Eq.(11) is given by

$$y(x) = \cosh(2x-1) - \cosh 1$$

Example 2

$$-\frac{d^2 y}{dx^2} = 9\sin(3x), \quad x \in [0,1] \tag{12}$$

And the exact solution for Eq.(12) is given by

$$y(x) = \cosh(2x-1) - \cosh 1$$

Example 3 [6]

$$\varepsilon y'' = y + \cos^2(\pi x) + 2\varepsilon\pi^2 \cos(2\pi x) \tag{13}$$

And the exact solution Eq.(13) is given by

$$y(x) = \exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})/[1+\exp(-1/\sqrt{\varepsilon})] - \cos^2(\pi x)$$

Following to above three examples, the results of numerical experiments obtained from implementation of the FSSOR, HSSOR, and QSSOR iterative methods have been recorded in Tables 1, 2 and 3, whereas Table 4 indicates depreciation percentage of number of iterations and execution time.

Table 1. Comparison of number of iterations K, the execution time (seconds) and maximum errors for the iterative methods using example 1.

M	FSSOR				HSSOR				QSSOR			
	ω	K	Time	Max Error	ω	K	Time	Max Error	ω	K	Time	Max Error
512	1.985772	1353	0.48	4.90E-7	1.971691	720	0.33	1.94E-6	1.946500	385	0.28	7.75E-6
1024	1.992876	2609	0.63	1.35E-7	1.985772	1353	0.48	4.90E-7	1.971691	720	0.43	1.94E-6
2048	1.996588	5214	1.66	3.43E-8	1.992876	2609	0.94	1.35E-7	1.985772	1353	0.46	4.90E-7
4096	1.998219	9158	3.98	8.29E-8	1.996436	4908	2.04	6.92E-8	1.992876	2609	0.72	1.35E-7
8192	1.999112	17378	16.83	2.34E-7	1.998219	9158	6.15	8.30E-8	1.996436	4908	2.06	6.92E-8

Table 2. Comparison of number of iterations K, the execution time (seconds) and maximum errors for the iterative methods using example 2.

M	FSSOR				HSSOR				QSSOR			
	ω	K	Time	Max Error	ω	K	Time	Max Error	ω	K	Time	Max Error
512	1.98794	1927	0.72	2.65E-6	1.976180	956	0.43	1.06E-5	1.952750	467	0.24	4.24E-5
1024	1.99391	3849	1.11	6.64E-7	1.987943	1927	0.53	2.65E-6	1.976180	956	0.38	1.06E-5
2048	1.99693	7653	2.07	1.70E-7	1.993912	3849	0.85	6.64E-7	1.987943	1927	0.62	2.65E-6
4096	1.99845	15164	7.62	5.33E-8	1.996930	7653	2.01	1.69E-7	1.993913	3849	0.98	6.64E-7
8192	1.99921	29962	29.98	5.01E-8	1.998450	15164	9.95	5.28E-8	1.996931	7653	3.31	1.68E-7

Table 3. Comparison of number of iterations K, the execution time (seconds) and maximum errors for the iterative methods using example 3.

M	FSSOR				HSSOR				QSSOR			
	ω	K	Time	Max Error	ω	K	Time	Max Error	ω	K	Tim e	Max Error
512	1.88578	323	0.29	5.83E-5	1.784500	168	0.12	2.34E-4	1.616000	87	0.09	9.38E-4
1024	1.94119	623	0.39	1.46E-5	1.885810	323	0.21	5.83E-5	1.784900	168	0.13	2.34E-4
2048	1.97017	1201	0.54	3.64E-6	1.941194	623	0.39	1.46E-5	1.885810	323	0.26	5.83E-5
4096	1.98498	2312	1.07	9.11E-7	1.970170	1201	0.65	3.64E-6	1.941194	623	0.47	1.46E-5
8192	1.99247	4444	3.95	2.28E-7	1.984982	2312	1.76	9.11E-7	1.970170	1201	1.22	3.65E-6

Table 4. Depreciation percentage of number of iterations and execution time for HSSOR and QSSOR iterative methods compared to FSSOR.

	HSSOR		QSSOR	
	Number of iterations	**Execution time**	**Number of iterations**	**Execution time**
Example 1	46.41– 49.96%	23.81– 63.45%	71.51– 74.05%	31.75– 87.75%
Example 2	49.39– 50.39%	40.27– 73.62%	74.46– 75.77%	65.77– 88.96%
Example 3	47.97– 48.15%	27.24– 58.62%	72.97– 73.10%	51.85– 69.11%

5 Computational complexity analysis

To compare the computational complexity of three proposed iterative methods, we need to calculate an estimation amount of the computational works needed for implementation of FSSOR, HSSOR and QSSOR methods. The computational work is evaluated by analysing arithmetic operation achieved per iteration for each of proposed iterative methods. Based on Algorithm 1, it can be seen that there is $4\left(\left(\frac{n+1}{p}\right)-1\right)$ addition/subtraction (ADD/SUB) and $5\left(\left(\frac{n+1}{p}\right)-1\right)$ Multiplication/division (MUL/DIV) operations in computing a value for each node point in the solution of cubic spline approximation equation. Based on the solid node points in Figure 1 and the order of coefficient matrix, A, the total number of arithmetic operation per iteration for the FSSOR, HSSOR and QSSOR iterative methods in solving Eq. (1) has been recorded in Table 5. Clearly it shows that the computational complexity of HSSOR and QSSOR methods is lesser than FSSOR method.

Table 5. Total number of arithmetic operations per iteration for FSSOR, HSSOR and QSSOR methods

Methods	**Arithmetic Operation**	
	ADD/SUB	**MUL/DIV**
FSSOR	$4(n-1)$	$5(n-1)$
HSSOR	$4\left(\frac{n}{2}-1\right)$	$5\left(\frac{n}{2}-1\right)$
QSSOR	$4\left(\frac{n}{4}-1\right)$	$5\left(\frac{n}{4}-1\right)$

6 Conclusion and Open Problems

The efficiency of the family of SOR iterative methods namely FSSOR, HSSOR and QSSOR with the corresponding cubic spline approximation equations was employed successfully for solving linear two-point boundary value problems. Clearly it can be observed that the general cubic spline approximation equation in Eq.(6) has been used to generate linear system which have been solved by the proposed iterative methods. Through numerical experiments results from Table 4, it clearly shows that QSSOR method managed to converge faster than FSSOR and HSSOR methods. This is because of the QSSOR method has less computational complexity. For future works, quarter-sweep iteration concept should be combine with a family of block [2,3,13,33,37] or two-stage [17,18,19,29,30,32,38] iterative methods to examining the performance of proposed methods in solving the generated linear system based on spline approach. Besides that the the proposed schemes in this work should be extended to solve fourth-order and sixth-order two point boundary value problems [8,35].

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