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Boundary Value Problems For Fractional Differential Equations

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Abstract

In this paper, we establish sufficient conditions for the existence of solutions for a boundary value problem for fractional differential equations of order $0 < \alpha < 1$ in Banach spaces. These results are obtained using Banach contraction fixed point theorem and Scheafer fixed point theorem.

1 Introduction

The fractional differential equations theory is a new branch of mathematics by which many physical phenomena in various fields of science and engineering can be modeled. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [7, 9, 10, 11, 12, 13, 14, 15]. For some papers dealing with these fractional operators, see [4, 5]. It is to note that there has been a significant development in fractional differential equations in recent years, see [1, 3]. We can also cite the papers of El-Sayed [6], Kilbas and Marzan [2], Mainardi [8], Momani and Hadid [16], Momani et al. [17] and the references therein. In this paper, we are concerned with the following problem

$$\begin{aligned} &D^{\alpha}y\left(t\right)=f\left(t,y\left(t\right)\right),t\in J:=[0,T],0<\alpha<1 \\ &ay(0)+by(T)=\textstyle\int_0^T k(\tau)y(\tau)d\tau,\ a,b\in\mathbb{R},a+b\neq 0, \end{aligned} \tag{1}$$

where D^{α} denotes the fractional derivative of order α in the sense of Caputo, and $f : J \times X \to X$ is continuous, such that $(X, ||.||)$ is a Banach space and $C(J, X)$ is the Banach space of all continuous functions from $J \to X$ endowed with a topology of uniform convergence with the norm denoted by $||.||$ and $k(t) \in L^1(\mathbb{R})$.

2 Preliminaries

In the following, we give the necessary notation and basic definitions which will be used in this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm.
$||y|| = \sup_{t \in J} y(t)$.

Definition 2.1:[11, 18] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as

$$\begin{gathered} J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \\ J^0 f(t) = f(t), \end{gathered} \tag{2}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2:[11, 18] The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as

$$D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in N^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{3}$$

Details on Caputo's derivative can be found in [11, 18].
We give also the following lemmas:

Lemma 2.1 *[14] For $\alpha > 0$, the general solution of the fractional differential $D^{\alpha} x = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... c_{n-1} t^{n-1}, \tag{4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, ... n-1, n = [\alpha] + 1$.

Lemma 2.2 *[14] Let $\alpha > 0$, then*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... c_{n-1} t^{n-1}, \tag{5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ... n-1, n = [\alpha] + 1$.

We give also, the following lemma:

Lemma 2.3 *Let $0 < \alpha < 1$. A solution of the the problem (1) is given by:*

$$y(t) = \frac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau \\ +J^{\alpha}f(t,y(t)). \tag{6}$$

Now, let us define the operator $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as follows:

$$Fy(t) = \frac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau \\ +J^{\alpha}f(t,y(t)). \tag{7}$$

3 Main Results

We prove the existence and the uniqueness of a solution for (1), by using the Banach fixed point theorem. The following conditions are essential to prove our results:

$$(H_1) : ||f(t,x) - f(t,y)|| \leq k_1||x-y||; k_1 > 0, x, y \in \mathbb{R}, t \in J,$$

$$||k|| = sup_{t\in J}|k(t)| \leq M < \infty.$$

(H_2) : The function f is continuous.
(H_3) : There exists a positive constant N such that

$$||f(t,x)|| \leq N, t \in J, x \in \mathbb{R}.$$

Our first result is given by:

Theorem 3.1 *Suppose that the conditions* (H_1) *and*

$$k_1\frac{(a+b)T^{\alpha} + MT(\alpha+1) + bT^{\alpha}}{(a+b)\Gamma(\alpha+1)} < 1, \tag{8}$$

are satisfied. Then the boundary value problem (1) *has a unique solution in* $C(J,\mathbb{R})$.

Proof: To prove this theorem, we need to prove that the operator F has a fixed point on $C(J,\mathbb{R})$. So, we shall prove that F is a contraction mapping on

$C(J,\mathbb{R})$.
Let $x, y \in C(J,\mathbb{R})$. Then we can write

$$\begin{aligned} ||F(y)-F(x)|| = ||\frac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau \\ -\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau \\ +J^\alpha f(t,y(t)) - \tfrac{1}{a+b}\int_0^T k(\tau)x(\tau)d\tau \\ +\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau \\ -J^\alpha f(t,x(t))||. \end{aligned} \tag{9}$$

We can estimate (9) as follows:

$$\begin{aligned} ||F(y)-F(x)|| \le \frac{1}{a+b}\int_0^T k(\tau)||y(\tau)-x(\tau)||d\tau \\ +\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}||f(\tau,y(\tau))-f(\tau,x(\tau))||d\tau \\ +\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}||f(\tau,y(\tau))-f(\tau,x(\tau))||d\tau. \end{aligned} \tag{10}$$

By (H_1), we get

$$||F(y)-F(x)|| \le k_1\frac{(a+b)T^\alpha + MT(\alpha+1) + bT^\alpha}{(a+b)\Gamma(\alpha+1)}||y-x||. \tag{11}$$

Using the condition (8), we conclude that F is a contraction mapping. Hence, by Banach fixed point theorem, there exists a unique fixed point $y^* \in C(J,\mathbb{R})$ which is a solution of (1).

Our second result is based on the existence of solution using Scheafer fixed point theorem. We have:

Theorem 3.2 *Suppose that the conditions (H_2) and (H_3) are satisfied and*

$$(a+b) \ge TM. \tag{12}$$

Then the boundary value problem (1) has at least a solution in $C(J,\mathbb{R})$.

Proof: We use Schaefer's fixed point theorem to prove that F has a fixed point on $C(J,\mathbb{R})$. Our proof is be given in four steps.
*Step*1 : F is continuous on $C(J,\mathbb{R})$.

Let y_n be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$. Then for each $t \in J$, we have :

$$\begin{aligned} ||F(y_n)-F(y)|| = ||\frac{1}{a+b}\int_0^T k(\tau)y_n(\tau)d\tau \\ -\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,y_n(\tau))d\tau \\ +J^\alpha f(t,y_n(t)) - \tfrac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau \\ +\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau \\ -J^\alpha f(t,y(t))||. \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} ||F(y_n)-F(y)|| \leq \frac{1}{a+b}\int_0^T k(\tau)||y_n(\tau)-y(\tau)||d\tau \\ +\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}||f(\tau,y_n(\tau))-f(\tau,y(\tau))||d\tau \\ +\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}||f(\tau,y_n(\tau))-f(\tau,y(\tau))||d\tau. \end{aligned} \tag{14}$$

Since f is a continuous function, we have

$$||F(y_n)-F(y)|| \to 0, n \to \infty. \tag{15}$$

$Step 2$: F maps bounded sets into bounded sets in $C(J,\mathbb{R})$. Indeed, it is enough to show that for any $v > 0$, there exists a positive constant m such that for each $y \in B_v = \{y \in C(J,R); ||y|| \leq v\}$, we have $||F(y)|| \leq m$.
It is clear that

$$\begin{aligned} ||Fy(t)|| = ||\frac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau \\ +J^\alpha f(t,y(t))||. \end{aligned} \tag{16}$$

And then,

$$\begin{aligned} ||Fy(t)|| \leq ||\frac{1}{a+b}\int_0^T k(\tau)||y(\tau)||d\tau \\ +\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}||f(\tau,y(\tau))||d\tau \\ +\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha}||f(t,y(t))||d\tau. \end{aligned} \tag{17}$$

Thanks to (H_3), we can write:

$$||Fy(t)|| \leq \frac{Mv}{a+b}\int_0^T d\tau + \frac{bN}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}d\tau$$
$$+\tfrac{N}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha}d\tau, t\in J. \quad (18)$$

Thus,

$$||Fy(t)|| \leq \frac{(a+b)NT^{\alpha}+MTv\Gamma(\alpha+1)+bNT^{\alpha}}{(a+b)\Gamma(\alpha+1)} := m, \quad (19)$$

and consequently,

$$||Fy(t)|| \leq m. \quad (20)$$

$Step3$: F maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1, t_2 \in J; t_1 < t_2$ and let $y \in B_v$. Then, we have

$$||Fy(t_2)-Fy(t_1)|| = ||\frac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau$$
$$-\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau$$
$$+J^{\alpha}f(t_2,y(t_2)) - \tfrac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau \quad (21)$$
$$+\tfrac{b}{(a+b)\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau$$
$$-J^{\alpha}f(t_1,y(t_1)||.$$

Therefore,

$$||Fy(t_2)-Fy(t_1) \leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}||f(\tau,y(\tau))||d\tau$$
$$+\tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}(t_2-\tau)^{\alpha-1}||f(\tau,y(\tau))||d\tau \quad (22)$$
$$\leq \tfrac{N}{\Gamma(\alpha)}\int_0^{t_1}(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}d\tau$$
$$+\tfrac{N}{\Gamma(\alpha)}\int_{t_2}^{t_1}(t_2-\tau)^{\alpha-1}d\tau$$

Thus

$$||Fy(t_2)-Fy(t_1)|| \leq \frac{2N}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} + \frac{N}{\Gamma(\alpha+1)}(t_1^{\alpha}-t_2^{\alpha}). \quad (23)$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Then, as a consequence of Steps 1,2,3 together with the Arzela-Ascoli theorem, we can conclude that F is completely continuous.

$Step 4$: Now, we prove that the set

$$\Omega = \{y \in C(J,\mathbb{R}), y = \lambda F(y), 0 < \lambda < 1\} \tag{24}$$

is bounded.

Let $y \in \Omega$, then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have :

$$y(t) = \lambda\Big[\frac{1}{a+b}\int_0^T k(\tau)y(\tau)d\tau - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1} f(\tau,y(\tau))d\tau$$
$$+\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-\tau)^{\alpha} f(t,y(t))d\tau\Big]. \tag{25}$$

Thanks to (H_3), we can write

$$\frac{1}{\lambda}||y(t)|| \le \lambda\Big[\frac{1}{a+b}\int_0^T k(\tau)||y(\tau)||d\tau - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}||f(\tau,y(\tau))||d\tau$$
$$+\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-\tau)^{\alpha}||f(t,y(t))||d\tau\Big]$$
$$\le \tfrac{MT||y(\tau)||}{a+b} + \tfrac{bNT^{\alpha}}{(a+b)\Gamma(\alpha+1)} + \tfrac{NT^{\alpha}}{\Gamma(\alpha+1)}. \tag{26}$$

Therefore,

$$||y|| \le \frac{\lambda(a+b)}{a+b-\lambda MT}\Big[\frac{bNT^{\alpha}}{(a+b)\Gamma(\alpha+1)} + \frac{NT^{\alpha}}{\Gamma(\alpha+1)}\Big]. \tag{27}$$

By the condition (11), we have

$$||Fy|| < \infty. \tag{28}$$

This shows that the set is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1).

4 Open Problems

At the end, we pose the following problems:
Open Problem 1. Using fractional differential operator of order α in the

sense of Caputo for a continuous function f on $J = [0, T] \times \mathbb{R}$ under what sufficient conditions do Theorem 3.1 and Theorem 3.2 hold for $1 < \alpha < 2$?
Open Problem 2. Is it possible to generalize Theorems 3.1 and Theorem 3.3 for $\alpha, n < \alpha < n + 1$?

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